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# The least squares $\eta$-Hermitian problems of quaternion matrix equation $A^H XA + B^H YB = C$

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**Abstract.** For any $A = A_1 + A_2 j \in \mathbf{Q}^{n\times n}$ and $\eta \in \{i, j, k\}$, denote $A^{\eta H} = -\eta A^H \eta$. If $A^{\eta H} = A$, $A$ is called an $\eta$-Hermitian matrix. If $A^{\eta H} = -A$, $A$ is called an $\eta$-anti-Hermitian matrix. Denote $\eta$-Hermitian matrices and $\eta$-anti -Hermitian matrices by $\eta\mathbf{HQ}^{n\times n}$ and $\eta\mathbf{AQ}^{n\times n}$, respectively.

In this paper, we consider the least squares $\eta$-Hermitian problems of quaternion matrix equation $A^H XA + B^H YB = C$ by using the complex representation of quaternion matrices, the Moore–Penrose generalized inverse and the Kronecker product of matrices. We derive the expressions of the least squares solution with the least norm of quaternion matrix equation $A^H XA + B^H YB = C$ over $[X, Y] \in \eta\mathbf{HQ}^{n\times n} \times \eta\mathbf{HQ}^{k\times k}$, $[X, Y] \in \eta\mathbf{AQ}^{n\times n} \times \eta\mathbf{AQ}^{k\times k}$, and $[X, Y] \in \eta\mathbf{HQ}^{n\times n} \times \eta\mathbf{AQ}^{k\times k}$, respectively.

## 1. Introduction

For convenience, we list some notations as follows:

| | |
|---|---|
| $\mathbf{R}^{m\times n}, \mathbf{C}^{m\times n}$ : | $m \times n$ real matrix set and $m \times n$ complex matrix set, respectively; |
| $\mathbf{SR}^{n\times n}$ : | $n \times n$ real symmetric matrix set; |
| $\mathbf{ASR}^{n\times n}$ : | $n \times n$ real anti-symmetric matrix set; |
| $\mathbf{Q}, \mathbf{Q}^{m\times n}$ : | the set of quaternions and $m \times n$ quaternion matrix set, respectively; |
| ReA : | real part of the complex matrix $A$; |
| ImA : | imaginary part of the complex matrix $A$; |
| $\overline{A}, A^T$ : | conjugate matrix and transpose matrix of $A$, respectively; |
| $A^H$ : | the conjugate transpose matrix of $A$, respectively; |
| $A^+$ : | the Moore-Penrose generalized inverse of $A$; |
| $0, I_n$ : | zero matrix of suitable size and identity matrix of order $n$, respectively; |
| $e_i$ : | the $i$-th column of $I_n$; |
| $A \otimes B$ : | Kronecker product of $A$ and $B$. |

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A quaternion $a$ can be uniquely expressed as $a = a_0 + a_1 i + a_2 j + a_3 k$ with real coefficients $a_0, a_1, a_2, a_3$, and $i^2 = j^2 = k^2 = -1, ij = -ji = k$, and $a$ can be uniquely expressed as $a = c_1 + c_2 j$, where $c_1$ and $c_2$ are complex numbers. The following quaternion involutions of a quaternion $a = a_0 + a_1 i + a_2 j + a_3 k$, defined as [3]

$$\begin{aligned} a^i = -iai &= a_0 + a_1 i - a_2 j - a_3 k, \\ a^j = -jaj &= a_0 - a_1 i + a_2 j - a_3 k, \\ a^k = -kak &= a_0 - a_1 i - a_2 j + a_3 k. \end{aligned}$$

For any $A \in \mathbf{Q}^{m\times n}$, $A$ can be uniquely expressed as $A = A_1 + A_2 j$, where $A_1, A_2 \in \mathbf{C}^{m\times n}$, and $A^H = (\mathrm{Re}A_1)^T - (\mathrm{Im}A_1)^T i - (\mathrm{Re}A_2)^T j - (\mathrm{Im}A_2)^T k$. Thus $A^H = A_1^H - A_2^T j$. The complex representation matrix of $A = A_1 + A_2 j \in \mathbf{Q}^{m\times n}$ is denoted by

$$f(A) = \begin{bmatrix} A_1 & A_2 \\ -\overline{A_2} & \overline{A_1} \end{bmatrix} \in \mathbf{C}^{2m\times 2n}. \tag{1}$$

Notice that $f(A)$ is uniquely determined by $A$. For $A \in \mathbf{Q}^{m\times n}, B \in \mathbf{Q}^{n\times s}$, we have $f(AB) = f(A)f(B)$ (see [37]). We define the inner product: $\langle A, B\rangle = \mathrm{tr}(B^H A)$ for all $A, B \in \mathbf{Q}^{m\times n}$. Then $\mathbf{Q}^{m\times n}$ is a right Hilbert inner product space and the norm of a matrix generated by this inner product is the quaternion matrix Frobenius norm $\|\cdot\|$. For matrix $A \in \mathbb{Q}^{m\times n}$, let $a_i = (a_{1i}, a_{2i}, \ldots, a_{mi})(i = 1, 2, \ldots, n)$, and denote by $\mathrm{vec}(A)$ the vector containing all the entries of matrix A:

$$\mathrm{vec}(A) = (a_1, a_2, \ldots, a_n)^T.$$

**Definition 1.1.** ([3, 5, 21, 35]) *A matrix $A \in \mathbf{Q}^{n\times n}$ is $\eta$-Hermitian if $A^{\eta H} = A$, and a matrix $A \in \mathbf{Q}^{n\times n}$ is $\eta$-anti-Hermitian if $A^{\eta H} = -A$, where $A^{\eta H} = -\eta A^H \eta$, $\eta \in \{i, j, k\}$. $\eta$-Hermitian matrices and $\eta$-anti -Hermitian matrices are denoted by $\eta\mathbf{HQ}^{n\times n}$ and $\eta\mathbf{AQ}^{n\times n}$, respectively.*

Denote the right linear space over the skew field of quaternions

$$\eta\mathbf{HQ}^{n\times n} \times \eta\mathbf{HQ}^{k\times k} = \{[X, Y] | X \in \eta\mathbf{HQ}^{n\times n}, Y \in \eta\mathbf{HQ}^{k\times k}\}.$$

Thus we can define the inner product as follows:

$$\langle [X_1, Y_1], [X_2, Y_2]\rangle = \mathrm{tr}[X_2^H X_1] + \mathrm{tr}[Y_2^H Y_1], \quad [X_i, Y_i] \in \eta\mathbf{HQ}^{n\times n} \times \eta\mathbf{HQ}^{k\times k}, (i = 1, 2).$$

Then $\eta\mathbf{HQ}^{n\times n} \times \eta\mathbf{HQ}^{k\times k}$ is a right Hilbert inner space.

