Filomat 28:7 (2014), 1425–1436
DOI 10.2298/FIL1407425U

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Oscillation Behavior Of Third-Order Quasilinear Neutral Delay Dynamic Equations On Time Scales

Nadide Utku[a], Mehmet Tamer Şenel[b]

[a]*Institute of Sciences, Erciyes University, 38039, Kayseri, Turkey*
[b]*Department of Mathematics, Faculty of Sciences, Erciyes University, 38039, Kayseri, Turkey*

Abstract. The aim of this paper is to give oscillation criteria for the third-order quasilinear neutral delay dynamic equation

$$\left[r(t)\Big([x(t)+p(t)x(\tau_0(t))]^{\Delta\Delta}\Big)^{\gamma}\right]^{\Delta}+q_1(t)x^{\alpha}(\tau_1(t))+q_2(t)x^{\beta}(\tau_2(t))=0,$$

on a time scale $\mathbb{T}$, where $0<\alpha<\gamma<\beta$. By using a generalized Riccati transformation and integral averaging technique, we establish some new sufficient conditions which ensure that every solution of this equation oscillates or converges to zero.

1. Introduction

In this paper, we deal with the oscillatory behavior of all solutions of the third-order quasilinear neutral delay dynamic equation

$$\left[r(t)\Big([x(t)+p(t)x(\tau_0(t))]^{\Delta\Delta}\Big)^{\gamma}\right]^{\Delta}+q_1(t)x^{\alpha}(\tau_1(t))+q_2(t)x^{\beta}(\tau_2(t))=0,\quad t\in\mathbb{T},\quad t\geq t_0. \tag{1}$$

In the sequel we will assume that the following conditions are satisfied:

(h1) γ,α,β are the ratio of positive odd integers such that $0<\alpha<\gamma<\beta$;
(h2) $r:\mathbb{T}\to(0,\infty)$ is a real valued rd-continuous function on $\mathbb{T}$ and

$$\int_{t_0}^{\infty}\left(\frac{1}{r(t)}\right)^{\frac{1}{\gamma}}\Delta t=\infty,\quad t_0\in\mathbb{T}; \tag{2}$$

(h3) q_1,q_2 are rd-continuous positive functions on $\mathbb{T}$ and $p(t)$ is real valued rd-continuous positive function on $\mathbb{T}$, $0\leq p(t)\leq P<1$;
(h4) $\tau_i:\mathbb{T}\to\mathbb{T}$ satisfied that $\tau_i(t)\leq t$ for $t\in\mathbb{T}$ and $\lim_{t\to\infty}\tau_i(t)=\infty$, for i=0, 1, 2 and there exists a function

---

2010 *Mathematics Subject Classification*. 34K11; 34N05
Keywords. Oscillation, third order quasilinear neutral delay dynamic equation, time scales
Received: 28 June 2013; Accepted: 15 September 2014
Communicated by Jelena Manojlović
Email addresses: nadideutku@gmail.com (Nadide Utku), senel@erciyes.edu.tr (Mehmet Tamer Şenel)

$\tau : \mathbb{T} \to \mathbb{T}$ which satisfies that $\tau(t) \le \tau_1(t), \ \ \tau(t) \le \tau_2(t)$ and $\lim_{t\to\infty} \tau(t) = \infty$.

Define the function by

$$z(t) = x(t) + p(t)x(\tau_0(t)). \tag{3}$$

Furthermore, the equation (1) can be written as

$$\left[r(t)\left([z(t)]^{\Delta\Delta}\right)^{\gamma}\right]^{\Delta} + q_1(t)x^{\alpha}(\tau_1(t)) + q_2(t)x^{\beta}(\tau_2(t)) = 0. \tag{4}$$

A solution $x(t)$ of (1) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is non-oscillatory. The theory of time scales, which has recently received a lot of attention, was introduced by Hilger [1], in order to unify continuous and discrete analysis. Since then, several authors have expounded on various aspects of this new theory; see the survey paper by R.P. Agarwal, M. Bohner, D. O'Regan and A. Peterson [2]. A book on the subject of time scales by M. Bohner and A. Peterson [3] also summarizes and organizes much of the time scale calculus. In the recent years, there has been increasing interest in obtaining sufficient conditions for the oscillation and non-oscillation of solutions of various equations on time-scales; we refer the reader to the papers [4-13].
To the best of our knowledge, it seems to have few oscillation results for the oscillation of third-order dynamic equations. Li, Han, Zhang, Sun [22] considered third-order nonlinear delay dynamic equation

