Unifying a Multitude of Common Fixed Point Theorems in Symmetric Spaces

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Abstract. The aim of this paper is to obtain some new common fixed point theorems for weakly compatible mappings in symmetric spaces satisfying an implicit function. Some illustrative examples to highlight the realized improvements are furnished. Our results generalize and extend some recent results contained in Ali and Imdad [Sarajevo J. Math. 4(17)(2008), 269-285] to symmetric spaces and consequently a host of metrical common fixed theorems are generalized and improved. We state an integral type fixed point theorem in symmetric space. In the process, we also derive a fixed point result on common fixed point in probabilistic symmetric spaces.

1. Introduction and Preliminaries

The celebrated Banach Contraction Principle is indeed the most fundamental result of metrical fixed point theory which is very effectively utilized to establish the existence of solutions of nonlinear Volterra integral equations, Fredholm integral equations, nonlinear integro-differential equations in Banach spaces besides supporting the convergence of algorithms in Computational Mathematics. However, sometimes one may come across situations wherein the full force of metric requirements are not used in the proofs of certain metrical fixed point theorems. Motivated by this fact, Hicks and Rhoades [16] proved some common fixed point theorems in symmetric spaces and showed that a general probabilistic structures admits a compatible symmetric or semi-metric. In 2006, Miheţ [29] pointed out that Hicks and Rhoades [16] have inadvertently used a triangle inequality in their results.

Jungck [24] generalized the idea of weakly commuting pair of mappings due to Sessa [40] by introducing the notion of compatible mappings and showed that compatible pair of mappings commute on the set of coincidence points of the involved mappings. The study of common fixed points for non-compatible mappings is equally interesting due to Pant [32]. In 1996, Jungck [25] introduced the notion of weakly compatible mappings in non-metric spaces. For more details on systematic comparisons and illustrations of these described notions, we refer to Singh and Tomar [41] and Murthy [30]. The study of fixed points

in metric spaces has been an area of vigorous research activity. To mention a few, we refer [3, 4, 36] and references cited therein.

In 2002, Aamri and El Moutawakil [1] introduced the notion of property (E.A) which is a special case of tangential property due to Sastry and Murthy [39]. The results contained in [31] are generalized and improved by Sastry and Murthy [39]. Later on, Liu et al. [27] initiated the notion of common property (E.A) for hybrid pairs of mappings which contained property (E.A). In this continuation, Imdad et al. [22] and Soliman et al. [43] extended the results of Sastry et al. [39] and Pant [31] to symmetric spaces by utilizing the weak compatible property with common property (E.A.). Since the notions of property (E.A) and common property (E.A) always requires the completeness (or closedness) of underlying subspaces for the existence of common fixed point, hence Sintunavarat and Kumam [42] coined the idea of 'common limit range property' which relaxes the requirement of completeness (or closedness) of the underlying subspace. Afterward, Imdad et al. [21] extended the notion of common limit range property to two pairs of self mappings and proved some fixed point theorems in Menger and metric spaces. Most recently, Karapınar et al. [26] utilized the notion of common limit range property and showed that the new notion buys certain typical conditions utilized by Pant [31] upto a pair of mappings on the cast of a relatively more natural absorbing property due to Gopal et al. [15].

In metric fixed point theory, implicit functions are generally utilized to cover several contraction conditions in one go rather than proving a separate theorem for each contraction condition. The first ever attempt to coin an implicit relation can be traced back to Popa [33–35]. In 2008, Ali and Imdad [3] introduced a new class of implicit functions which covers several classes of contraction conditions such as: Ćirić quasi-contractions, generalized contractions, ϕ-type contractions, rational inequalities and various others. Thereafter, many researchers utilized various implicit relations to prove a number of fixed point theorems in different settings.

In 2002, Branciari [8] firstly studied an integral analogue of Banach contraction principle for a pair of self mappings. Since then, a number of fixed point theorems have been established by several mathematicians employing different integral type contraction condition. For details, we refer the reader to [6, 7, 28, 37, 38, 46]. In an interesting paper of Suzuki [44], it is showed that a Meir-Keeler contraction of integral type is still a Meir-Keeler contraction. Turkoglu and Altun [45] established a new class of implicit function and proved an integral type fixed point theorem for weakly compatible mappings with property (E.A) in symmetric spaces.

The object of this manuscript is to prove some common fixed point theorems for two pairs of non-self weakly compatible mappings with common limit range property satisfying an implicit function in symmetric spaces. We furnish some illustrative examples to highlight the superiority of our results over several results existing in the literature. As an extension of our main result, we state some fixed point theorems for five mappings, six mappings and four finite families of mappings in symmetric spaces by using the notion of the pairwise commuting mappings which is studied by Imdad et al. [19]. We derive an integral analogue of our main result. Inspired by the work of Hicks and Rhoades [16], we state a fixed point theorem in probabilistic symmetric space.

The following definitions and results will be needed in the sequel.

Definition 1.1. *A symmetric on a set X is a function $d : X \times X \to [0, \infty)$ satisfying the following conditions:*

1. *$d(x, y) = 0$ if and only if $x = y$ for $x, y \in X$,*
2. *$d(x, y) = d(y, x)$ for all $x, y \in X$.*

Let d be a symmetric on a set X. For $x \in X$ and $\epsilon > 0$, let $\mathcal{B}(x, \epsilon) = \{y \in X : d(x, y) < \epsilon\}$. A topology $\tau(d)$ on X is defined as follows: $U \in \tau(d)$ if and only if for each $x \in U$, there exists an $\epsilon > 0$ such that $\mathcal{B}(x, \epsilon) \subset U$. A subset S of X is a neighbourhood of $x \in X$ if there exists $U \in \tau(d)$ such that $x \in U \subset S$. A symmetric d is a semimetric if for each $x \in X$ and each $\epsilon > 0$, $\mathcal{B}(x, \epsilon)$ is a neighbourhood of x in the topology $\tau(d)$. A symmetric (resp., semimetric) space (X, d) is a topological space whose topology $\tau(d)$ on X is induced by symmetric (resp., semi-metric) d. The difference of a symmetric and a metric comes from the triangle inequality. Since a symmetric space is not essentially Hausdorff, therefore in order to prove fixed point

theorems some additional axioms are required. The following axioms, which are available in Wilson [47], Aliouche [5] and Imdad and Soliman [22], are relevant to this presentation.

From now on symmetric space will be denoted by (X, d) where as a non-empty arbitrary set will be denoted by Y.

(W_3) [47] Given $\{x_n\}$, x and y in X, $\lim_{n\to\infty} d(x_n, x) = 0$ and $\lim_{n\to\infty} d(x_n, y) = 0$ imply $x = y$.

(W_4) [47] Given $\{x_n\}$, $\{y_n\}$ and x in X, $\lim_{n\to\infty} d(x_n, x) = 0$ and $\lim_{n\to\infty} d(x_n, y_n) = 0$ imply $\lim_{n\to\infty} d(y_n, x) = 0$.

(HE) [5] Given $\{x_n\}$, $\{y_n\}$ and x in X, $\lim_{n\to\infty} d(x_n, x) = 0$ and $\lim_{n\to\infty} d(y_n, x) = 0$ imply $\lim_{n\to\infty} d(x_n, y_n) = 0$.

$(1C)$ [12] A symmetric d is said to be 1-continuous if $\lim_{n\to\infty} d(x_n, x) = 0$ implies $\lim_{n\to\infty} d(x_n, y) = d(x, y)$ where $\{x_n\}$ is a sequence in X and $x, y \in X$.

(CC) [12] A symmetric d is said to be continuous if $\lim_{n\to\infty} d(x_n, x) = 0$ and $\lim_{n\to\infty} d(y_n, y) = 0$ imply $\lim_{n\to\infty} d(x_n, y_n) = d(x, y)$ where $\{x_n\}$, $\{y_n\}$ are sequences in X and $x, y \in X$.

Here, it is observed that $(CC) \implies (1C)$, $(W_4) \implies (W_3)$, and $(1C) \implies (W_3)$ but the converse implications are not true. In general, all other possible implications amongst (W_3), $(1C)$, and (HE) are not true. For detailed description, we refer an interesting note of Cho et al. [11] which contained some illustrative examples. However, (CC) implies all the remaining four conditions namely: (W_3), (W_4), (HE) and $(1C)$. Employing these axioms, several authors proved common fixed point theorems in framework of symmetric spaces (see [2, 10, 13, 14, 17, 18, 20, 26, 45]).

Let (A, S) be a pair of self mappings defined on a non-empty set X equipped with a symmetric d. Then for the pair (A, S), we recall some relevant concepts as follows:

Definition 1.2. *[24] A pair (A, S) of self mappings is said to be compatible if $\lim_{n\to\infty} d(ASx_n, SAx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ for some $z \in X$.*

Definition 1.3. *[32] A pair (A, S) of self mappings is said to be non-compatible if there exists at least one sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ for some $z \in X$ but $\lim_{n\to\infty} d(ASx_n, SAx_n)$ is either non-zero or non-existent.*

Definition 1.4. *[25] A pair (A, S) of self mappings is said to be weakly compatible (or partially commuting or coincidentally commuting) if the pair commutes on the set of coincidence points, that is, $Ax = Sx$ for some $x \in X$ implies $ASx = SAx$.*

Definition 1.5. *[2, 39] A pair (A, S) of self mappings is said to satisfy the property (E.A) if there exists a sequence $\{x_n\}$ in X such that*

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z,$$

for some $z \in X$.