Similarly, $\eta\mathbf{AQ}^{n\times n} \times \eta\mathbf{AQ}^{k\times k}$ and $\eta\mathbf{HQ}^{n\times n} \times \eta\mathbf{AQ}^{k\times k}$ are also the right linear spaces and the right Hilbert inner space over the skew field of quaternions. The associated Frobenius norms of matrix pairs $[X, Y] \in \eta\mathbf{HQ}^{n\times n} \times \eta\mathbf{HQ}^{k\times k}$, $[X, Y] \in \eta\mathbf{AQ}^{n\times n} \times \eta\mathbf{AQ}^{k\times k}$ and $[X, Y] \in \eta\mathbf{HQ}^{n\times n} \times \eta\mathbf{AQ}^{k\times k}$ can be described as follows:

$$\|[X, Y]\| = \langle [X, Y], [X, Y]\rangle^{\frac{1}{2}} = (\mathrm{tr}[X^H X] + \mathrm{tr}[Y^H Y])^{\frac{1}{2}} = (\|X\|^2 + \|Y\|^2)^{\frac{1}{2}}.$$

Many authors have devoted to the study of the real, complex, and quaternion matrix equations such as $AXB = C, AX + XB = C, AXB + CXD = E, (AXB, CXD) = (E, F), AXB + CYD = E$ and $X - A\overline{X}B = C$, and we refer to [4, 6–12, 16–19, 24–36]. For the real matrix equation

$$AXA^T + BYB^T = C, \tag{2}$$

there are many important results about their solutions. For example, Chang and Wang [2] studied the necessary and sufficient conditions and derived and the expressions for the symmetric solutions of matrix equation (2). Liao and Bai [13] studied the least squares symmetric problem of matrix equation (2) by using the canonical correlation decomposition of matrix pairs. Furthermore, Liao and Bai [14] studied the least squares symmetric solution of matrix equation (2) with the least norm by using the singular value decomposition and generalized singular value decomposition. For the complex matrix equation

$$A^H XA + B^H YB = C, \tag{3}$$

Zhang [38] investigated the necessary and sufficient conditions and derived the expressions for the Hermitian nonnegative-definite and positive-definite solutions of matrix equation (3). Recently, different constrained solutions to multi-variables real and quaternion matrix equations are concerned by some authors. See [30, 36] for details.

In this paper, we consider the least squares constrained problems of quaternion matrix equation (3). Our motives are twofold: (i) $\eta\mathbf{HQ}^{n\times n}$ is an important class of matrices applied in widely linear modelling and convergence analysis in statistical signal processing due to the quaternion involution properties (see [20–23] for details). (ii) Motivated by the work mentioned above and the recent increasing interesting in $\eta-$Hermitian matrices, this article can extend the results for the least-squares problems of real matrix equation (3) to the least-squares problem of quaternion matrix equation (3). we describe the related problems as follows.

**Problem I.** Given $A \in \mathbf{Q}^{n\times s}$, $B \in \mathbf{Q}^{k\times s}$, and $C \in \mathbf{Q}^{s\times s}$, let

$$\begin{aligned} H_L &= \{[X,Y] | X \in \eta\mathbf{HQ}^{n\times n}, Y \in \eta\mathbf{HQ}^{k\times k}, \\ &\quad \|A^H XA + B^H YB - C\| = \min_{X_0\in\eta\mathbf{HQ}^{n\times n}, Y_0\in\eta\mathbf{HQ}^{k\times k}} \|A^H X_0 A + B^H Y_0 B - C\|\}. \end{aligned}$$

Find $[X_H, Y_H] \in H_L$ such that

$$\|[X_H, Y_H]\|^2 = \|X_H\|^2 + \|Y_H\|^2 = \min_{[X,Y]\in H_L} (\|X\|^2 + \|Y\|^2). \tag{4}$$

**Problem II.** Given $A \in \mathbf{Q}^{n\times s}$, $B \in \mathbf{Q}^{k\times s}$, and $C \in \mathbf{Q}^{s\times s}$, let

$$\begin{aligned} A_L &= \{[X,Y] | X \in \eta\mathbf{AQ}^{n\times n}, Y \in \eta\mathbf{AQ}^{k\times k}, \\ &\quad \|A^H XA + B^H YB - C\| = \min_{X_0\in\eta\mathbf{AQ}^{n\times n}, Y_0\in\eta\mathbf{AQ}^{k\times k}} \|A^H X_0 A + B^H Y_0 B - C\|\}. \end{aligned}$$

Find $[X_A, Y_A] \in A_L$ such that

$$\|[X_A, Y_A]\|^2 = \|X_A\|^2 + \|Y_A\|^2 = \min_{[X,Y]\in A_L} (\|X\|^2 + \|Y\|^2). \tag{5}$$

**Problem III.** Given $A \in \mathbf{Q}^{n\times s}$, $B \in \mathbf{Q}^{k\times s}$, and $C \in \mathbf{Q}^{s\times s}$, let

$$\begin{aligned} S_L &= \{[X,Y] | X \in \eta\mathbf{HQ}^{n\times n}, Y \in \eta\mathbf{AQ}^{k\times k}, \\ &\quad \|A^H XA + B^H YB - C\| = \min_{X_0\in\eta\mathbf{HQ}^{n\times n}, Y_0\in\eta\mathbf{AQ}^{k\times k}} \|A^H X_0 A + B^H Y_0 B - C\|\}. \end{aligned}$$

Find $[X_H, Y_A] \in S_L$ such that

$$\|[X_H, Y_A]\|^2 = \|X_H\|^2 + \|Y_A\|^2 = \min_{[X,Y]\in S_L} (\|X\|^2 + \|Y\|^2). \tag{6}$$

Our approach to solving the problem is to make use of the complex representation of quaternion matrices, the Moore–Penrose generalized inverse, the Kronecker product of matrices, and the matrix structures of $\eta\mathbf{HQ}^{n\times n}$ and $\eta\mathbf{AQ}^{n\times n}$ in [35, 36], and turns Problems I, II, III into the least squares unconstrained problems of a real matrix equation, respectively.

This paper is organized as follows. In Section 2, we give some preliminary lemmas for the solutions of Problems I, II, III. In Sections 3, 4, and 5, we derive the explicit expression of the solutions of Problems I, II, and III, respectively.