$$x^{\Delta^3} + p(t)x^{\gamma}(\tau(t)) = 0,$$

on a time scale $\mathbb{T}$, where $\gamma > 0$ is quotient of odd positive integers.
Li, Han, Sun, Zhao [16] considered third-order nonlinear delay dynamic equation

$$(a(t)([r(t)x^{\Delta}(t)]^{\Delta})^{\gamma})^{\Delta} + f(t, x(\tau(t))) = 0,$$

on a time scale $\mathbb{T}$, where $\gamma > 0$ is quotient of odd positive integers.
Şenel[19] considered a third order dynamic equation

$$\left(c(ax^{\Delta})^{\Delta}\right)^{\Delta}(t) + P(t, x(t), x^{\Delta}(t)) + f(t, x(t)) = 0.$$

Saker and Graef[20] and Zhang [21] considered a third order half-linear neutral dynamic equation

$$\left(r_1(t)((r_2(t)(x(t) + a(t)x(\tau(t)))^{\Delta})^{\Delta})^{\gamma})\right)^{\Delta} + p(t)x^{\gamma}(\delta(t)) = 0.$$

Han, Li, Sun, Zhang [14] and Grace, Graef, El-Beltagy [15] considered third-order neutral delay dynamic equation

$$(r(t)(x(t) - a(t)x(\tau(t)))^{\Delta\Delta})^{\Delta} + p(t)x^{\gamma}(\delta(t)) = 0,$$

on a time scale $\mathbb{T}$.
In this paper, we consider third-order quasilinear neutral delay dynamic equation on time scales which is not in literature. We obtain some conclusions which contribute to oscillation theory of third order quasilinear neutral delay dynamic equations.

2. Several Lemmas

Before stating our main results, we begin with the following lemmas which play an important role in the proof of the main results. Throughout this paper, we let

$\eta_+(t) := \max\{0, \eta(t)\}$, $\eta_-(t) := \max\{0, -\eta(t)\}$,

$\varphi := min\{\frac{\beta-\alpha}{\beta-\gamma}, \frac{\beta-\alpha}{\gamma-\alpha}\}$,

$\kappa := min\{k^\alpha, k^\beta\}$,

$\Phi(t) = \varphi(q_1(t)(1-P)^\alpha)^{(\beta-\gamma)/(\beta-\alpha)}(q_2(t)(1-P)^\beta)^{(\gamma-\alpha)/(\beta-\alpha)}(\frac{\tau(t)}{\sigma(t)})^\gamma$,

$\beta(t) := \frac{t}{\sigma(t)},\ 0 < \gamma \leq 1,\ \ \beta(t) := (\frac{t}{\sigma(t)})^\gamma,\ \gamma > 1$,

$R(t, t_*) := \int_{t_*}^{t} \left(\frac{1}{r(s)}\right)^{\frac{1}{\gamma}} \Delta s$,

where, for sufficiently large $t_* \in [t_0, \infty)_{\mathbb{T}}$.

Lemma 2.1. *Let $x(t)$ be a positive solution of (1), $z(t)$ is defined as in (3). Then $z(t)$ has only one of the following two properties:*

(1) $z(t) > 0$, $z^\Delta(t) > 0$, $z^{\Delta\Delta}(t) > 0$;
(2) $z(t) > 0$, $z^\Delta(t) < 0$, $z^{\Delta\Delta}(t) > 0$,
where $t \geq t_1$, t_1 sufficiently large.

Proof. Let $x(t)$ be a positive solution of (1) on $[t_0, \infty)$, so that $z(t) > x(t) > 0$, and

$$\left[r(t)(z^{\Delta\Delta}(t))^\gamma\right]^\Delta = -q_1(t)x^\alpha(\tau_1(t)) - q_2(t)x^\beta(\tau_2(t)) < 0.$$

Then $r(t)([z(t)]^{\Delta\Delta})^\gamma$ is a decreasing function and therefore eventually of one sign, so $z^{\Delta\Delta}(t)$ is either eventually positive or eventually negative on $t \geq t_1 \geq t_0$. We assert that $z^{\Delta\Delta}(t) > 0$ on $t \geq t_1 \geq t_0$. Otherwise, assume that $z^{\Delta\Delta}(t) < 0$, then there exists a constant $M > 0$, such that

$$r(t)(z^{\Delta\Delta}(t))^\gamma \leq -M < 0.$$

By integrating the last inequality from t_1 to t, we obtain

$$z^\Delta(t) \leq z^\Delta(t_1) - M^{\frac{1}{\gamma}} \int_{t_1}^{t} (\frac{1}{r(s)})^{\frac{1}{\gamma}} \Delta s.$$