Clearly a pair of non-compatible mappings satisfies the property (E.A).

Now we define the following definitions in non-self arena.

Definition 1.6. *[27] Let Y be an arbitrary set and X be a non-empty set equipped with symmetric d. Then the pairs (A, S) and (B, T) of mappings from Y into X are said to share the common property (E.A), if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in Y such that*

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = z,$$

for some $z \in X$.

Definition 1.7. *[42] Let Y be an arbitrary set and X be a non-empty set equipped with symmetric d. Then the pair (A, S) of mappings from Y into X is said to have the common limit range property with respect to the mapping S (denoted by (CLR_S)) if there exists a sequence $\{x_n\}$ in Y such that*

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z,$$

where $z \in S(Y)$.

Definition 1.8. *[21] Let Y be an arbitrary set and X be a non-empty set equipped with symmetric d. Then the pairs (A, S) and (B, T) of mappings from Y into X are said to have the common limit range property (with respect to mappings S and T), often denoted by (CLR_{ST}) if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in Y such that*

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = z,$$

where $z \in S(Y) \cap T(Y)$.

Remark 1.9.   1. *If $A = B$ and $S = T$ then Definition 1.8 implies (CLR_S) property due to Sintunavarat and Kumam [42].*

2. *It is clear that (CLR_{ST}) property implies the common property (E.A) but converse is not true. For this regard, see the following example.*

Example 1.10. *Consider $X = Y = [2, 11]$ equipped with the symmetric $d(x, y) = (x - y)^2$ for all $x, y \in X$ which also satisfies (1C) and (HE). Define self mappings A, B, S and T on X as*

$$A(x) = \begin{cases} 3, & \text{if } x = 2; \\ 5, & \text{if } 2 < x \le 7; \\ \frac{x+1}{2}, & \text{if } x > 7. \end{cases} \quad B(x) = \begin{cases} 6, & \text{if } x = 2; \\ \frac{x+6}{2}, & \text{if } 2 < x \le 7; \\ 7, & \text{if } x > 7. \end{cases}$$

$$S(x) = \begin{cases} 3, & \text{if } x = 2; \\ 2, & \text{if } 2 < x \le 7; \\ \frac{2x+6}{5}, & \text{if } x > 7. \end{cases} \quad T(x) = \begin{cases} 7, & \text{if } x = 2; \\ x + 2, & \text{if } 2 < x \le 7; \\ 8, & \text{if } x > 7. \end{cases}$$

Then $A(X) = \{3\} \cup \left(4, 6\right]$, $B(X) = \left(4, \frac{13}{2}\right] \cup \{7\}$, $S(X) = \{2, 3\} \cup \left(4, \frac{28}{5}\right]$ and $T(X) = (4, 9]$. Now consider two sequences $\{x_n\} = \{7 + \frac{1}{n}\}$ and $\{y_n\} = \{2 + \frac{1}{n}\}$ in X. Then clearly

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = 4 \in X,$$

It is noted here that $4 \notin S(X) \cap T(X)$. Hence the pairs (A, S) and (B, T) do not satisfy the (CLR_{ST}) property while they satisfy the common property (E.A).

Definition 1.11. *Two finite families of self mappings $\{A_i\}_{i=1}^{m}$ and $\{S_k\}_{k=1}^{n}$ of a non-empty set X are said to be pairwise commuting if*

1. *$A_i A_j = A_j A_i$, $i, j \in \{1, 2, \dots, m\}$,*
2. *$S_k S_l = S_l S_k$, $k, l \in \{1, 2, \dots, n\}$,*
3. *$A_i S_k = S_k A_i$, $i \in \{1, 2, \dots, m\}$ and $k \in \{1, 2, \dots, n\}$.*

2. Implicit Function

In this section, we define an implicit function and furnish a variety of examples which include most of the well known contractions of the existing literature besides admitting several new ones. Here it is fascinating to note that some of the presented examples are of nonexpansive type and Lipschitzian type. Here, it may be pointed out that most of the following examples do not meet the requirements of implicit function due to Popa [35]. In order to describe our implicit function, let Ψ be the family of lower semi-continuous functions $\psi : \mathcal{R}_+^6 \to \mathcal{R}$ satisfying the following conditions:

(ψ_1) $\psi(u, 0, u, 0, 0, u) > 0$, for all $u > 0$;

(ψ_2) $\psi(u, 0, 0, u, u, 0) > 0$, for all $u > 0$;

(ψ_3) $\psi(u, u, 0, 0, u, u) > 0$, for all $u > 0$.

Example 2.1. *Define* $\psi(u_1, u_2, u_3, u_4, u_5, u_6) : \mathcal{R}_+^6 \to \mathcal{R}$ *as*

$$\psi(u_1, u_2, \ldots, u_6) = u_1 - k \max \left\{ u_2, u_3, u_4, \frac{u_5 + u_6}{2} \right\}, \tag{1}$$

where $k \in [0, 1)$.

Example 2.2. *Define* $\psi(u_1, u_2, u_3, u_4, u_5, u_6) : \mathcal{R}_+^6 \to \mathcal{R}$ *as*

$$\psi(u_1, u_2, \ldots, u_6) = u_1^2 - a \max \left\{ u_2^2, u_3^2, u_4^2, \frac{u_5^2 + u_6^2}{2} \right\} - L \min \{ u_2 u_3, u_3 u_4, u_4 u_5, u_5 u_6 \}, \tag{2}$$

where $L \geq 0$ *and* $a \in [0, 1)$.

Example 2.3. *Define* $\psi(u_1, u_2, u_3, u_4, u_5, u_6) : \mathcal{R}_+^6 \to \mathcal{R}$ *as*

$$\psi(u_1, u_2, \ldots, u_6) = u_1 - \alpha u_2 - \beta u_3 - \gamma u_4 - \eta (u_5 + u_6), \tag{3}$$

where $\alpha, \beta, \gamma, \eta \geq 0$ *and* $\alpha + \beta + \gamma + 2\eta < 1$.

Example 2.4. *Define* $\psi(u_1, u_2, u_3, u_4, u_5, u_6) : \mathcal{R}_+^6 \to \mathcal{R}$ *as*

$$\psi(u_1, u_2, \ldots, u_6) = u_1^2 - \alpha u_2^2 - \beta u_2 u_3 - \gamma u_2 (u_4 + u_5) - \delta u_2 u_6, \tag{4}$$

where $\alpha, \beta, \gamma, \delta \geq 0$ *and* $\alpha + \gamma + \delta < 1$.

Example 2.5. *Define* $\psi(u_1, u_2, u_3, u_4, u_5, u_6) : \mathcal{R}_+^6 \to \mathcal{R}$ *as*

$$\psi(u_1, u_2, \ldots, u_6) = u_1^3 - k \max \left\{ u_2^3, u_3^3, u_4^3, u_5^3, u_6^3 \right\}, \tag{5}$$

where $k \in [0, 1)$.

Example 2.6. *Define* $\psi(u_1, u_2, u_3, u_4, u_5, u_6) : \mathcal{R}_+^6 \to \mathcal{R}$ *as*

1. $\psi(u_1, u_2, \ldots, u_6) = u_1 - a u_2$, *where* $a \in [0, 1)$;
2. $\psi(u_1, u_2, \ldots, u_6) = u_1 - b(u_3 + u_4)$, *where* $b \in [0, 1)$;
3. $\psi(u_1, u_2, \ldots, u_6) = u_1 - c(u_5 + u_6)$, *where* $c \in \left[0, \frac{1}{2}\right)$.

Example 2.7. *Define* $\psi(u_1, u_2, u_3, u_4, u_5, u_6) : \mathcal{R}_+^6 \to \mathcal{R}$ *as*

$$\psi(u_1, u_2, \ldots, u_6) = \begin{cases} u_1 - a \max \left\{ u_3, u_4, u_2 \left[\dfrac{u_5 + u_6}{u_3 + u_4} \right] \right\}, & \text{if } u_3 + u_4 \neq 0; \\ u_1, & \text{if } u_3 + u_4 = 0, \end{cases} \tag{6}$$

where $a \in [0, 1)$.

Example 2.8. *Define* $\psi(u_1, u_2, u_3, u_4, u_5, u_6) : \mathcal{R}_+^6 \to \mathcal{R}$ *as*

$$\psi(u_1, u_2, \ldots, u_6) = \begin{cases} u_1 - \alpha u_2 - \beta u_2^2 \dfrac{\sqrt{u_4 u_6 + 1}}{u_4 + u_6}, & \text{if } u_4 + u_6 \neq 0; \\ u_1, & \text{if } u_4 + u_6 = 0, \end{cases} \tag{7}$$

where $\alpha, \beta \geq 0$ *and* $\alpha + \beta < 1$.