## 2. Preliminary lemmas

In order to study the solution of problems I, II, III, we first introduce the structures of $\eta\mathbf{HQ}^{n\times n}$ and $\eta\mathbf{AQ}^{n\times n}$ and give some preliminary lemmas in this section.

**Definition 2.1.** *For matrix $A \in \mathbf{Q}^{n\times n}$, let $a_1 = (a_{11}, \sqrt{2}a_{21}, \ldots, \sqrt{2}a_{n1})$, $a_2 = (a_{22}, \sqrt{2}a_{32}, \ldots, \sqrt{2}a_{n2})$, $\ldots$, $a_{n-1} = (a_{(n-1)(n-1)}, \sqrt{2}a_{n(n-1)})$, $a_n = a_{nn}$, and denote by $\mathrm{vec}_S(A)$ the following vector:*

$$\mathrm{vec}_S(A) = (a_1, a_2, \ldots, a_{n-1}, a_n)^T \in \mathbf{Q}^{\frac{n(n+1)}{2}}. \tag{7}$$

**Definition 2.2.** *For matrix $B \in \mathbf{Q}^{n\times n}$, let $b_1 = (b_{21}, b_{31}, \ldots, b_{n1})$, $b_2 = (b_{32}, b_{42}, \ldots, b_{n2})$, $\ldots$, $b_{n-2} = (b_{(n-1)(n-2)}, b_{n(n-2)})$, $b_{n-1} = b_{n(n-1)}$, and denote by $\mathrm{vec}_A(B)$ the following vector:*

$$\mathrm{vec}_A(B) = \sqrt{2}(b_1, b_2, \ldots, b_{n-2}, b_{n-1})^T \in \mathbf{Q}^{\frac{n(n-1)}{2}}. \tag{8}$$

**Lemma 2.3.** ([34]) *Suppose $X \in \mathbf{R}^{n\times n}$, then*

$$(i) \quad X \in \mathbf{SR}^{n\times n} \Longleftrightarrow \mathrm{vec}(X) = K_S \mathrm{vec}_S(X), \tag{9}$$

*where $\mathrm{vec}_S(X)$ is represented as (7), and the matrix $K_S \in \mathbf{R}^{n^2\times\frac{n(n+1)}{2}}$ is of the following form*
$K_S = \frac{1}{\sqrt{2}}$

$$\begin{bmatrix} \sqrt{2}e_1 & e_2 & \cdots & e_{n-1} & e_n & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & e_1 & \cdots & 0 & 0 & \sqrt{2}e_2 & e_3 & \cdots & e_{n-1} & e_n & \cdots & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & e_2 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & e_1 & 0 & 0 & 0 & \cdots & e_2 & 0 & \cdots & \sqrt{2}e_{n-1} & e_n & 0 \\ 0 & 0 & \cdots & 0 & e_1 & 0 & 0 & \cdots & 0 & e_2 & \cdots & 0 & e_{n-1} & \sqrt{2}e_n \end{bmatrix},$$

$$(ii) \quad X \in \mathbf{ASR}^{n\times n} \Longleftrightarrow \mathrm{vec}(X) = K_A \mathrm{vec}_A(X), \tag{10}$$

*where $\mathrm{vec}_A(X)$ is represented as (8), and the matrix $K_A \in \mathbf{R}^{n^2\times\frac{n(n-1)}{2}}$ is of the following form*

$$K_A = \frac{1}{\sqrt{2}} \begin{bmatrix} e_2 & e_3 & \cdots & e_{n-1} & e_n & 0 & \cdots & 0 & 0 & \cdots & 0 \\ -e_1 & 0 & \cdots & 0 & 0 & e_3 & \cdots & e_{n-1} & e_n & \cdots & 0 \\ 0 & -e_1 & \cdots & 0 & 0 & -e_2 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & 0 & 0 & & 0 \\ 0 & 0 & \cdots & -e_1 & 0 & 0 & \cdots & -e_2 & 0 & \cdots & e_n \\ 0 & 0 & \cdots & 0 & -e_1 & 0 & \cdots & 0 & -e_2 & \cdots & -e_{n-1} \end{bmatrix}.$$

*Obviously, $K_S^T K_S = I_{\frac{n(n+1)}{2}}$, $K_A^T K_A = I_{\frac{n(n-1)}{2}}$.*

We identify $q \in \mathbf{Q}$ with a complex vector $\vec{q} \in \mathbf{C}^2$, and denote such an identification by the symbol $\cong$, that is,

$$c_1 + c_2 j = q \cong \vec{q} = (c_1, c_2).$$

Similarly, for $A = A_1 + A_2 j \in \mathbf{Q}^{m\times n}$, denote $\Phi_A = (A_1, A_2)$, we have $A \cong \Phi_A$,

$$\|A\| = \|\Phi_A\| = \sqrt{\|\mathrm{Re}A_1\|^2 + \|\mathrm{Im}A_1\|^2 + \|\mathrm{Re}A_2\|^2 + \|\mathrm{Im}A_2\|^2},$$

and $\Phi_{A+B} = \Phi_A + \Phi_B$. Furthermore,

$$\mathrm{vec}(A) = \mathrm{vec}(A_1 + A_2 j) = \mathrm{vec}(A_1) + \mathrm{vec}(A_2) j,$$

thus we have

$$\mathrm{vec}(A) \cong \mathrm{vec}(\Phi_A) = \begin{bmatrix} \mathrm{vec}(A_1) \\ \mathrm{vec}(A_2) \end{bmatrix}$$

and

$$\|\mathrm{vec}(A)\| = \|\mathrm{vec}(\Phi_A)\| = \left\| \begin{bmatrix} \mathrm{vec}(A_1) \\ \mathrm{vec}(A_2) \end{bmatrix} \right\|.$$

We denote $\overrightarrow{A} = (\mathrm{Re}A_1, \mathrm{Im}A_1, \mathrm{Re}A_2, \mathrm{Im}A_2)$,

$$\mathrm{vec}(\overrightarrow{A}) = \begin{bmatrix} \mathrm{vec}(\mathrm{Re}A_1) \\ \mathrm{vec}(\mathrm{Im}A_1) \\ \mathrm{vec}(\mathrm{Re}A_2) \\ \mathrm{vec}(\mathrm{Im}A_2) \end{bmatrix}.$$