Let $t \to \infty$. Then from (4) , we have $(z(t))^\Delta \to -\infty$, and therefore eventually $z^\Delta(t) < 0$.
Since $z^{\Delta\Delta}(t) < 0$ and $z^\Delta(t) < 0$, we have $z(t) < 0$, which contradicts our assumption $z(t) > 0$. Therefore, z(t) has only one of the two properties (1) and (2).
This completes the proof. □

Lemma 2.2. *Let x(t) be a positive solution of (1), correspondingly $z(t)$ has the property (2). If*

$$\int_{t_0}^{\infty} \int_{v}^{\infty} [\frac{1}{r(u)} \int_{u}^{\infty} (q_1(s) + q_2(s))\Delta s]^{\frac{1}{\gamma}} \Delta u \Delta v = \infty, \tag{5}$$

then $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} z(t) = 0$.

Proof. Let $x(t)$ be a positive solution of (1). Since $z(t)$ has the property (2), then there exists finite $\lim_{t\to\infty} z(t) = \ell$. We shall prove that $\ell = 0$. Assume that $\ell > 0$, then for any $\epsilon > 0$, we have $\ell + \epsilon > z(t) > \ell$, eventually. Choosing $0 < \epsilon < \frac{\ell(1-p)}{p}$, we obtain from (3)

$$x(t) \;=\; z(t) - p(t)x(\tau_0(t)) > \ell - p(t)z(\tau_0(t)) > \ell - p(t)(\ell + \epsilon) = k(\ell + \epsilon) > kz(t),$$

where $k = \frac{\ell - P(1+\epsilon)}{\ell+\epsilon} > 0$. Using (4), (h1) and (h4), we obtain

$$\begin{aligned}
[r(t)(z^{\Delta\Delta}(t))^{\gamma}]^{\Delta} &= -q_1(t)x^{\alpha}(\tau_1(t)) - q_2(t)x^{\beta}(\tau_2(t)) \\
&< -q_1(t)k^{\alpha}z^{\alpha}(\tau_1(t)) - q_2(t)k^{\beta}z^{\beta}(\tau_2(t)) \\
&\leq -q_1(t)k^{\alpha}z^{\alpha}(t) - q_2(t)k^{\beta}z^{\beta}(t) \\
&\leq -q_1(t)\kappa z^{\alpha}(t) - q_2(t)\kappa z^{\alpha}(t).
\end{aligned}$$

Then

$$[r(t)(z^{\Delta\Delta}(t))^{\gamma}]^{\Delta} \leq -\kappa z^{\alpha}(t)(q_1(t) + q_2(t)). \tag{6}$$

Integrating inequality (6) from t to ∞, we obtain

$$r(t)(z^{\Delta\Delta}(t))^{\gamma} \geq \kappa \int_t^{\infty} z^{\alpha}(s)(q_1(s) + q_2(s))\Delta s.$$

Using $z^{\alpha}(s) \geq \ell^{\alpha}$, we obtain

$$z^{\Delta\Delta}(t) \geq \frac{\kappa^{1/\gamma}\ell^{\alpha/\gamma}}{r^{\frac{1}{\gamma}}(t)}\Big[\int_t^{\infty}(q_1(s) + q_2(s))\Delta s\Big]^{\frac{1}{\gamma}}. \tag{7}$$

Integrating inequality (7) from t to ∞, we have

$$-z^{\Delta}(t) \geq \kappa^{1/\gamma}\ell^{\alpha/\gamma}\int_t^{\infty}[\frac{1}{r(u)}\int_u^{\infty}(q_1(s) + q_2(s))\Delta s]^{\frac{1}{\gamma}}\Delta u.$$

Integrating the last inequality from t_1 to ∞, we obtain

$$z(t_1) \geq \kappa^{1/\gamma}\ell^{\alpha/\gamma}\int_{t_1}^{\infty}\int_v^{\infty}[\frac{1}{r(u)}\int_u^{\infty}(q_1(s) + q_2(s))\Delta s]^{\frac{1}{\gamma}}\Delta u\Delta v. \tag{8}$$

The last inequality contradict (5), we have $\ell = 0$. And since $0 \leq x(t) \leq z(t)$, then $\lim_{t\to\infty} x(t) = 0$. This completes the proof. □

Lemma 2.3. *Assume that* $x(t)$ *is a positive solution of equation (1),* $z(t)$ *is defined as in (3) such that* $z^{\Delta\Delta}(t) > 0, z^{\Delta}(t) > 0$, *on* $[t_*, \infty)_{\mathbb{T}}$, $t_* \geq 0$. *Then*

$$z^{\Delta}(t) \geq R(t, t_*)r^{\frac{1}{\gamma}}(t)z^{\Delta\Delta}(t). \tag{9}$$