Example 2.9. *Define* $\psi(u_1, u_2, u_3, u_4, u_5, u_6) : \mathcal{R}_+^6 \to \mathcal{R}$ *as*

$$\psi(u_1, u_2, \ldots, u_6) = \begin{cases} u_1 - \alpha u_2 - \dfrac{\beta u_2 u_4 + \gamma u_5 u_6}{u_4 + u_6}, & \text{if } u_4 + u_6 \neq 0; \\ u_1, & \text{if } u_4 + u_6 = 0, \end{cases} \tag{8}$$

where $\alpha, \beta, \gamma \geq 0$ *and* $\alpha + \gamma < 1$.

Example 2.10. *Define* $\psi(u_1, u_2, u_3, u_4, u_5, u_6) : \mathcal{R}_+^6 \to \mathcal{R}$ *as*

$$\psi(u_1, u_2, \ldots, u_6) = u_1^3 - u_2 u_3 u_4 - u_3 u_4 u_5 - u_4 u_5 u_6. \tag{9}$$

Example 2.11. *Define* $\psi(u_1, u_2, u_3, u_4, u_5, u_6) : \mathcal{R}_+^6 \to \mathcal{R}$ *as*

$$\psi(u_1, u_2, \ldots, u_6) = u_1 - \alpha(u_2 + u_3) - \beta(u_3 + u_4) - \gamma(u_4 + u_5) - \delta(u_5 + u_6), \tag{10}$$

where $\alpha, \beta, \gamma, \delta \geq 0$ *with* $\alpha + \beta + 2\gamma + 2\delta < 1$.

Example 2.12. *Define* $\psi(u_1, u_2, u_3, u_4, u_5, u_6) : \mathcal{R}_+^6 \to \mathcal{R}$ *as*

$$\psi(u_1, u_2, \ldots, u_6) = u_1^2 - \alpha(u_2 + u_3)u_4 - \beta(u_3 + u_4)u_5 - \gamma(u_4 + u_5)u_6, \tag{11}$$

where $\alpha \geq 0$ *and* $\beta, \gamma \in [0, 1)$.

Example 2.13. *Define* $\psi(u_1, u_2, u_3, u_4, u_5, u_6) : \mathcal{R}_+^6 \to \mathcal{R}$ *as*

$$\psi(u_1, u_2, \ldots, u_6) = \begin{cases} u_1 - \alpha u_2 - \beta \dfrac{u_3 u_6 + u_4 u_5}{u_3 + u_5}, & \text{if } u_3 + u_5 \neq 0; \\ u_1, & \text{if } u_3 + u_5 = 0, \end{cases} \tag{12}$$

where $\alpha, \beta \in [0, 1)$.

Example 2.14. *Define* $\psi(u_1, u_2, u_3, u_4, u_5, u_6) : \mathcal{R}_+^6 \to \mathcal{R}$ *as*

$$\psi(u_1, u_2, \ldots, u_6) = u_1^3 - k \max\{u_2 u_3 u_4, u_3 u_4 u_5, u_4 u_5 u_6\}, \tag{13}$$

where $k \in [0, \infty)$.

Example 2.15. *Define* $\psi(u_1, u_2, u_3, u_4, u_5, u_6) : \mathcal{R}_+^6 \to \mathcal{R}$ *as*

$$\psi(u_1, u_2, \ldots, u_6) = u_1^2 - \phi\left(\max\{u_2 u_3, u_3 u_4, u_4 u_5, u_5 u_6\}\right), \tag{14}$$

where $\phi : \mathcal{R}_+ \to \mathcal{R}$ *is an upper semi-continuous function with* $\phi(0) = 0$ *and* $\phi(t) < t$ *for all* $t > 0$.

Example 2.16. *Define* $\psi(u_1, u_2, u_3, u_4, u_5, u_6) : \mathcal{R}_+^6 \to \mathcal{R}$ *as*

$$\psi(u_1, u_2, \ldots, u_6) = \begin{cases} u_1^2 - \dfrac{u_1 u_2 u_3 + u_2 u_3 u_4 + u_3 u_4 u_5 + u_4 u_5 u_6}{u_2 + u_3 + u_4 + u_5 + u_6}, \\ \qquad\qquad \text{if } u_2 + u_3 + u_4 + u_5 + u_6 \neq 0; \\ u_1, \qquad \text{if } u_2 + u_3 + u_4 + u_5 + u_6 = 0. \end{cases} \tag{15}$$

Example 2.17. *Define* $\psi(u_1, u_2, u_3, u_4, u_5, u_6) : \mathcal{R}_+^6 \to \mathcal{R}$ *as*

$$\psi(u_1, u_2, \ldots, u_6) = \begin{cases} u_1 - \dfrac{u_3 u_5 + u_4 u_6}{1 + u_2} - \alpha \dfrac{u_2 u_3 + u_4 u_5}{u_5 + u_6}, & \text{if } u_5 + u_6 \neq 0; \\ u_1, & \text{if } u_5 + u_6 = 0, \end{cases} \tag{16}$$

where $\alpha \in [0, 1)$.

Example 2.18. *Define* $\psi(u_1, u_2, u_3, u_4, u_5, u_6) : \mathcal{R}_+^6 \to \mathcal{R}$ *as*

$$\psi(u_1, u_2, \ldots, u_6) = \begin{cases} u_1 - \alpha \dfrac{u_2^2 + u_3^2}{u_2 + u_3} - \beta \dfrac{u_4^2 + u_5^2}{u_4 + u_5} - \gamma u_6, \\ \\ \qquad\qquad\quad if\ u_2 + u_3 \neq 0,\ u_4 + u_5 \neq 0; \\ u_1, \qquad\quad if\ u_2 + u_3 = 0,\ u_4 + u_5 = 0, \end{cases} \tag{17}$$

where $\alpha, \beta, \gamma \geq 0$ *with* $\alpha + \beta + \gamma < 1$.

Example 2.19. *Define* $\psi(u_1, u_2, u_3, u_4, u_5, u_6) : \mathcal{R}_+^6 \to \mathcal{R}$ *as*

$$\psi(u_1, u_2, \ldots, u_6) = \begin{cases} u_1 - k u_2 - \dfrac{u_3 u_4 + u_5 u_6}{u_3 + u_4} - \dfrac{u_3 u_5 + u_4 u_6}{u_5 + u_6}, \\ \\ \qquad\qquad\quad if\ u_3 + u_4 \neq 0,\ u_5 + u_6 \neq 0; \\ u_1, \qquad\quad if\ u_3 + u_4 = 0,\ u_5 + u_6 = 0, \end{cases} \tag{18}$$

where $0 \leq k < \infty$.

Example 2.20. *Define* $\psi(u_1, u_2, u_3, u_4, u_5, u_6) : \mathcal{R}_+^6 \to \mathcal{R}$ *as*

$$F(u_1, u_2, \ldots, u_6) = u_1^2 - \phi\left(u_2^2, u_3(u_4 + u_5), u_4(u_5 + u_6), u_5(u_4 + u_6), u_6(u_3 + u_5)\right), \tag{19}$$

where $\phi : \mathcal{R}_+^5 \to \mathcal{R}$ *is an upper semi-continuous function such that*

$$\max\{\phi(0, 0, 0, 0, t), \phi(0, 0, t, t, 0), \phi(t, 0, 0, t, t)\} < 0,$$

for each $t > 0$.

Since verification of requirements ψ_1, ψ_2 and ψ_3 for Examples 2.1-2.20 are easy. Hence details are not included. For more extensive collection of contraction conditions one can be referred to Ali and Imdad [3] and references cited therein.

3. Results

A simple and natural way to unify and prove in a simple manner several metrical fixed point theorems is to consider an implicit contraction type condition instead of the usual explicit contractive conditions. Popa [33, 34] initiated this direction of research which produced so far a consistent literature (that cannot be completely cited here) on fixed point, common fixed point, and coincidence point theorems, for both single valued and multi-valued mappings, in various ambient spaces.

We begin with the following observation:

Lemma 3.1. *Let* (X, d) *be a symmetric space wherein* d *satisfies the conditions* (1C) *and* (HE) *whereas* Y *be an arbitrary non-empty set with* $A, B, S, T : Y \to X$. *Suppose that the following hypotheses hold:*

1. *the pair* (A, S) *satisfies the* (CLR$_S$) *property* $\left(\text{or the pair } (B, T) \text{ satisfies the } (CLR_T) \text{ property}\right)$,

2. $A(Y) \subset T(Y)$ $\left(\text{or } B(Y) \subset S(Y)\right)$,

3. $T(Y)$ $\left(\text{or } S(Y)\right)$ *is a closed subset of* X,

4. *for all* $x, y \in Y$ *and* $\psi \in \Psi$

$$\psi\left(d(Ax, By), d(Sx, Ty), d(Ax, Sx), d(By, Ty), d(Sx, By), d(Ty, Ax)\right) \leq 0. \tag{20}$$

Then the pairs (A, S) and (B, T) share the (CLR_{ST}) property.