Notice that $\|\mathrm{vec}(\Phi_A)\| = \|\mathrm{vec}(\overrightarrow{A})\|$. In particular, for $A = A_1 + A_2 i \in \mathbf{C}^{m\times n}$ with $A_1, A_2 \in \mathbf{R}^{m\times n}$, we have $A \cong \overrightarrow{A} = (A_1, A_2)$, and

$$\mathrm{vec}(A_1) + \mathrm{vec}(A_2) i = \mathrm{vec}(A) \cong \mathrm{vec}(\overrightarrow{A}) = \begin{bmatrix} \mathrm{vec}(A_1) \\ \mathrm{vec}(A_2) \end{bmatrix}.$$

Addition of two quaternion matrices $A = A_1 + A_2 j$ and $B = B_1 + B_2 j$ satisfies

$$(A_1 + B_1) + (A_2 + B_2) j = (A + B) \cong \Phi_A + \Phi_B = (A_1 + B_1, A_2 + B_2),$$

whereas multiplication satisfies

$$AB = (A_1 + A_2 j)(B_1 + B_2 j) = (A_1 B_1 - A_2 \overline{B_2}) + (A_1 B_2 + A_2 \overline{B_1}) j.$$

So $AB \cong \Phi_{AB}$, moreover, $\Phi_{AB}$ can be expressed as

$$\begin{aligned} \Phi_{AB} &= (A_1 B_1 - A_2 \overline{B_2}, A_1 B_2 + A_2 \overline{B_1}) \\ &= (A_1, A_2) \begin{bmatrix} B_1 & B_2 \\ -\overline{B_2} & \overline{B_1} \end{bmatrix} \\ &= \Phi_A f(B). \end{aligned}$$

**Lemma 2.4.** ([36]) *If* $X = X_1 + X_2 j \in \mathbf{Q}^{n\times n}$, *then*

$$X \in \eta\mathbf{HQ}^{n\times n} \Longleftrightarrow \mathrm{vec}(\overrightarrow{X}) = K_{\eta H}^{(n)} \mathrm{vec}_{\eta H}^{(n)}(\overrightarrow{X}), \tag{11}$$

*where*

$$K_{iH}^{(n)} = \begin{bmatrix} K_S & 0 & 0 & 0 \\ 0 & K_A & 0 & 0 \\ 0 & 0 & K_S & 0 \\ 0 & 0 & 0 & K_S \end{bmatrix}, \quad \mathrm{vec}_{iH}^{(n)}(\overrightarrow{X}) = \begin{bmatrix} \mathrm{vec}_S(\mathrm{Re}X_1) \\ \mathrm{vec}_A(\mathrm{Im}X_1) \\ \mathrm{vec}_S(\mathrm{Re}X_2) \\ \mathrm{vec}_S(\mathrm{Im}X_2) \end{bmatrix},$$

$$K_{jH}^{(n)} = \begin{bmatrix} K_S & 0 & 0 & 0 \\ 0 & K_S & 0 & 0 \\ 0 & 0 & K_A & 0 \\ 0 & 0 & 0 & K_S \end{bmatrix}, \quad \mathrm{vec}_{jH}^{(n)}(\overrightarrow{X}) = \begin{bmatrix} \mathrm{vec}_S(\mathrm{Re}X_1) \\ \mathrm{vec}_S(\mathrm{Im}X_1) \\ \mathrm{vec}_A(\mathrm{Re}X_2) \\ \mathrm{vec}_S(\mathrm{Im}X_2) \end{bmatrix},$$

$$K_{kH}^{(n)} = \begin{bmatrix} K_S & 0 & 0 & 0 \\ 0 & K_S & 0 & 0 \\ 0 & 0 & K_S & 0 \\ 0 & 0 & 0 & K_A \end{bmatrix}, \quad \mathrm{vec}_{kH}^{(n)}(\overrightarrow{X}) = \begin{bmatrix} \mathrm{vec}_S(\mathrm{Re}X_1) \\ \mathrm{vec}_S(\mathrm{Im}X_1) \\ \mathrm{vec}_S(\mathrm{Re}X_2) \\ \mathrm{vec}_A(\mathrm{Im}X_2) \end{bmatrix}.$$

**Lemma 2.5.** ([36]) *If $Y = Y_1 + Y_2 j \in \mathbf{Q}^{k\times k}$, then*

$$Y \in \eta\mathbf{AQ}^{k\times k} \Longleftrightarrow \mathrm{vec}(\overrightarrow{Y}) = K_{\eta A}^{(k)}\mathrm{vec}_{\eta A}^{(k)}(\overrightarrow{Y}), \tag{12}$$

*where*

$$K_{iA}^{(k)} = \begin{bmatrix} K_A & 0 & 0 & 0 \\ 0 & K_S & 0 & 0 \\ 0 & 0 & K_A & 0 \\ 0 & 0 & 0 & K_A \end{bmatrix}, \quad \mathrm{vec}_{iA}^{(k)}(\overrightarrow{Y}) = \begin{bmatrix} \mathrm{vec}_A(\mathrm{Re}Y_1) \\ \mathrm{vec}_S(\mathrm{Im}Y_1) \\ \mathrm{vec}_A(\mathrm{Re}Y_2) \\ \mathrm{vec}_A(\mathrm{Im}Y_2) \end{bmatrix},$$

$$K_{jA}^{(k)} = \begin{bmatrix} K_A & 0 & 0 & 0 \\ 0 & K_A & 0 & 0 \\ 0 & 0 & K_S & 0 \\ 0 & 0 & 0 & K_A \end{bmatrix}, \quad \mathrm{vec}_{jA}^{(k)}(\overrightarrow{Y}) = \begin{bmatrix} \mathrm{vec}_A(\mathrm{Re}Y_1) \\ \mathrm{vec}_A(\mathrm{Im}Y_1) \\ \mathrm{vec}_S(\mathrm{Re}Y_2) \\ \mathrm{vec}_A(\mathrm{Im}Y_2) \end{bmatrix},$$

$$K_{kA}^{(k)} = \begin{bmatrix} K_A & 0 & 0 & 0 \\ 0 & K_A & 0 & 0 \\ 0 & 0 & K_A & 0 \\ 0 & 0 & 0 & K_S \end{bmatrix}, \quad \mathrm{vec}_{kA}^{(k)}(\overrightarrow{Y}) = \begin{bmatrix} \mathrm{vec}_A(\mathrm{Re}Y_1) \\ \mathrm{vec}_A(\mathrm{Im}Y_1) \\ \mathrm{vec}_A(\mathrm{Re}Y_2) \\ \mathrm{vec}_S(\mathrm{Im}Y_2) \end{bmatrix}.$$