Proof. Since $r(t)(z^{\Delta\Delta}(t))^{\gamma}$ is strictly decreasing on $[t_*, \infty)_{\mathbb{T}}$, we get for $t \in [t_*, \infty)_{\mathbb{T}}$

$$\begin{aligned} z^{\Delta}(t) &> z^{\Delta}(t) - z^{\Delta}(t_*) \\ &= \int_{t_*}^{t} \frac{(r(s)(z^{\Delta\Delta}(t))^{\gamma})^{\frac{1}{\gamma}}}{r^{\frac{1}{\gamma}}(s)} \Delta s \\ &\geq (r(t)(z^{\Delta\Delta}(t))^{\gamma})^{\frac{1}{\gamma}} \int_{t_*}^{t} \left(\frac{1}{r(s)}\right)^{\frac{1}{\gamma}} \Delta s \end{aligned}$$

and, hence

$z^{\Delta}(t) > R(t, t_*) r^{\frac{1}{\gamma}}(t) z^{\Delta\Delta}(t)$ on $[t_*, \infty)_{\mathbb{T}}$. □

Lemma 2.4. *Assume that $x(t)$ is a positive solution of equation (1), correspondingly $z(t)$ has the property (1). Such that $z^{\Delta}(t) > 0$, $z^{\Delta\Delta}(t) > 0$, on $[t_*, \infty)_{\mathbb{T}}$, $t_* \geq t_0$. Furthermore,*

$$\int_{t_0}^{t} (q_1(s) + q_2(s)) \tau^{\alpha}(s) \Delta s = \infty. \tag{10}$$

Then there exists a $T \in [t_*, \infty)_{\mathbb{T}}$, sufficiently large, so that

$$z(t) > t z^{\Delta}(t),$$

$z(t)/t$ is strictly decreasing, $t \in [T, \infty)_{\mathbb{T}}$.

Proof. Let $U(t) = z(t) - t z^{\Delta}(t)$. Hence $U^{\Delta}(t) = -\sigma(t) z^{\Delta\Delta}(t) < 0$. We claim there exists a $t_1 \in [t_*, \infty)_{\mathbb{T}}$ such that $U(t) > 0$, $z(\tau(t)) > 0$ on $[t_*, \infty)_{\mathbb{T}}$. Assume not. Then $U(t) < 0$ on $[t_*, \infty)_{\mathbb{T}}$. Therefore,

$$\left(\frac{z(t)}{t}\right)^{\Delta} = \frac{t z^{\Delta}(t) - z(t)}{t \sigma(t)} = -\frac{U(t)}{t \sigma(t)} > 0,$$

which implies that $z(t)/t$ is strictly increasing. Pick $t_2 \in [t_1, \infty)_{\mathbb{T}}$ so that $\tau(t) \geq \tau(t_*)$, for $t \geq t_2$. Then

$$\frac{z(\tau(t))}{\tau(t)} \geq \frac{z(\tau(t_*))}{\tau(t_*)} = d > 0,$$

so that $z(\tau(t)) > d\tau(t)$, for $t \geq t_2$. By (3) and (h3), we obtain

$$\begin{aligned} x(t) &= z(t) - p(t) x(\tau_0(t)) > z(t) - p(t) z(\tau_0(t)) \geq (1 - p(t)) z(t) \\ &\geq (1 - P) z(t). \end{aligned} \tag{11}$$

Using (4) and (11), we have

$$\begin{aligned} [r(t)([z(t)]^{\Delta\Delta})^{\gamma}]^{\Delta} &= -q_1(t) x^{\alpha}(\tau_1(t)) - q_2(t) x^{\beta}(\tau_2(t)) \\ &\leq -q_1(t)(1 - P)^{\alpha} z^{\alpha}(\tau_1(t)) - q_2(t)(1 - P)^{\beta} z^{\beta}(\tau_2(t)) \end{aligned} \tag{12}$$

Using (h1) and (h4), we have

$$\begin{aligned} &\leq -q_1(t)(1-P)^{\alpha} z^{\alpha}(\tau(t)) - q_2(t)(1-P)^{\beta} z^{\beta}(\tau(t)) \\ &\leq -q_1(t)(1-P)^{\beta} z^{\alpha}(\tau(t)) - q_2(t)(1-P)^{\beta} z^{\alpha}(\tau(t)) \\ &\leq -(1-P)^{\beta} z^{\alpha}(\tau(t))(q_1(t)+q_2(t)). \end{aligned}$$