Proof. Since the pair (A, S) enjoys the (CLR_S) property with respect to mapping S, there exists a sequence $\{x_n\}$ in Y such that

$$\lim_{n \to \infty} d(Ax_n, z) = \lim_{n \to \infty} d(Sx_n, z) = 0,$$

where $z \in S(Y)$. Therefore, by (HE) we have $\lim_{n \to \infty} d(Ax_n, Sx_n) = 0$. Since $A(Y) \subset T(Y)$, for each sequence $\{x_n\}$ there exists a sequence $\{y_n\}$ in Y such that $Ax_n = Ty_n$. Therefore, due to closedness of $T(Y), z \in S(Y) \cap T(Y)$. Thus in all, we have

$$\lim_{n \to \infty} d(Ax_n, z) = \lim_{n \to \infty} d(Sx_n, z) = \lim_{n \to \infty} d(Ty_n, z) = 0,$$

where $z \in S(Y) \cap T(Y)$. Hence by (1C), we have $\lim_{n \to \infty} d(Sx_n, By_n) = d(z, \lim_{n \to \infty} By_n)$ and $\lim_{n \to \infty} d(Ax_n, By_n) = d(z, \lim_{n \to \infty} By_n)$. Now, we show that $\lim_{n \to \infty} d(By_n, z) = 0$. If not, then using inequality (20) with $x = x_n$, $y = y_n$, we have

$$\psi\left(d(Ax_n, By_n), d(Sx_n, Ty_n), d(Ax_n, Sx_n), d(By_n, Ty_n), d(Sx_n, By_n), d(Ty_n, Ax_n)\right) \leq 0,$$

which on making $n \to \infty$, reduces to

$$\psi\left(d(z, \lim_{n \to \infty} By_n), d(z, z), d(z, z), d(\lim_{n \to \infty} By_n, z), d(z, \lim_{n \to \infty} By_n), d(z, z)\right) \leq 0,$$

and so

$$\psi\left(d(z, \lim_{n \to \infty} By_n), 0, 0, d(\lim_{n \to \infty} By_n, z), d(z, \lim_{n \to \infty} By_n), 0\right) \leq 0,$$

a contradiction to (ψ_2). Hence $By_n \to z$ as $n \to \infty$. Hence the pairs (A, S) and (B, T) enjoy the (CLR_{ST}) property. This concludes the proof. $\square$

In general, the converse of Lemma 3.1 is not true (see Example 3.3).

Theorem 3.2. *Let (X, d) be a symmetric space wherein d satisfies the conditions (1C) and (HE) whereas Y be an arbitrary non-empty set with $A, B, S, T : Y \to X$. Suppose that the inequality (20) of Lemma 3.1 holds. If the pairs (A, S) and (B, T) enjoy the (CLR_{ST}) property, then (A, S) and (B, T) have a coincidence point each. Moreover if $Y = X$, then A, B, S and T have a unique common fixed point provided both the pairs (A, S) and (B, T) are weakly compatible.*

Proof. If the pairs (A, S) and (B, T) enjoy the (CLR_{ST}) property, then there exist two sequences $\{x_n\}$ and $\{y_n\}$ in Y such that

$$\lim_{n \to \infty} d(Ax_n, z) = \lim_{n \to \infty} d(Sx_n, z) = \lim_{n \to \infty} d(By_n, z) = \lim_{n \to \infty} d(Ty_n, z) = 0,$$

where $z \in S(Y) \cap T(Y)$. Since $z \in S(Y)$, there exists a point $w \in Y$ such that $Sw = z$. We assert that $Aw = z$. If not, then using inequality (20) with $x = w$ and $y = y_n$, one obtains

$$\psi\left(d(Aw, By_n), d(Sw, Ty_n), d(Aw, Sw), d(By_n, Ty_n), d(Sw, By_n), d(Ty_n, Aw)\right) \leq 0,$$

Taking limit as $n \to \infty$ and using property (1C) and (HE), we get

$$\psi\left(d(Aw, z), 0, d(Aw, z), 0, 0, d(z, Aw)\right) \leq 0,$$

a contradiction to (ψ_1). Then we have $z = Aw = Sw$ which shows that w is a coincidence point of the pair (A, S).

Also $z \in T(Y)$, there exists a point $v \in Y$ such that $Tv = z$. We assert that $Bv = z$. If not, then using inequality (20) with $x = w$, $y = v$, we get

$$\psi\left(d(Aw, Bv), d(Sw, Tv), d(Aw, Sw), d(Bv, Tv), d(Sw, Bv), d(Tv, Aw)\right) \le 0,$$

or, equivalently,

$$\psi\left(d(z, Bv), d(z, z), d(z, z), d(Bv, z), d(z, Bv), d(z, z)\right) \le 0,$$

which implies

$$\psi\left(d(z, Bv), 0, 0, d(Bv, z), d(z, Bv), 0\right) \le 0,$$

a contradiction to (ψ_2). Hence $z = Bv = Tv$ which shows that v is a coincidence point of the pair (B, T). Thus we have $Aw = Sw = Bv = Tv = z$.

Now consider $Y = X$. Since the pair (A, S) is weakly compatible and $Aw = Sw$ hence $Az = ASw = SAw = Sz$. Now we prove that z is a common fixed point of the pair (A, S). Suppose that $Az \ne z$, then using inequality (20) with $x = z$, $y = v$, we have

$$\psi\left(d(Az, Bv), d(Sz, Tv), d(Az, Sz), d(Bv, Tv), d(Sz, Bv), d(Tv, Az)\right) \le 0,$$

and so

$$\psi\left(d(Az, z), d(Az, z), d(Az, Az), d(z, z), d(Az, z), d(z, Az)\right) \le 0,$$

which implies

$$\psi\left(d(Az, z), d(Az, z), 0, 0, d(Az, z), d(z, Az)\right) \le 0,$$

a contradiction to (ψ_3). Hence we have $Az = z = Sz$ which shows that z is a common fixed point of the pair (A, S).

Also the pair (B, T) is weakly compatible and $Bv = Tv$, then $Bz = BTw = TBw = Tz$. If not, then using inequality (20) with $x = w$, $y = z$, we have

$$\psi\left(d(Aw, Bz), d(Sw, Tz), d(Aw, Sw), d(Bz, Tz), d(Sw, Bz), d(Tz, Aw)\right) \le 0,$$

which reduces it to

$$\psi\left(d(z, Bz), d(z, Bz), d(Au, Au), d(Bz, Bz), d(z, Bz), d(Bz, z)\right) \le 0,$$

that is,

$$\psi\left(d(z, Bz), d(z, Bz), 0, 0, d(z, Bz), d(Bz, z)\right) \le 0,$$

a contradiction. Therefore, $Bz = z = Tz$ which shows that z is a common fixed point of the pair (B, T). Hence z is a common fixed point of both the pairs (A, S) and (B, T).

For uniqueness, let us consider that $z'(\ne z)$ be another common fixed point of the mappings A, B, S and T. Then using inequality (20) with $x = z'$, $y = z$, we have

$$\psi\left(d(Az', Bz), d(Sz', Tz), d(Az', Sz'), d(Bz, Tz), d(Sz', Bz), d(Tz, Az')\right) \le 0,$$

and so

$$\psi\left(d(z', z), d(z', z), d(z', z'), d(z, z), d(z', z), d(z, z')\right) \le 0,$$

which reduces it to

$$\psi\left(d(z', z), d(z', z), 0, 0, d(z', z), d(z, z')\right) \le 0,$$

a contradiction. Hence $z' = z$. Thus all the involved mappings A, B, S and T have a unique common fixed point. $\square$

Now, we furnish an illustrative example which demonstrates the validity of the hypotheses and degree of generality of our main result over comparable ones from the existing literature.

Example 3.3. *Let $X = Y = [2, 11)$ equipped with the symmetric $d(x, y) = (x − y)^2$ for all $x, y \in X$ which also satisfies (1C) and (HE). Define the mappings A, B, S and T by*

$$Ax = \begin{cases} 2, & if\ x \in \{2\} \cup (7, 11); \\ 8, & if\ 2 < x \le 7. \end{cases} \quad Bx = \begin{cases} 2, & if\ x \in \{2\} \cup (7, 11); \\ 10, & if\ 2 < x \le 7. \end{cases}$$

$$Sx = \begin{cases} 2, & if\ x = 2; \\ 6, & if\ 2 < x \le 7; \\ x - 5, & if\ 7 < x < 11. \end{cases} \quad Tx = \begin{cases} 2, & if\ x = 2; \\ 9, & if\ 2 < x \le 7; \\ \frac{x+1}{4}, & if\ 7 < x < 11. \end{cases}$$