**Lemma 2.6.** ([34]) *Let $A = A_1 + A_2 j \in \mathrm{Q}^{m\times n}$, $B = B_1 + B_2 j \in \mathrm{Q}^{n\times s}$, and $C = C_1 + C_2 j \in \mathrm{Q}^{s\times t}$ be given. Then*

$$\mathrm{vec}(\Phi_{ABC}) = (f(C)^T \otimes A_1, f(Cj)^H \otimes A_2)\begin{pmatrix} \mathrm{vec}(\Phi_B) \\ \mathrm{vec}(-\Phi_{jBj}) \end{pmatrix}. \tag{13}$$

Let

$$W_n = \begin{bmatrix} I_{n^2} & iI_{n^2} & 0 & 0 \\ 0 & 0 & I_{n^2} & iI_{n^2} \\ I_{n^2} & -iI_{n^2} & 0 & 0 \\ 0 & 0 & I_{n^2} & -iI_{n^2} \end{bmatrix}. \tag{14}$$

We now get the structures of $\mathrm{vec}(\Phi_{A^H XA})$ over $X \in \eta\mathbf{HQ}^{n\times n}$ and $\mathrm{vec}(\Phi_{B^H YB})$ over $Y \in \eta\mathbf{AQ}^{k\times k}$.

**Lemma 2.7.** *Given $A \in \mathbf{Q}^{n\times s}$, let $K_{\eta H}^{(n)}$ and $\mathrm{vec}_{\eta H}^{(n)}(\overrightarrow{X})$ are in the form of* (11)*, and $W_n$ is in the form of* (14)*. Then*

$$\mathrm{vec}(\Phi_{A^H XA}) = (f(A)^T \otimes A_1^H, -f(Aj)^H \otimes A_2^T)W_n K_{\eta H}^{(n)}\mathrm{vec}_{\eta H}^{(n)}(\overrightarrow{X}). \tag{15}$$

*Proof.* For $A = A_1 + A_2 j$, $A^H = A_1^H - A_2^T j$. By (11), (13), and Lemma 2.6, we have

$$\begin{aligned}\mathrm{vec}(\Phi_{A^HXA}) &= (f(A)^T \otimes A_1^H, -f(Aj)^H \otimes A_2^T)\begin{pmatrix}\mathrm{vec}(\Phi_X)\\ \mathrm{vec}(-\Phi_{jXj})\end{pmatrix}\\ &= (f(A)^T \otimes A_1^H, -f(Aj)^H \otimes A_2^T)W_n\mathrm{vec}(\overrightarrow{X})\\ &= (f(A)^T \otimes A_1^H, -f(Aj)^H \otimes A_2^T)W_nK_{\eta H}^{(n)}\mathrm{vec}_{\eta H}^{(n)}(\overrightarrow{X}). \qquad \blacksquare\end{aligned}$$

**Lemma 2.8.** *If $B = B_1 + B_2 j \in Q^{k\times s}$, $K_{\eta A}^{(k)}$ and $\mathrm{vec}_{\eta A}^{(k)}(\overrightarrow{Y})$ are in the form of* (12), *and $W_k$ is in the form of* (14). *Then*

$$\mathrm{vec}(\Phi_{B^HYB}) = (f(B)^T \otimes B_1^H, -f(Bj)^H \otimes B_2^T)W_kK_{\eta A}^{(k)}\mathrm{vec}_{\eta A}^{(k)}(\overrightarrow{Y}). \tag{16}$$

**Lemma 2.9.** ([1]) *The matrix equation $Ax = b$, with $A \in \mathbf{R}^{m\times n}$ and $b \in \mathbf{R}^n$, has a solution $x \in \mathbf{R}^n$ if and only if*

$$AA^+b = b, \tag{17}$$

*in this case it has the general solution*

$$x = A^+b + (I - A^+A)y, \tag{18}$$

*where $y \in \mathbf{R}^n$ is an arbitrary vector.*

**Lemma 2.10.** ([1]) *The least squares solutions of the matrix equation $Ax = b$, with $A \in \mathbf{R}^{m\times n}$ and $b \in \mathbf{R}^n$, can be represented as*

$$x = A^+b + (I - A^+A)y, \tag{19}$$

*where $y \in \mathbf{R}^n$ is an arbitrary vector, and the least squares solution of the matrix equation $Ax = b$ with the least norm is $x = A^+b$.*

## 3. The solution of problem I

Based on our earlier discussions, we now turn our attention to Problem I. The following notations are necessary for deriving the solutions of Problem I. For $A = A_1 + A_2 j \in Q^{n\times s}$, $B = B_1 + B_2 j \in Q^{k\times s}$, $C \in Q^{s\times s}$, set

$$P = [f(A)^T \otimes A_1^H, -f(Aj)^H \otimes A_2^T]W_nK_{\eta H}^{(n)},$$
$$Q = [f(B)^T \otimes B_1^H, -f(Bj)^H \otimes B_2^T]W_kK_{\eta H}^{(k)}.$$

Let

$$T_1 = [\mathrm{Re}P, \mathrm{Re}Q], \quad T_2 = [\mathrm{Im}P, \mathrm{Im}Q], \quad e = \begin{bmatrix}\mathrm{vec}(\mathrm{Re}\Phi_C)\\ \mathrm{vec}(\mathrm{Im}\Phi_C)\end{bmatrix}, \tag{20}$$

and

$$\begin{aligned}R &= (I_{2n^2+n+2k^2+k} - T_1^+T_1)T_2^T,\\ Z &= (I_{2s^2} + (I_{2s^2} - R^+R)T_2T_1^+T_1^{+T}T_2^T(I_{2s^2} - R^+R))^{-1},\\ H &= R^+ + (I_{2s^2} - R^+R)ZT_2T_1^+T_1^{+T}(I_{2n^2+n+2k^2+k} - T_2^TR^+),\\ S_{11} &= I_{2s^2} - T_1T_1^+ + T_1^{+T}T_2^TZ(I_{2s^2} - R^+R)T_2T_1^+,\\ S_{12} &= -T_1^{+T}T_2^T(I_{2s^2} - R^+R)Z,\\ S_{22} &= (I_{2s^2} - R^+R)Z.\end{aligned}$$

From the results in [15], we have

$$\begin{bmatrix} T_1 \\ T_2 \end{bmatrix}^+ = (T_1^+ - H^T T_2 T_1^+, H^T), \quad \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}^+ \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = T_1^+ T_1 + RR^+,$$

$$I_{2n^2+n+2k^2+k} - \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}^+ = \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{bmatrix}.$$