Now by integrating both sides of last equation from t_2 to t, we have

$$r(t)(z^{\Delta\Delta}(t))^{\gamma} - r(t_2)(z^{\Delta\Delta}(t_2))^{\gamma} + \int_{t_2}^{t} (1-P)^{\beta}(q_1(t)+q_2(t)) z^{\alpha}(\tau(t))\Delta s \leq 0.$$

This implies that

$$\begin{aligned} r(t_2)(z^{\Delta\Delta}(t_2))^{\gamma} &\geq \int_{t_2}^{t} (1-P)^{\beta}(q_1(s)+q_2(s)) z^{\alpha}(\tau(s))\Delta s \\ &\geq d^{\alpha}(1-P)^{\beta} \int_{t_2}^{t} (q_1(s)+q_2(s))\tau^{\alpha}(s)\Delta s, \end{aligned}$$

which contradicts (10). So $U(t) > 0$ on $t \in [t_1, \infty)_{\mathbb{T}}$ and consequently,

$$\left(\frac{z(t)}{t}\right)^{\Delta} = \frac{t z^{\Delta}(t) - z(t)}{t\sigma(t)} = -\frac{U(t)}{t\sigma(t)} < 0, \ t \in [t_1, \infty)_{\mathbb{T}},$$

and we have that $z(t)/t$ is strictly decreasing on $t \in [t_1, \infty)_{\mathbb{T}}$. The proof is now complete. □

3. Main Results

In this section we give some new oscillation criteria for (1).

Theorem 3.1. *Assume that (2), (5) and (10) hold and that, for all sufficiently large $T_1 \in [t_0, \infty)_{\mathbb{T}}$, there is a $T > T_1$ such that*

$$\limsup_{t\to\infty} \int_{T}^{t} \left[\rho^{\sigma}(s)\Phi(s) - \frac{((\rho^{\Delta}(s))_{+})^{\gamma+1}}{(\gamma+1)^{\gamma+1}(\beta(s)\rho^{\sigma}(s)R(s,t_*))^{\gamma}}\right]\Delta s = \infty, \tag{13}$$

where the function $\rho \in C^1_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$ is a nonnegative function. Then every solution of equation (1) is either oscillatory or tends to zero.

Proof. Assume (1) has a nonoscillatory solution $x(t)$ on $[t_0, \infty)_{\mathbb{T}}$. Then, without loss of generality that $x(t) > 0$, $x(\tau_0(t)) > 0$, $x(\tau_1(t)) > 0$, $x(\tau_2(t)) > 0$ for $t \geq t_1$. $z(t)$ is defined as in (3). We shall consider only $z(t) > 0$, since the proof when $z(t)$ is eventually negative is similar. Therefore Lemma 2.1 and Lemma 2.2, we have

$$\left[r(t)([z(t)]^{\Delta\Delta})^{\gamma}\right]^{\Delta} < 0, \ z^{\Delta\Delta}(t) > 0, \ t \in [t_1, \infty)_{\mathbb{T}},$$

and either $z^{\Delta}(t) > 0$ for $t \geq t_2 \geq t_1$ or $\lim_{t\to\infty} z(t) = \lim_{t\to\infty} x(t) = 0$. Let $z^{\Delta}(t) > 0$ on $[t_2, \infty)_{\mathbb{T}}$.
Define the function $w(t)$ by Riccati substitution

$$w(t) = \rho(t)\frac{r(t)([z(t)]^{\Delta\Delta})^{\gamma}}{z^{\gamma}(t)}. \tag{14}$$

Then

$$\begin{aligned} w^{\Delta}(t) &= \rho^{\Delta}(t)\frac{r(t)([z(t)]^{\Delta\Delta})^{\gamma}}{z^{\gamma}(t)} + \rho^{\sigma}(t)\Big[\frac{r(t)([z(t)]^{\Delta\Delta})^{\gamma}}{z^{\gamma}(t)}\Big]^{\Delta} \\ &= \rho^{\Delta}(t)\frac{r(t)([z(t)]^{\Delta\Delta})^{\gamma}}{z^{\gamma}(t)} + \rho^{\sigma}(t)\frac{[r(t)([z(t)]^{\Delta\Delta})^{\gamma}]^{\Delta}}{z^{\gamma\sigma}(t)} - \rho^{\sigma}(t)\frac{r(t)([z(t)]^{\Delta\Delta})^{\gamma}(z^{\gamma}(t))^{\Delta}}{z^{\gamma}(t)z^{\gamma\sigma}(t)}. \end{aligned}$$