Then we have $A(X) = \{2, 8\} \not\subseteq [2, 3) \cup \{9\} = T(X)$ and $B(X) = \{2, 10\} \not\subseteq [2, 7) = S(X)$. Consider an implicit function $\psi(u_1, u_2, u_3, u_4, u_5, u_6) : \mathcal{R}_+^6 \to \mathcal{R}$ as

$$\psi(u_1, u_2, \dots, u_6) = \begin{cases} u_1 - ku_2 - \dfrac{u_3 u_4 + u_5 u_6}{u_3 + u_4} - \dfrac{u_3 u_5 + u_4 u_6}{u_5 + u_6}, \\ \qquad\qquad if\ u_3 + u_4 \ne 0,\ u_5 + u_6 \ne 0; \\ u_1, \qquad\qquad if\ u_3 + u_4 = 0,\ u_5 + u_6 = 0, \end{cases}$$

where $0 \le k < \infty$ and $\psi \in \Psi$. If we choose two sequences as $\{x_n\} = \left\{7 + \frac{1}{n}\right\}_{n \in \mathbb{N}}$, $\{y_n\} = \{2\}$ $\left(or\ \{x_n\} = \{2\},\ \{y_n\} = \left\{7 + \frac{1}{n}\right\}_{n \in \mathbb{N}}\right)$, then both the pairs (A, S) and (B, T) enjoy the (CLR_{ST}) property:

$$\lim_{n \to \infty} d(Ax_n, 2) = \lim_{n \to \infty} d(Sx_n, 2) = \lim_{n \to \infty} d(By_n, 2) = \lim_{n \to \infty} d(Ty_n, 2) = 0,$$

where $2 \in S(X) \cap T(X)$. Hence the subspaces $S(X)$ and $T(X)$ are not closed subspaces of X. By a routine calculation, one can verify the inequality (20). Thus all the conditions of Theorem 3.2 are satisfied and 2 is a unique common fixed point of the pairs (A, S) and (B, T) which also remains a point of coincidence as well. Here, one may notice that all the involved mappings are discontinuous at their unique common fixed point 2.

Corollary 3.4. *Let A, B, S and T be self mappings of a symmetric (semi-metric) space (X, d) satisfying (1C) and (HE). Suppose that the conditions (1)-(4) of Lemma 3.1 hold then A, B, S and T have a unique common fixed point provided both the pairs (A, S) and (B, T) are weakly compatible.*

Proof. Owing to Lemma 3.1, it follows that the pairs (A, S) and (B, T) share the (CLR_{ST}) property. Hence, the conditions of Theorem 3.2 are satisfied, and A, B, S and T have a unique common fixed point provided both the pairs (A, S) and (B, T) are weakly compatible. $\square$

It is pointed out that Example 3.3 cannot be obtained using Corollary 3.8, since conditions (2) and (3) of Lemma 3.1 are not fulfilled. We present another example, showing the situation where the conclusion can be reached using Corollary 3.8.

Example 3.5. *In the setting of Example 3.3, replace the mappings A, B, S and T by the following: besides retaining the rest:*

$$Ax = \begin{cases} 2, & if\ x \in \{2\} \cup (7, 11); \\ \frac{15}{2}, & if\ 2 < x \le 7. \end{cases} \quad Bx = \begin{cases} 2, & if\ x \in \{2\} \cup (7, 11); \\ 8, & if\ 2 < x \le 7. \end{cases}$$

$$Sx = \begin{cases} 2, & if\ x = 2; \\ x + 1, & if\ 2 < x \le 7; \\ x - 5, & if\ 7 < x < 11. \end{cases} \quad Tx = \begin{cases} 2, & if\ x = 2; \\ x + \frac{1}{2}, & if\ 2 < x \le 7; \\ \frac{x+1}{4}, & if\ 7 < x < 11, \end{cases}$$

where $A(X) = \left\{2, \frac{15}{2}\right\} \subset \left[2, \frac{15}{2}\right] = T(X)$ and $B(X) = \{2, 8\} \subset [2, 8] = S(X)$. Here $S(X)$ and $T(X)$ are closed subsets of X. Consider an implicit function as defined in Example 3.3. Consider two sequences as $\{x_n\} = \left\{7 + \frac{1}{n}\right\}_{n \in \mathbb{N}}$, $\{y_n\} = \{2\}$ $\left(or \{x_n\} = \{2\}, \{y_n\} = \left\{7 + \frac{1}{n}\right\}_{n \in \mathbb{N}}\right)$, hence the pairs (A, S) and (B, T) satisfy the (CLR_{ST}) property:

$$\lim_{n \to \infty} d(Ax_n, 2) = \lim_{n \to \infty} d(Sx_n, 2) = \lim_{n \to \infty} d(By_n, 2) = \lim_{n \to \infty} d(Ty_n, 2) = 0,$$

where $2 \in S(X) \cap T(X)$. Thus all the conditions of Corollary 3.4 are satisfied and 2 is a unique common fixed point of the pairs (A, S) and (B, T) which also remains a point of coincidence as well.

Notice that Theorem 3.2 is not applicable to this example as both $S(X)$, $T(X)$ are closed subsets of X which demonstrates the situational utility of Corollary 3.4 over Theorem 3.2.

Theorem 3.6. *Let (X, d) be a symmetric space wherein d satisfies the conditions (1C) and (HE) whereas Y be an arbitrary non-empty set with $A, B, S, T : Y \to X$. Suppose that the inequality (20) and the following hypotheses hold:*

1. *the pairs (A, S) and (B, T) satisfy the common property (E.A),*
2. *$S(Y)$ and $T(Y)$ are closed subsets of X.*

Then (A, S) and (B, T) have a coincidence point each. Moreover if $Y = X$, then A, B, S and T have a unique common fixed point provided both the pairs (A, S) and (B, T) are weakly compatible.

Proof. Since the pairs (A, S) and (B, T) enjoy the common property (E.A), there exist two sequences $\{x_n\}$ and $\{y_n\}$ in Y such that

$$\lim_{n \to \infty} d(Ax_n, z) = \lim_{n \to \infty} d(Sx_n, z) = \lim_{n \to \infty} d(By_n, z) = \lim_{n \to \infty} d(Ty_n, z) = 0,$$

for some $z \in X$. Since $S(Y)$ is closed, $\lim_{n \to \infty} Sx_n = z = Su$ for some $u \in Y$. Also $T(Y)$ is closed, then $\lim_{n \to \infty} Ty_n = z = Tv$ for some $v \in Y$. The rest of the proof runs on the lines of the proof of Theorem 3.2. □

Example 3.7. *In the setting of Example 3.3, replace the mappings S and T by the following: besides retaining the rest:*

$$Sx = \begin{cases} 2, & \text{if } x = 2; \\ x + 1, & \text{if } 2 < x \leq 7; \\ x - 5, & \text{if } 7 < x < 11. \end{cases} \quad Tx = \begin{cases} 2, & \text{if } x = 2; \\ 3, & \text{if } 2 < x \leq 7; \\ \frac{x+1}{4}, & \text{if } 7 < x < 11. \end{cases}$$

Consider two sequences $\{x_n\}$ and $\{y_n\}$ as in Example 3.3, one can obtain

$$\lim_{n \to \infty} d(Ax_n, 2) = \lim_{n \to \infty} d(Sx_n, 2) = \lim_{n \to \infty} d(By_n, 2) = \lim_{n \to \infty} d(Ty_n, 2) = 0,$$

where $2 \in X$. Hence both the pairs (A, S) and (B, T) satisfy the common property (E.A). $A(X) = \{2, 8\} \nsubseteq [2, 3] = T(X)$ and $B(X) = \{2, 10\} \nsubseteq [2, 8] = S(X)$, that is, the subspaces $S(X)$ and $T(X)$ are closed subspaces of X. By a routine calculation, one can easily verify the inequality (20). Thus all the conditions of Theorem 3.6 are satisfied and 2 is a unique common fixed point of the pairs (A, S) and (B, T) which also remains a point of coincidence as well. Also, one may notice that all the involved mappings are discontinuous at their unique common fixed point 2.

Corollary 3.8. *The conclusions of Theorem 3.6 remain true if condition (2) of Theorem 3.6 is replaced by the following:*

$(2)'$ *$\overline{A(Y)} \subset T(Y)$ and $\overline{B(Y)} \subset S(Y)$,*

where $\overline{A(Y)}$ and $\overline{B(Y)}$ denote the closure of ranges of the mappings A and B.

Corollary 3.9. *The conclusions of Theorem 3.6 and Corollary 3.8 remain true if the conditions (2) and $(2)'$ are replaced by the following:*

$(2)''$ *$A(Y)$ and $B(Y)$ are closed subsets of X provided $A(Y) \subset T(Y)$ and $B(Y) \subset S(Y)$.*

Example 3.10. *In the setting of Example 3.3, replace the mappings S and T by the following: besides retaining the rest:*

$$Sx = \begin{cases} 2, & \text{if } x = 2; \\ 10, & \text{if } 2 < x \leq 7; \\ x - 5, & \text{if } 7 < x < 11. \end{cases} \quad Tx = \begin{cases} 2, & \text{if } x = 2; \\ 8, & \text{if } 2 < x \leq 7; \\ \frac{x+1}{4}, & \text{if } 7 < x < 11. \end{cases}$$

Clearly, assuming the sequences as defined in Example 3.3, both the pairs (A, S) and (B, T) enjoy the common property (E.A). Also, $A(X) = \{2, 8\} \subset [2, 3) \cup \{8\} = T(X)$ and $B(X) = \{2, 10\} \subset [2, 7) \cup \{10\} = S(X)$. By a routine calculation, one can verify the inequality (20). Thus all the conditions of Corollary 3.8 and Corollary 3.9 are satisfied and 2 is a unique common fixed point of the pairs (A, S) and (B, T) which also remains a point of coincidence as well. Here, it is worth noting that Theorem 3.6 cannot be used in the context of this example as $S(X)$ and $T(X)$ are not closed subsets of X.