**Theorem 3.1.** *Let $A \in \mathbf{Q}^{n\times s}, B \in \mathbf{Q}^{k\times s}$, $C \in \mathbf{Q}^{s\times s}$. Let $M = \mathrm{diag}(K_{\eta H}^{(n)}, K_{\eta H}^{(k)})$, and $T_1, T_2, e$ be as in* (20). *Then*

$$H_L = \left\{ [X, Y] \middle| \begin{bmatrix} \mathrm{vec}(\overrightarrow{X}) \\ \mathrm{vec}(\overrightarrow{Y}) \end{bmatrix} = M[T_1^+ - H^T T_2 T_1^+, H^T]e + M[I_{2n^2+n+2k^2+k} - T_1^+ T_1 - RR^+]z \right\}, \tag{21}$$

*where $z \in \mathbf{R}^{2n^2+n+2k^2+k}$ is an arbitrary vector.*

*Proof.* By Lemmas 2.7, 2.8, we have

$$\begin{aligned}
& \|A^H XA + B^H YB - C\|^2 \\
= \; & \|\Phi_{A^H XA} + \Phi_{B^H YB} - \Phi_C\|^2 \\
= \; & \|\mathrm{vec}(\Phi_{A^H XA}) + \mathrm{vec}(\Phi_{B^H YB}) - \mathrm{vec}(\Phi_C)\|^2 \\
= \; & \|P\mathrm{vec}_{\eta H}^{(n)}(\overrightarrow{X}) + Q\mathrm{vec}_{\eta H}^{(k)}(\overrightarrow{Y}) - \mathrm{vec}(\Phi_C)\|^2 \\
= \; & \|[\mathrm{Re}P + i\mathrm{Im}P]\mathrm{vec}_{\eta H}^{(n)}(\overrightarrow{X}) + [\mathrm{Re}Q + i\mathrm{Im}Q]\mathrm{vec}_{\eta H}^{(k)}(\overrightarrow{Y}) - [\mathrm{vec}(\mathrm{Re}\Phi_C) + i\mathrm{vec}(\mathrm{Im}\Phi_C)]\|^2 \\
= \; & \left\| \begin{bmatrix} \mathrm{Re}P & \mathrm{Re}Q \\ \mathrm{Im}P & \mathrm{Im}Q \end{bmatrix} \begin{bmatrix} \mathrm{vec}_{\eta\mathrm{H}}{}^{(n)}(\overrightarrow{X}) \\ \mathrm{vec}_{\eta\mathrm{H}}{}^{(k)}(\overrightarrow{Y}) \end{bmatrix} - \begin{bmatrix} \mathrm{vec}(\mathrm{Re}\Phi_C) \\ \mathrm{vec}(\mathrm{Im}\Phi_C) \end{bmatrix} \right\|^2 \\
= \; & \left\| \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} \begin{bmatrix} \mathrm{vec}_{\eta\mathrm{H}}{}^{(n)}(\overrightarrow{X}) \\ \mathrm{vec}_{\eta\mathrm{H}}{}^{(k)}(\overrightarrow{Y}) \end{bmatrix} - e \right\|^2.
\end{aligned}$$

By Lemma 2.10, it follows that

$$\begin{bmatrix} \mathrm{vec}_{\eta\mathrm{H}}{}^{(n)}(\overrightarrow{X}) \\ \mathrm{vec}_{\eta\mathrm{H}}{}^{(k)}(\overrightarrow{Y}) \end{bmatrix} = \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}^+ e + \left[ I - \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}^+ \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} \right] z.$$

Thus

$$\begin{bmatrix} \mathrm{vec}(\overrightarrow{X}) \\ \mathrm{vec}(\overrightarrow{Y}) \end{bmatrix} = M(T_1^+ - H^T T_2 T_1^+, H^T)e + M(I - T_1^+ T_1 - RR^+)z.$$

■

By Lemma 2.9 and Theorem 3.1, we get the following conclusion.

**Corollary 3.2.** *The quaternion matrix equation* (3) *has a solution $X \in \eta\mathbf{HQ}^{n\times n}$, $Y \in \eta\mathbf{HQ}^{k\times k}$ if and only if*

$$\begin{bmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{bmatrix} e = 0. \tag{22}$$

*In this case, denote by $H_E$ the solution set of* (3). *Then*

$$H_E = \left\{ [X, Y] \middle| \begin{bmatrix} \mathrm{vec}(\overrightarrow{X}) \\ \mathrm{vec}(\overrightarrow{Y}) \end{bmatrix} = M[(T_1^+ - H^T T_2 T_1^+, H^T)e + (I_{2n^2+n+2k^2+k} - T_1^+ T_1 - RR^+)z] \right\}, \tag{23}$$

*where* $z \in \mathbf{R}^{2n^2+n+2k^2+k}$ *is an arbitrary vector.*

*Furthermore, if* (22) *holds, then the quaternion matrix equation* (3) *has a unique solution* $[X, Y] \in H_E$ *if and only if*

$$\operatorname{rank}\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = 2n^2 + n + 2k^2 + k. \tag{24}$$

*In this case,*

$$H_E = \left\{ [X, Y] \middle| \begin{bmatrix} \operatorname{vec}(\overrightarrow{X}) \\ \operatorname{vec}(\overrightarrow{Y}) \end{bmatrix} = M(T_1^+ - H^T T_2 T_1^+, H^T)e \right\}. \tag{25}$$

**Theorem 3.3.** *Problem I has a unique solution* $[X_H, Y_H] \in H_L$*. This solution satisfies*

$$\begin{bmatrix} \operatorname{vec}(\overrightarrow{X_H}) \\ \operatorname{vec}(\overrightarrow{Y_H}) \end{bmatrix} = M(T_1^+ - H^T T_2 T_1^+, H^T)e. \tag{26}$$

*Proof.* From (21), it is easy to verify that the solution set $H_L$ is nonempty and is a closed convex set. Hence, Problem I has a unique solution $[X_H, Y_H] \in H_L$.