By (4) and (11), we have

$$[r(t)([z(t)]^{\Delta\Delta})^{\gamma}]^{\Delta} \le -q_1(t)(1-P)^{\alpha}z^{\alpha}(\tau_1(t)) - q_2(t)(1-P)^{\beta}z^{\beta}(\tau_2(t)).$$

From the definition of $w(t)$ and the last inequality, we have,

$$\begin{aligned} w^{\Delta}(t) &\le \frac{\rho^{\Delta}(t)}{\rho(t)}w(t) - \rho^{\sigma}(t)q_1(t)(1-P)^{\alpha}\frac{z^{\alpha}(\tau_1(t))}{z^{\gamma}(\sigma(t))} - \rho^{\sigma}(t)q_2(t)(1-P)^{\beta}\frac{z^{\beta}(\tau_2(t))}{z^{\gamma}(\sigma(t))} \\ &- \rho^{\sigma}(t)\frac{r(t)([z(t)]^{\Delta\Delta})^{\gamma}(z^{\gamma}(t))^{\Delta}}{z^{\gamma}(t)z^{\gamma\sigma}(t)}. \end{aligned} \tag{15}$$

By Young's inequality

$$|ab| \le \frac{1}{p}|a|^p + \frac{1}{q}|b|^q,\ a,b\in\mathbb{R},\ p>1,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1,$$

we have

$$\begin{aligned} &\frac{\beta-\gamma}{\beta-\alpha}q_1(t)(1-P)^{\alpha}\frac{z^{\alpha}(\tau_1(t))}{z^{\gamma}(\sigma(t))} + \frac{\gamma-\alpha}{\beta-\alpha}q_2(t)(1-P)^{\beta}\frac{z^{\beta}(\tau_2(t))}{z^{\gamma}(\sigma(t))} \\ &\ge \left[q_1(t)(1-P)^{\alpha}\frac{z^{\alpha}(\tau_1(t))}{z^{\gamma}(\sigma(t))}\right]^{(\beta-\gamma)/(\beta-\alpha)}\left[q_2(t)(1-P)^{\beta}\frac{z^{\beta}(\tau_2(t))}{z^{\gamma}(\sigma(t))}\right]^{(\gamma-\alpha)/(\beta-\alpha)} \\ &= (q_1(t)(1-P)^{\alpha})^{(\beta-\gamma)/(\beta-\alpha)}(q_2(t)(1-P)^{\beta})^{(\gamma-\alpha)/(\beta-\alpha)}(\frac{z^{\alpha}(\tau_1(t))}{z^{\gamma}(\sigma(t))})^{(\beta-\gamma)/(\beta-\alpha)}(\frac{z^{\beta}(\tau_2(t))}{z^{\gamma}(\sigma(t))})^{(\gamma-\alpha)/(\beta-\alpha)} \\ &\ge (q_1(t)(1-P)^{\alpha})^{(\beta-\gamma)/(\beta-\alpha)}(q_2(t)(1-P)^{\beta})^{(\gamma-\alpha)/(\beta-\alpha)}(\frac{z(\tau(t)}{z(\sigma(t))})^{\gamma}. \end{aligned} \tag{16}$$

Hence, by (15) and (16) and using the fact that $z(t)/t$ is decreasing, we obtain

$$\begin{aligned} w^{\Delta}(t) &\le \frac{\rho^{\Delta}(t)}{\rho(t)}w(t) - \varphi\rho^{\sigma}(t)(q_1(t)(1-P)^{\alpha})^{(\beta-\gamma)/(\beta-\alpha)}(q_2(t)(1-P)^{\beta})^{(\gamma-\alpha)/(\beta-\alpha)}(\frac{(\tau(t)}{(\sigma(t))})^{\gamma} \\ &\quad -\rho^{\sigma}(t)\frac{r(t)([z(t)]^{\Delta\Delta})^{\gamma}(z^{\gamma}(t))^{\Delta}}{z^{\gamma}(t)z^{\gamma\sigma}(t)}. \end{aligned}$$