Corollary 3.11. *The conclusions of Lemma 3.1, Theorem 3.2, Theorem 3.6, Corollary 3.4, Corollary 3.8 and Corollary 3.9 remain true if inequality (20) is replaced by one of the following contraction conditions: for all $x, y \in X$*

$$d(Ax, By) \leq k \max \left\{ d(Sx, Ty), d(Ax, Sx), d(By, Ty), \frac{d(Sx, By) + d(Ty, Ax)}{2} \right\}, \tag{21}$$

where $k \in [0, 1)$.

$$\begin{aligned} d^2(Ax, By) \quad \leq \quad & a \max \left\{ d^2(Sx, Ty), d^2(Ax, Sx), d^2(By, Ty), \frac{d^2(Sx, By) + d^2(Ty, Ax)}{2} \right\} \\ & -L \min \left\{ \begin{array}{l} d(Sx, Ty)d(Ax, Sx), d(Ax, Sx)d(By, Ty), \\ d(By, Ty)d(Sx, By), d(Sx, By)d(Ty, Ax) \end{array} \right\}, \end{aligned} \tag{22}$$

where $L \geq 0$ and $a \in [0, 1)$.

$$d(Ax, By) \leq \alpha d(Sx, Ty) - \beta d(Ax, Sx) - \gamma d(By, Ty) - \eta(d(Sx, By) + d(Ty, Ax)), \tag{23}$$

where $\alpha, \beta, \gamma, \eta \geq 0$ and $\alpha + \beta + \gamma + 2\eta < 1$.

$$\begin{aligned} d^2(Ax, By) \quad \leq \quad & \alpha d^2(Sx, Ty) - \beta d(Sx, Ty)d(Ax, Sx) - \delta d(Sx, Ty)d(Ty, Ax) \\ & -\gamma d(Sx, Ty)(d(By, Ty) + d(Sx, By)), \end{aligned} \tag{24}$$

where $\alpha, \beta, \gamma, \delta \geq 0$ with $\alpha + \gamma + \delta < 1$.

$$d^3(Ax, By) \leq k \max\{d^3(Sx, Ty), d^3(Ax, Sx), d^3(By, Ty), d^3(Sx, By), d^3(Ty, Ax)\}, \tag{25}$$

where $k \in [0, 1)$.

1. *$d(Ax, By) \leq ad(Sx, Ty)$, where $a \in [0, 1)$,*
2. *$d(Ax, By) \leq b(d(Ax, Sx) + d(By, Ty))$, where $b \in [0, 1)$,*
3. *$d(Ax, By) \leq c(d(Sx, By) + d(Ty, Ax))$, where $c \in \left[0, \frac{1}{2}\right)$.*

$$d(Ax, By) \leq \begin{cases} a \max \left\{ \begin{array}{c} d(Ax, Sx), d(By, Ty), \\ d(Sx, Ty)\left(\dfrac{d(Sx, Ty) + d(Ty, Ax)}{d(Ax, Sx) + d(By, Ty)}\right) \end{array} \right\}, \\ \qquad\qquad\qquad \text{if } d(Ax, Sx) + d(By, Ty) \neq 0; \\ 0, \qquad\qquad\qquad \text{if } d(Ax, Sx) + d(By, Ty) = 0, \end{cases} \tag{26}$$

where $a \in [0, 1)$.

$$d(Ax, By) \leq \begin{cases} \alpha d(Sx, Ty) + \beta d^2(Sx, Ty) \dfrac{\sqrt{d(By, Ty)d(Ty, Ax) + 1}}{d(By, Ty) + d(Ty, Ax)}, \\ \qquad\qquad\qquad\quad \text{if } d(By, Ty) + d(Ty, Ax) \neq 0; \\ 0, \qquad\qquad\quad \text{if } d(By, Ty) + d(Ty, Ax) = 0, \end{cases} \tag{27}$$

where $\alpha, \beta \geq 0$ and $\alpha + \beta < 1$.

$$d(Ax, By) \leq \begin{cases} \alpha d(Sx, Ty) + \dfrac{\beta d(Sx, Ty)d(By, Ty) + \gamma d(Sx, By)d(Ty, Ax)}{d(By, Ty) + d(Ty, Ax)}, \\ \qquad\qquad\qquad\quad \text{if } d(By, Ty) + d(Ty, Ax) \neq 0; \\ 0, \qquad\qquad\quad \text{if } d(By, Ty) + d(Ty, Ax) = 0, \end{cases} \tag{28}$$

where $\alpha, \beta, \gamma \geq 0$ and $\alpha + \gamma < 1$.

$$\begin{aligned} d^3(Ax, By) \quad \leq \quad & d(Sx, Ty)d(Ax, Sx)d(By, Ty) + d(Ax, Sx)d(By, Ty)d(Sx, By) \\ & + d(By, Ty)d(Sx, By)d(Ty, Ax). \end{aligned} \tag{29}$$

$$\begin{aligned} d(Ax, By) \quad \leq \quad & \alpha(d(Sx, Ty) + d(Ax, Sx)) + \beta(d(Ax, Sx) + d(By, Ty)) \\ & + \gamma(d(By, Ty) + d(Sx, By)) + \delta(d(Sx, By) + d(Ty, Ax)) \end{aligned} \tag{30}$$

where $\alpha, \beta, \gamma, \delta \geq 0$ with $\alpha + \beta + \delta < 1$, $\beta + 2\gamma + \delta < 1$ and $\alpha + \gamma + 2\delta < 1$.

$$\begin{aligned} d^2(Ax, By) \quad \leq \quad & \alpha(d(Sx, Ty) + d(Ax, Sx))d(By, Ty) + \beta(d(Ax, Sx) + d(By, Ty)) \\ & d(Sx, By) + \gamma(d(By, Ty) + d(Sx, By))d(Ty, Ax), \end{aligned} \tag{31}$$

where $\alpha \geq 0$ and $\beta, \gamma \in [0, 1)$.

$$d(Ax, By) \leq \begin{cases} \alpha d(Sx, Ty) + \beta \dfrac{d(Ax, Sx)d(Ty, Ax) + d(By, Ty)d(Sx, By)}{d(Ax, Sx) + d(Sx, By)}, \\ \qquad\qquad\qquad\quad \text{if } d(Ax, Sx) + d(Sx, By) \neq 0; \\ 0, \qquad\qquad\quad \text{if } d(Ax, Sx) + d(Sx, By) = 0, \end{cases} \tag{32}$$

$$d^3(Ax, By) \quad \leq \quad k \max \left\{ \begin{array}{l} d(Sx, Ty)d(Ax, Sx)d(By, Ty), \\ d(Ax, Sx)d(By, Ty)d(Sx, By), \\ d(By, Ty)d(Sx, By)d(Ty, Ax) \end{array} \right\}, \tag{33}$$

where $0 \leq k < \infty$.

$$d^2(Ax, By) \quad \leq \quad \phi\left(\max \left\{ \begin{array}{l} d(Sx, Ty)d(Ax, Sx), d(Ax, Sx)d(By, Ty), \\ d(By, Ty)d(Sx, By), d(Sx, By)d(Ty, Ax) \end{array} \right\} \right) \tag{34}$$

where $\phi : \mathcal{R}_+ \to \mathcal{R}$ is an upper semi-continuous function with $\phi(0) = 0$ and $\phi(t) < t$ for all $t > 0$.
Let we denote $P = d(Sx, Ty) + d(Ax, Sx) + d(By, Ty) + d(Sx, By) + d(Ty, Ax)$ in the following contraction condition:

$$d^2(Ax, By) \leq \begin{cases} \dfrac{d(fx, By)d(Sx, Ty)d(Ax, Sx) + d(Sx, Ty)d(Ax, Sx)d(By, Ty)}{d(Sx, Ty) + d(Ax, Sx) + d(By, Ty) + d(Sx, By) + d(Ty, Ax)} \\[2ex] +\dfrac{d(Ax, Sx)d(By, Ty)d(Sx, By) + d(By, Ty)d(Sx, By), d(Ty, Ax)}{d(Sx, Ty) + d(Ax, Sx) + d(By, Ty) + d(Sx, By) + d(Ty, Ax)}, \\[3ex] \qquad\qquad\qquad\qquad\qquad\qquad\quad if\ P \neq 0; \\ 0, \qquad\qquad\qquad\qquad\qquad\quad\ if\ P = 0. \end{cases} \tag{35}$$