We now prove that the solution $[X_H, Y_H]$ can be expressed as (26). Since

$$\begin{aligned} \min_{[X,Y]\in H_L} (\|[X, Y]\|^2) &= \min_{[X,Y]\in H_L} (\|X\|^2 + \|Y\|^2) \\ &= \min_{[X,Y]\in H_L} (\|\operatorname{vec}(\overrightarrow{X})\|^2 + \|\operatorname{vec}(\overrightarrow{Y})\|^2) \\ &= \min_{[X,Y]\in H_L} \left\| \begin{bmatrix} \operatorname{vec}(\overrightarrow{X}) \\ \operatorname{vec}(\overrightarrow{Y}) \end{bmatrix} \right\|^2, \end{aligned}$$

by Lemma 2.10 and (21), we obtain

$$\begin{bmatrix} \operatorname{vec}(\overrightarrow{X_H}) \\ \operatorname{vec}(\overrightarrow{Y_H}) \end{bmatrix} = M \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}^+ e.$$

Thus,

$$\begin{bmatrix} \operatorname{vec}(\overrightarrow{X_H}) \\ \operatorname{vec}(\overrightarrow{Y_H}) \end{bmatrix} = M(T_1^+ - H^T T_2 T_1^+, H^T)e.$$

∎

**Corollary 3.4.** *The least norm problem*

$$\|[X_H, Y_H]\|^2 = \|X_H\|^2 + \|Y_H\|^2 = \min_{[X,Y]\in H_E} (\|X\|^2 + \|Y\|^2)$$

*has a unique solution* $[X_H, Y_H] \in H_E$ *and* $[X_H, Y_H]$ *can be expressed as* (26).

## 4. The solution of problem II

Based on our earlier discussions, we now turn our attention to Problem II. The following notations are necessary for deriving the solutions of Problem II. For $A = A_1 + A_2 j \in Q^{n\times m}, B = B_1 + B_2 j \in Q^{k\times s}, C \in Q^{s\times s}$, set

$$P' = [f(A)^T \otimes A_1^H, -f(Aj)^H \otimes A_2^T]W_n K_{\eta A}^{(n)},$$
$$Q' = [f(B)^T \otimes B_1^H, -f(Bj)^H \otimes B_2^T]W_k K_{\eta A}^{(k)}.$$

Let

$$Q_1 = [\mathrm{Re}P', \mathrm{Re}Q'], \quad Q_2 = [\mathrm{Im}P', \mathrm{Im}Q'], \quad e = \begin{bmatrix} \mathrm{vec}(\mathrm{Re}\Phi_C) \\ \mathrm{vec}(\mathrm{Im}\Phi_C) \end{bmatrix}, \tag{27}$$

and

$$\begin{aligned}
R_1 &= (I_{2n^2-n+2k^2-k} - Q_1^+ Q_1)Q_2^T, \\
Z_1 &= (I_{2s^2} + (I_{2s^2} - R_1^+ R_1)Q_2 Q_1^+ Q_1^{+T} Q_2^T (I_{2s^2} - R_1^+ R_1))^{-1}, \\
H_1 &= R_1^+ + (I_{2s^2} - R_1^+ R_1)Z_1 Q_2 Q_1^+ Q_1^{+T}(I_{2n^2-n+2k^2-k} - Q_2^T R_1^+), \\
\Delta_{11} &= I_{2s^2} - Q_1 Q_1^+ + Q_1^{+T} Q_2^T Z_1 (I_{2s^2} - R_1^+ R_1) Q_2 Q_1^+, \\
\Delta_{12} &= -Q_1^{+T} Q_2^T (I_{2s^2} - R^+ R) Z, \\
\Delta_{22} &= (I_{2s^2} - R_1^+ R_1) Z_1.
\end{aligned}$$

From the results in [15], we have

$$\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}^+ = (Q_1^+ - H_1^T Q_2 Q_1^+, H_1^T), \quad \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}^+ \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = Q_1^+ Q_1 + R_1 R_1^+,$$
$$I_{2n^2-n+2k^2-k} - \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}^+ = \begin{bmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{12}^T & \Delta_{22} \end{bmatrix}.$$

**Theorem 4.1.** *Let $A \in \mathbf{Q}^{n\times s}, B \in \mathbf{Q}^{k\times s}, C \in \mathbf{Q}^{s\times s}$. Let $M_1 = \mathrm{diag}(K_{\eta A}^{(n)}, K_{\eta A}^{(k)})$, and $Q_1, Q_2, e$ be as in* (27). *Then*

$$A_L = \left\{ [X, Y] \middle| \begin{bmatrix} \mathrm{vec}(\overrightarrow{X}) \\ \mathrm{vec}(\overrightarrow{Y}) \end{bmatrix} = M_1[Q_1^+ - H_1^T Q_2 Q_1^+, H_1^T]e + M_1[I_{2n^2+n+2k^2+k} - T_1^+ T_1 - RR^+]z \right\}, \tag{28}$$

*where $z \in \mathbf{R}^{2n^2-n+2k^2-k}$ is an arbitrary vector.*

**Corollary 4.2.** *The quaternion matrix equation* (3) *has a solution $X \in \eta\mathbf{AQ}^{n\times n}$, $Y \in \eta\mathbf{AQ}^{k\times k}$ if and only if*

$$\begin{bmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{12}^T & \Delta_{22} \end{bmatrix} e = 0. \tag{29}$$

*In this case, denote by $A_E$ the solution set of* (3). *Then*

$$A_E = \left\{ [X, Y] \middle| \begin{bmatrix} \mathrm{vec}(\overrightarrow{X}) \\ \mathrm{vec}(\overrightarrow{Y}) \end{bmatrix} = M_1[(Q_1^+ - H_1^T Q_2 Q_1^+, H_1^T)e + (I_{2n^2-n+2k^2-k} - Q_1^+ Q_1 - R_1 R_1^+)z] \right\}, \tag{30}$$

*where $z \in \mathbf{R}^{2n^2-n+2k^2-k}$ is an arbitrary vector.*

*Furthermore, if* (29) *holds, then the quaternion matrix equation* (3) *has a unique solution* $[X, Y] \in A_E$ *if and only if*

$$\mathrm{rank}\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = 2n^2 - n + 2k^2 - k. \tag{31}$$

*In this case,*

$$A_E = \left\{ [X, Y] \middle| \begin{bmatrix} \mathrm{vec}(\overrightarrow{X}) \\ \mathrm{vec}(\overrightarrow{Y}) \end{bmatrix} = M_1(Q_1^+ - H_1^T Q_2 Q_1^+, H_1^T)e \right\}. \tag{32}$$

**Theorem 4.3.** *Problem II has a unique solution* $[X_A, Y_A] \in A_L$*. This solution satisfies*

$$\begin{bmatrix} \mathrm{vec}(\overrightarrow{X_A}) \\ \mathrm{vec}(\overrightarrow{Y_A}) \end{bmatrix} = M_1(Q_1^+ - H_1^T Q_2 Q_1^+, H_1^T)e. \tag{33}$$

**Corollary 4.4.** *The least norm problem*

$$\|[X_A, Y_A]\|^2 = \|X_A\|^2 + \|Y_A\|^2 = \min_{[X,Y]\in A_E} (\|X\|^2 + \|Y\|^2)$$

*has a unique solution* $[X_A, Y_A] \in A_E$ *and* $[X_A, Y_A]$ *can be expressed as* (33).