In the first case $0<\gamma\le 1$. Using the Keller's chain rule(see [3]), we have

$$(z^{\gamma}(t))^{\Delta} = \gamma\int_0^1 [hz^{\sigma} + (1-h)z]^{\gamma-1}z^{\Delta}(t)dh \ge \gamma(z^{\sigma}(t))^{\gamma-1}z^{\Delta}(t), \tag{17}$$

in view of (17), Lemma 2.2, Lemma 2.3, (9) and using the fact that $z(t)/t$ is decreasing, we have

$$\begin{aligned} w^{\Delta}(t) &\leq -\rho^{\sigma}(t)\Phi(t) + \frac{(\rho^{\Delta}(t))_+}{\rho(t)}w(t) - \gamma\rho^{\sigma}(t)\frac{r(t)(z^{\Delta\Delta}(t))^{\gamma}z^{\Delta}(t)z(t)}{z^{\gamma+1}(t)z^{\sigma}(t)} \\ &\leq -\rho^{\sigma}(t)\Phi(t) + \frac{(\rho^{\Delta}(t))_+}{\rho(t)}w(t) - \gamma\rho^{\sigma}(t)R(t,t_*)\frac{r^{\frac{\gamma+1}{\gamma}}(t)(z^{\Delta\Delta}(t))^{\gamma+1}z(t)}{z^{\gamma+1}(t)z(\sigma(t))} \\ &\leq -\rho^{\sigma}(t)\Phi(t) + \frac{(\rho^{\Delta}(t))_+}{\rho(t)}w(t) - \gamma\rho^{\sigma}(t)R(t,t_*)\frac{t}{\sigma(t)}\frac{w^{\frac{\gamma+1}{\gamma}}(t)}{\rho^{\frac{\gamma+1}{\gamma}}(t)}. \end{aligned} \tag{18}$$

Let $\gamma > 1$. Applying the Keller's chain rule , we have

$$(z^{\gamma}(t))^{\Delta} = \gamma\int_0^1 [hz^{\sigma} + (1-h)z]^{\gamma-1}z^{\Delta}(t)dh \geq \gamma(z(t))^{\gamma-1}z^{\Delta}(t), \tag{19}$$

in the view of (19), Lemma 2.2, Lemma 2.3 and (9), we have

$$w^{\Delta}(t) \leq -\rho^{\sigma}(t)\Phi(t) + \frac{(\rho^{\Delta}(t))_+}{\rho(t)}w(t) - \gamma\rho^{\sigma}(t)\frac{r(t)[z(t)]^{\Delta\Delta})^{\gamma}z^{\Delta}(t)z^{\gamma}(t)}{z^{\gamma+1}(t)z^{\gamma\sigma}(t)}$$

$$w^{\Delta}(t) \leq -\rho^{\sigma}(t)\Phi(t) + \frac{(\rho^{\Delta}(t))_+}{\rho(t)}w(t) - \gamma\rho^{\sigma}(t)(\frac{t}{\sigma(t)})^{\gamma}R(t,t_*)\frac{w^{\frac{\gamma+1}{\gamma}}(t)}{\rho^{\frac{\gamma+1}{\gamma}}(t)}. \tag{20}$$

By (18), (20) and the definition of $\beta(t)$, we have, for $\gamma > 0$,

$$w^{\Delta}(t) \leq -\rho^{\sigma}(t)\Phi(t) + \frac{(\rho^{\Delta}(t))_+}{\rho(t)}w(t) - \gamma\rho^{\sigma}(t)\beta(t)R(t,t_*)\frac{w^{\lambda}(t)}{\rho^{\lambda}(t)}, \tag{21}$$

where $\lambda := \frac{\gamma+1}{\gamma}$.

Define $A \geq 0$ and $B \geq 0$ by

$A^{\lambda} := \gamma\rho^{\sigma}(t)\beta(t)R(t,t_*)\frac{w^{\lambda}(t)}{\rho^{\lambda}(t)}$,

$B^{\lambda-1} := \frac{\rho^{\Delta}(t)}{\lambda(\gamma\rho^{\sigma}(t)\beta(t)R(t,t_*))^{\frac{1}{\lambda}}}$.

Then using the inequality [18]

$$\lambda AB^{\lambda-1} - A^{\lambda} \leq (\lambda-1)B^{\lambda} \tag{22}$$

which yields

$$\frac{(\rho^{\Delta}(t))_+}{\rho(t)}w(t) - \gamma\rho^{\sigma}(t)\beta(t)R(t,t_*)\frac{w^{\lambda}(t)}{\rho^{\lambda}(t)} \leq \frac{((\rho^{\Delta}(t))_+)^{\gamma+1}}{(\gamma+1)^{\gamma+1}(\beta(t)\rho^{\sigma}(t)R(t,t_*))^{\gamma}}.$$

From this last inequality and (21), we find

$$w^{\Delta}(t) \leq -\rho^{\sigma}(t)\Phi(t) + \frac{((\rho^{\Delta}(t))_+)^{\gamma+1}}{(\gamma+1)^{\gamma+1}(\beta(t)\rho^{\sigma}(t)R(t,t_*))^{\gamma}}.$$