Let we denote $Q = d(Sx, By) + d(Ty, Ax)$ in the following contraction condition:

$$d(Ax, By) \leq \begin{cases} \dfrac{d(Ax, Sx)d(Sx, By) + d(By, Ty)d(Ty, Ax)}{1 + d(Sx, Ty)} + \\[2ex] +\alpha \dfrac{d(Sx, Ty)d(Ax, Sx) + d(By, Ty)d(Sx, By)}{d(Sx, By) + d(Ty, Ax)}, \\[3ex] \qquad\qquad\qquad\qquad\qquad if\ Q \neq 0; \\ 0, \qquad\qquad\qquad\qquad\quad if\ Q = 0. \end{cases} \tag{36}$$

Let we denote $P_1 = d(Sx, Ty) + d(Ax, Sx)$ and $P_2 = d(By, Ty) + d(Sx, By)$ in the following contraction condition:

$$d(Ax, By) \leq \begin{cases} \alpha \dfrac{d^2(Sx, Ty) + d^2(Ax, Sx)}{d(Sx, Ty) + d(Ax, Sx)} + \beta \dfrac{d^2(By, Ty)d^2(Sx, By)}{d(By, Ty) + d(Sx, By)} + \gamma d(Ty, Sx), \\[3ex] \qquad\qquad\qquad\qquad\qquad if\ P_1 \neq 0, P_2 \neq 0; \\ 0, \qquad\qquad\qquad\qquad\quad if\ P_1 = 0, P_2 = 0, \end{cases} \tag{37}$$

where $\alpha, \beta, \gamma \geq 0$ with $\alpha + \beta + \gamma < 1$.
Let we denote $Q_1 = d(Ax, Sx) + d(By, Ty)$ and $Q_2 = d(Sx, By) + d(Ty, Ax)$ in the following contraction condition:

$$d(Ax, By) \leq \begin{cases} kd(Sx, Ty) + \dfrac{d(Ax, Sx)d(By, Ty) + d(Sx, By)d(Ty, Ax)}{d(Ax, Sx) + d(By, Ty)} \\[2ex] +\dfrac{d(Ax, Sx)d(Sx, By) + d(By, Ty)d(Ty, Ax)}{d(Sx, By) + d(Ty, Ax)}, \\[3ex] \qquad\qquad\qquad\qquad\qquad if\ Q_1 \neq 0, Q_2 \neq 0; \\ 0, \qquad\qquad\qquad\qquad\quad if\ Q_1 = 0, Q_2 = 0, \end{cases} \tag{38}$$

where $0 \leq k < \infty$.

$$d^2(Ax, By) \leq \phi \begin{pmatrix} d^2(Sx, Ty), d(Ax, Sx)(d(By, Ty) + d(Sx, By)), \\ d(By, Ty)(d(Sx, By) + d(Ty, Ax)), \\ d(Sx, By)(d(By, Ty) + d(Ty, Ax)), \\ d(Ty, Ax)(d(Ax, Sx) + d(Sx, By)) \end{pmatrix} \tag{39}$$

where $\phi : \mathcal{R}_+^5 \to \mathcal{R}$ is an upper semi-continuous function such that

$$\max\{\phi(0, 0, 0, 0, t), \phi(0, 0, t, t, 0), \phi(t, 0, 0, t, t)\} < 0,$$

for each $t > 0$.

Proof. The proof of each inequality (21)-(39) easily follows from Theorem 3.2 in view of Examples 2.1-2.20. □

Remark 3.12. *Corollaries corresponding to contraction conditions* (21) *to* (39) *are new results as these never require any condition on containment of ranges.*

By choosing A, B, S and T suitably, we can deduce corollaries involving two as well as three self mappings. For the sake of naturality, we only derive the following corollary involving a pair of self mappings:

Corollary 3.13. *Let* (X, d) *be a symmetric space wherein d satisfies the conditions* (1C) *and* (HE) *whereas Y be an arbitrary non-empty set with* $A, S : Y \to X$. *Suppose that the following hypotheses hold:*

1. *the pair* (A, S) *enjoys the* (CLR_S) *property,*
2. *for all* $x, y \in Y$ *and* $\psi \in \Psi$

$$\psi\left(d(Ax, Ay), d(Sx, Sy), d(Ax, Sx), d(Ay, Sy), d(Sx, Ay), d(Sy, Ax)\right) \le 0. \tag{40}$$

Then (A, S) *has a coincidence point. Moreover if* $Y = X$, *then A and S have a unique common fixed point provided the pair* (A, S) *is weakly compatible.*

As an application of Theorem 3.2, we have the following result for four finite families of self mappings.

Theorem 3.14. *Let* (X, d) *be a symmetric space wherein d satisfies the conditions* (1C) *and* (HE) *whereas Y be an arbitrary non-empty set. Let* $\{A_1, A_2, \ldots, A_m\}$, $\{B_1, B_2, \ldots, B_p\}$, $\{S_1, S_2, \ldots, S_n\}$ *and* $\{T_1, T_2, \ldots, T_q\}$ *be four finite families with* $A = A_1 A_2 \ldots A_m$, $B = B_1 B_2 \ldots B_p$, $S = S_1 S_2, \ldots S_n$ *and* $T = T_1 T_2 \ldots T_q$ *satisfying condition* (20) *and the pairs* (A, S) *and* (B, T) *satisfy the* (CLR_{ST}) *property. Then the pairs* (A, S) *and* (B, T) *have a coincidence point each.*

Moreover, if $Y = X$, *then* $\{A_i\}_{i=1}^m, \{B_j\}_{j=1}^n, \{S_k\}_{k=1}^n$ *and* $\{T_l\}_{l=1}^q$ *have a unique common fixed point provided the families* $(\{A_i\}, \{S_k\})$ *and* $(\{B_j\}, \{T_l\})$ *commute pairwise wherein* $i \in \{1, 2, \ldots, m\}, k \in \{1, 2, \ldots, n\}, j \in \{1, 2, \ldots, p\}$ *and* $l \in \{1, 2, \ldots, q\}$.

Proof. The proof of this theorem can be completed on the lines of Theorem of Imdad et al. [18, Theorem 2.2]. □

Remark 3.15.

1. *A result similar to Theorem 3.14 can be outlined in respect of Theorem 3.2.*
2. *Theorem 3.14 improves and extends the results of Gopal et al.* [14] *and Imdad and Ali* [17].

Now, we indicate that Theorem 3.14 can be utilized to derive common fixed point theorems for any finite number of mappings. As a sample for five mappings, we can derive the following by setting one family of two members while the remaining three of single members:

Corollary 3.16. *Let* (X, d) *be a symmetric space wherein d satisfies the conditions* (1C) *and* (HE) *whereas Y be an arbitrary non-empty set with* $A, B, R, S, T : Y \to X$. *Suppose that the following hypotheses hold:*

1. *the pairs* (A, SR) *and* (B, T) *share the* $(CLR_{(SR)(T)})$ *property,*
2. *for all* $x, y \in Y$ *and* $\psi \in \Psi$

$$\psi\left(d(Ax, By), d(SRx, Ty), d(Ax, SRx), d(By, Ty), d(SRx, By), d(Ty, Ax)\right) \le 0. \tag{41}$$

Then (A, SR) *and* (B, T) *have a coincidence point each. Moreover, if* $Y = X$, *then* A, B, R, S *and* T *have a unique common fixed point provided both the pairs* (A, SR) *and* (B, T) *commute pairwise, that is,* $AS = SA$, $AR = RA$, $SR = RS$, $BT = TB$.

Similarly, we can derive a common fixed point theorem for six mappings by setting two families of two members while the rest two of single members:

Corollary 3.17. *Let (X, d) be a symmetric space wherein d satisfies the conditions (1C) and (HE) whereas Y be an arbitrary non-empty set with $A, B, H, R, S, T : Y \to X$. Suppose that the following hypotheses hold:*

1. *the pairs (A, SR) and (B, TH) share the $(CLR_{(SR)(TH)})$ property,*
2. *for all $x, y \in Y$ and $\psi \in \Psi$*

$$\psi\left(d(Ax, By), d(SRx, THy), d(Ax, SRx), d(By, THy), d(SRx, By), d(THy, Ax)\right) \leq 0. \tag{42}$$

Then (A, SR) and (B, TH) have a coincidence point each. Moreover, if $Y = X$, then A, B, H, R, S and T have a unique common fixed point provided both the pairs (A, SR) and (B, TH) commute pairwise, that is, $AS = SA$, $AR = RA$, $SR = RS$, $BT = TB$, $BH = HB$ and $TH = HT$.

By setting $A_1 = A_2 = \ldots = A_m = A$, $B_1 = B_2 = \ldots = B_p = B$, $S_1 = S_2 = \ldots = S_n = S$ and $T_1 = T_2 = \ldots = T_q = T$ in Theorem 3.14, we deduce the following:

Corollary 3.18. *Let (X, d) be a symmetric space wherein d satisfies the conditions (1C) and (HE) whereas Y be an arbitrary non-empty set with $A, B, S, T : Y \to X$. Suppose that the following hypotheses hold:*

1. *the pairs (A^m, S^n) and (B^p, T^q) share the $(CLR_{S^n T^q})$ property,*
2. *for all $x, y \in Y$ and $\psi \in \Psi$*

$$\psi\left(\begin{array}{c} d(A^m x, B^p y), d(S^n x, T^q y), d(A^m x, S^n x), \\ d(B^p y, T^q y), d(S^n x, B^p y), d(T^q y, A^m x) \end{array}\right) \leq 0, \tag{43}$$

where m, n, p and q are fixed positive integers.