## 5. The solution of problem III

In this section, we now turn our attention to Problem III. The following notations are necessary for deriving the solutions of Problem III. For $A = A_1 + A_2 j \in \mathrm{Q}^{n\times s}, B = B_1 + B_2 j \in \mathrm{Q}^{k\times s}, C \in \mathrm{Q}^{s\times s}$, set

$$P'' = [f(A)^T \otimes A_1^H, -f(Aj)^H \otimes A_2^T] W_n K_{\eta H}^{(n)},$$
$$Q'' = [f(B)^T \otimes B_1^H, -f(Bj)^H \otimes B_2^T] W_k K_{\eta A}^{(k)}.$$

Let

$$Q_3 = [\mathrm{Re}P'', \mathrm{Re}Q''], \quad Q_4 = [\mathrm{Im}P'', \mathrm{Im}Q''], \quad e = \begin{bmatrix} \mathrm{vec}(\mathrm{Re}\Phi_C) \\ \mathrm{vec}(\mathrm{Im}\Phi_C) \end{bmatrix}, \tag{34}$$

and

$$\begin{aligned}
R_2 &= (I_{2n^2+n+2k^2-k} - Q_3^+ Q_4) Q_4^T, \\
Z_2 &= (I_{2s^2} + (I_{2s^2} - R_2^+ R_2) Q_4 Q_3^+ Q_3^{+T} Q_4^T (I_{2s^2} - R_2^+ R_2))^{-1}, \\
H_2 &= R_2^+ + (I_{2s^2} - R_2^+ R_2) Z_2 Q_4 Q_3^+ Q_3^{+T} (I_{2n^2+n+2k^2-k} - Q_4^T R_2^+), \\
\Lambda_{11} &= I_{2s^2} - Q_3 Q_3^+ + Q_3^{+T} Q_4^T Z_2 (I_{2s^2} - R_2^+ R_2) Q_4 Q_3^+, \\
\Lambda_{12} &= -Q_3^{+T} Q_4^T (I_{2s^2} - R_2^+ R_2) Z_2, \\
\Lambda_{22} &= (I_{2s^2} - R_1^+ R_1) Z_1.
\end{aligned}$$

From the results in [15], we have

$$\begin{bmatrix} Q_3 \\ Q_4 \end{bmatrix}^+ = (Q_3^+ - H_2^T Q_4 Q_3^+, H_2^T), \quad \begin{bmatrix} Q_3 \\ Q_4 \end{bmatrix}^+ \begin{bmatrix} Q_3 \\ Q_4 \end{bmatrix} = Q_3^+ Q_3 + R_2 R_2^+,$$

$$I_{2n^2+n+2k^2-k} - \begin{bmatrix} Q_3 \\ Q_4 \end{bmatrix} \begin{bmatrix} Q_3 \\ Q_4 \end{bmatrix}^+ = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{12}^T & \Lambda_{22} \end{bmatrix}.$$

**Theorem 5.1.** *Let $A \in \mathbf{Q}^{n\times s}, B \in \mathbf{Q}^{k\times s}$, $C \in \mathbf{Q}^{s\times s}$. Let $M_2 = \mathrm{diag}(K_{\eta H}^{(n)}, K_{\eta A}^{(k)})$, and $Q_3, Q_4, e_2$ be as in* (34). *Then*

$$S_L = \left\{ [X,Y] \left| \begin{bmatrix} \mathrm{vec}(\overrightarrow{X}) \\ \mathrm{vec}(\overrightarrow{Y}) \end{bmatrix} = M_2[Q_3^+ - H_2^T Q_4 Q_3^+, H_2^T]e + M_2[I_{2n^2+n+2k^2-k} - Q_3^+ Q_3 - R_2 R_2^+]z \right. \right\}, \tag{35}$$

*where $z \in \mathbf{R}^{2n^2+n+2k^2-k}$ is an arbitrary vector.*

**Corollary 5.2.** *The quaternion matrix equation* (3) *has a solution $X \in \eta\mathbf{HQ}^{n\times n}$, $Y \in \eta\mathbf{AQ}^{k\times k}$ if and only if*

$$\begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{12}^T & \Lambda_{22} \end{bmatrix} e = 0. \tag{36}$$

*In this case, denote by $S_E$ the solution set of* (3). *Then*

$$S_E = \left\{ [X,Y] \left| \begin{bmatrix} \mathrm{vec}(\overrightarrow{X}) \\ \mathrm{vec}(\overrightarrow{Y}) \end{bmatrix} = M_2[(Q_3^+ - H_2^T Q_4 Q_3^+, H_2^T)e + (I_{2n^2+n+2k^2-k} - Q_3^+ Q_3 - R_2 R_2^+)z] \right. \right\}, \tag{37}$$

*where $z \in \mathbf{R}^{2n^2+n+2k^2-k}$ is an arbitrary vector.*

*Furthermore, if* (36) *holds, then the quaternion matrix equation* (3) *has a unique solution $[X,Y] \in S_E$ if and only if*

$$\mathrm{rank} \begin{bmatrix} Q_3 \\ Q_4 \end{bmatrix} = 2n^2 + n + 2k^2 - k. \tag{38}$$

*In this case,*

$$S_E = \left\{ [X,Y] \left| \begin{bmatrix} \mathrm{vec}(\overrightarrow{X}) \\ \mathrm{vec}(\overrightarrow{Y}) \end{bmatrix} = M_2(Q_3^+ - H_2^T Q_4 Q_3^+, H_2^T)e \right. \right\}. \tag{39}$$

**Theorem 5.3.** *Problem I has a unique solution $[X_H, Y_A] \in S_L$. This solution satisfies*

$$\begin{bmatrix} \mathrm{vec}(\overrightarrow{X_H}) \\ \mathrm{vec}(\overrightarrow{Y_A}) \end{bmatrix} = M_2(Q_3^+ - H_2^T Q_4 Q_3^+, H_2^T)e. \tag{40}$$

**Corollary 5.4.** *The least norm problem*

$$\|[X_H, Y_A]\|^2 = \|X_H\|^2 + \|Y_A\|^2 = \min_{[X,Y]\in S_E} (\|X\|^2 + \|Y\|^2)$$

*has a unique solution $[X_H, Y_A] \in S_E$ and $[X_H, Y_A]$ can be expressed as* (40).

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