Moreover, if $Y = X$, then A, B, S and T have a unique common fixed point provided $AS = SA$ and $BT = TB$.

Remark 3.19. *Corollary 3.18 is a slight but partial generalization of Theorem 3.2 as the commutativity requirements (that is, $AS = SA$ and $BT = TB$) in this corollary are relatively stronger as compared to weak compatibility in Theorem 3.2.*

Remark 3.20. *Results similar to Corollary 3.18 can be derived from Theorem 3.2 and Corollary 3.11.*

Now we state and prove an integral analogue of Theorem 3.2 as follows:

Theorem 3.21. *Let (X, d) be a symmetric space wherein d satisfies the conditions (1C) and (HE) whereas Y be an arbitrary non-empty set with $A, B, S, T : Y \to X$. Assume that there exists a Lebesgue integrable function $\varphi : \mathcal{R} \to [0, \infty)$ and a function $G : \mathcal{R}_+^6 \to \mathcal{R}$ such that, for all $x, y \in Y$,*

$$\int_0^{G\left(d(Ax,By),d(Sx,Ty),d(Ax,Sx),d(By,Ty),d(Sx,By),d(Ty,Ax)\right)} \varphi(s)\,ds \leq 0, \tag{44}$$

and for all $u > 0$,

$$\int_0^{G(u,0,u,0,0,u)} \varphi(s)\,ds \leq 0, \tag{45}$$

$$\int_0^{G(u,0,0,u,u,0)} \varphi(s)\,ds \leq 0, \tag{46}$$

$$\int_0^{G(u,u,0,0,u,u)} \varphi(s)\,ds \leq 0. \tag{47}$$

Suppose that the pairs (A, S) and (B, T) satisfy the (CLR_{ST}) property. Then (A, S) and (B, T) have a coincidence point each. Moreover if $Y = X$, then A, B, S and T have a unique common fixed point provided both the pairs (A, S) and (B, T) are weakly compatible.

Proof. The function $\psi : \mathcal{R}_+^6 \to \mathcal{R}$ defined by

$$\psi(u_1, u_2, u_3, u_4, u_5, u_6) = \int_0^{G(u_1, u_2, u_3, u_4, u_5, u_6)} \varphi(s) ds \tag{48}$$

belongs to Ψ for conditions (45)-(47) and so condition (44) is a special case of condition (20). Thus, the result follows immediately from Theorem 3.2. □

4. Fixed Point Results in Probabilistic Symmetric Spaces

A real valued function f on the set of real numbers is called a distribution function if it is nondecreasing, left continuous with $\inf_{t \in \mathcal{R}} f(t) = 0$ and $\sup_{t \in \mathcal{R}} f(t) = 1$.

We denote by $\mathfrak{I}$ the set of all distribution functions defined on the set of real numbers and by $\mathfrak{I}^+ = \{f \in \mathfrak{I}, f(0) = 0\}$.

We shall use the Heaviside distribution function defined by

$$H(t) = \begin{cases} 0, & \text{if } t \leq 0; \\ 1, & \text{if } t > 0. \end{cases}$$

In the sequel, we need the following definitions and results which are given in [16].

Definition 4.1. *A probabilistic symmetric on a non-empty set X is a mapping F from $X \times X$ into $\mathfrak{I}^+$ satisfying the following conditions:*

1. *$F_{x,y}(t) = H(t)$ iff $x = y$,*
2. *$F_{x,y}(t) = F_{y,x}(t)$ for all $x, y \in X$ and $t \in \mathcal{R}$.*

The pair (X, F) is a probabilistic symmetric space. Let F be a probabilistic symmetric on a set X and $\epsilon > 0$, we write $\mathcal{B}(x, \epsilon) = \{y \in X : F_{x,y}(\epsilon) > 1 - \epsilon\}$. A $\mathcal{T}_1$ topology $\tau(F)$ on X is obtained as follows: $U \in \tau(F)$ if for each $x \in U$, there exists $\epsilon > 0$ such that $\mathcal{B}(x, \epsilon) \subset U$. Now $\mathcal{B}(x, \epsilon)$ may not be a $\tau(F)$ neighborhood of x. If it is so, then $\tau(F)$ is said to be topological.

Definition 4.2. *A probabilistic symmetric space (X, F) is complete if for every Cauchy sequence $\{x_n\}$ convergent in X, that is, every sequence such that for all $t > 0$ $n \geq m$, $\lim_{n,m \to \infty} F_{x_n,x_m}(t) = 1$, there exists some $x \in X$ with $\lim_{n \to \infty} F_{x_n,x}(t) = 1$ for all $t > 0$.*

Hicks and Rhoades [16] proved that each probabilistic symmetric space (X, F) admits a compatible symmetric d such that the probabilistic symmetric F is related to the symmetric d. To be precise:

Theorem 4.3. *[16] Let (X, F) be a probabilistic symmetric space. Define $d : X \times X \to [0, 1)$ as*

$$d(x, y) = \begin{cases} 0, & \text{if } y \in U_x(t, t) \text{ for all } t > 0; \\ \sup\{t : y \notin U_x(t, t), 0 < t < 1\}, & \text{otherwise.} \end{cases}$$

Then

1. *$d(x, y) < t$ iff $F_{x,y}(t) > 1 - t$,*
2. *d is a compatible symmetric for $\tau(F)$,*
3. *(X, F) is complete iff (X, d) is S-complete,*
4. *if $\tau(F)$ is topological, d is a semi-metric.*

The conditions (HE) and $(1C)$ for compatible symmetric d are equivalent to the following conditions:

(PHE) [14] For all $t > 0$, $F_{x_n,x}(t) \to 1$ and $F_{y_n,x}(t) \to 1$ imply $F_{x_n,y_n}(t) \to 1$.

(*P1C*) [14] For all $t > 0$, $F_{x_n,x}(t) \to 1$ implies $F_{x_n,y}(t) \to F_{x,y}(t)$ for all $y \in X$.

Remark 4.4. *Let (X, F) be a probabilistic symmetric space and d the compatible symmetric for $\tau(F)$. If (X, F) satisfies the condition*

1. *(HE), then (X, d) satisfies the condition (HE);*
2. *(1C), then (X, d) also satisfies the condition (1C).*

Following Imdad et al. [23], let Θ be the set of all continuous functions $\zeta(u_1, u_2, \ldots, u_6) : [0, 1]^6 \to \mathcal{R}$ satisfying the following conditions:

(ζ_1) $\zeta(u, 1, u, 1, 1, u) < 0$, for all $u \in (0, 1)$,

(ζ_2) $\zeta(u, 1, 1, u, u, 1) < 0$, for all $u \in (0, 1)$,

(ζ_3) $\zeta(u, u, 1, 1, u, u) < 0$, for all $u \in (0, 1)$.

Example 4.5. *[23] Define $\zeta(u_1, u_2, \ldots, u_6) : [0, 1]^6 \to \mathcal{R}$ as*

$$\zeta(u_1, u_2, \ldots, u_6) = u_1 - \sigma(\min\{u_2, u_3, u_4, u_5, u_6\}), \tag{49}$$

where $\sigma : [0, 1] \to [0, 1]$ is increasing and continuous function such that $\sigma(t) > t$ for all $t \in (0, 1)$. Notice that

(ζ_1) $\zeta(u, 1, u, 1, 1, u) = u - \sigma(u) < 0$, *for all $u \in (0, 1)$,*

(ζ_2) $\zeta(u, 1, 1, u, u, 1) = u - \sigma(u) < 0$, *for all $u \in (0, 1)$,*

(ζ_3) $\zeta(u, u, 1, 1, u, u) = u - \sigma(u) < 0$, *for all $u \in (0, 1)$.*

Theorem 4.6. *Let (X, F) be a probabilistic semi-metric space wherein F satisfies the conditions (1C) and (HE) whereas Y be an arbitrary non-empty set with $A, B, S, T : Y \to X$. Suppose that the following hypotheses hold:*

1. *the pairs (A, S) and (B, T) satisfy the (CLR_{ST}) property,*
2. *for all $x, y \in Y$, $t > 0$ and $\zeta \in \Theta$*

$$\zeta\left(F_{Ax,By}(t), F_{Sx,Ty}(t), F_{Ax,Sx}(t), F_{By,Ty}(t), F_{Sx,By}(t), F_{Ty,Ax}(t)\right) \geq 0. \tag{50}$$

Then (A, S) and (B, T) have a coincidence point each. Moreover if $Y = X$, then A, B, S and T have a unique common fixed point provided both the pairs (A, S) and (B, T) are weakly compatible.

Proof. In view of Theorem 6 contained in Gopal et al. [14], one can show that Theorem 4.6 reduces to Theorem 3.2. Hence the result follows. $\square$

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