Integrating both sides from T to t , we get

$$\int_T^t [\rho^{\sigma}(s)\Phi(s) - \frac{((\rho^{\Delta}(s))_+)^{\gamma+1}}{(\gamma+1)^{\gamma+1}(\beta(s)\rho^{\sigma}(s)R(s,t_*))^{\gamma}}]\Delta s \leq w(T) - w(t) \leq w(T),$$

which contradicts to assumption (13). This completes the proof of Theorem 3.1. □

Theorem 3.2. *Assume that (2), (5) and (10) hold. Furthermore, suppose that there exist functions $H, h \in C_{rd}(\mathbb{D},\mathbb{R})$, where $\mathbb{D} \equiv (t,s) : t \geq s \geq t_0$ such that*
$H(t,t) = 0,\ t \geq 0$
$H(t,s) > 0,\ t > s \geq t_0$,
and H has a nonpositive continous Δ-partial derivative $H^{\Delta s}(t,s)$ with respect to the second variable and satisfies

$$H^{\Delta s}(\sigma(t),s) + H(\sigma(t),\sigma(s))\frac{\rho^{\Delta}(s)}{\rho(s)} = -\frac{h(t,s)}{\rho(s)}H(\sigma(t),\sigma(s))^{\frac{\gamma}{\gamma+1}}, \tag{23}$$

and for all sufficiently large $T_1 \in [t_0,\infty)_{\mathbb{T}}$, there is a $T > T_1$ such that

$$\limsup_{t\to\infty} \frac{1}{H(\sigma(t),T)} \int_T^{\sigma(t)} \chi(t,s)\Delta s = \infty, \tag{24}$$

where ρ is a positive Δ-differentiable function and

$$\chi(t,s) = H(\sigma(t),\sigma(s))\rho^{\sigma}(s)\Phi(s) - \frac{(h_-(t,s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}(\beta(s)\rho^{\sigma}(s)R(s,T_1))^{\gamma}}.$$

Then every solution of equation (1) is either oscillatory or tends to zero.

Proof. Suppose that $x(t)$ is a nonoscillatory solution of (1) and $z(t)$ is defined as in (2). Without loss of generality, we may assume that there is a $t_1 \in [t_0,\infty)_{\mathbb{T}}$ sufficiently large so that the conclusions of Lemma 2.1. hold and (23) holds for $t_2 > t_1$. If case (I) of Lemma 2.1. holds then proceeding as in the proof of Theorem 3.1. , we see that (21) holds for $t > t_2$. Multiply both sides of (21) by $H(\sigma(t),\sigma(s))$ and integrating from T to $\sigma(t)$, we get

$$\begin{aligned}\int_T^{\sigma(t)} H(\sigma(t),\sigma(s))\rho^{\sigma}(s)\Phi(s)\Delta s \quad &\leq \quad -\int_T^{\sigma(t)} H(\sigma(t),\sigma(s))w^{\Delta}(s)\Delta s + \int_T^{\sigma(t)} H(\sigma(t),\sigma(s))\frac{\rho^{\Delta}(s)}{\rho(s)}w(s)\Delta s \\ &- \quad \int_T^{\sigma(t)} H(\sigma(t),\sigma(s))\gamma\rho^{\sigma}(s)\beta(s)R(s,T_1)\frac{w^{\lambda}(s)}{\rho^{\lambda}(s)}\Delta s,\ \left(\lambda = \frac{\gamma+1}{\gamma}\right).\end{aligned} \tag{25}$$

Integrating by parts and using $H(t,t) = 0$, we obtain

$$\int_T^{\sigma(t)} H(\sigma(t),\sigma(s))w^{\Delta}(s)\Delta s = -H(\sigma(t),T)w(T) - \int_T^{\sigma(t)} H^{\Delta s}(\sigma(t),s)w(s)\Delta s.$$

It then follows from (25) that

$$\int_T^{\sigma(t)} H(\sigma(t),\sigma(s))\rho^\sigma(s)\Phi(s)\Delta s \le H(\sigma(t),T)w(T) + \int_T^{\sigma(t)} H^{\Delta s}(\sigma(t),s)w(s)\Delta s$$

$$+ \int_T^{\sigma(t)} H(\sigma(t),\sigma(s))\frac{\rho^\Delta(s)}{\rho(s)}w(s)\Delta s - \int_T^{\sigma(t)} H(\sigma(t),\sigma(s))\gamma\rho^\sigma(s)\beta(s)R(s,T_1)\frac{w^\lambda(s)}{\rho^\lambda(s)}\Delta s.$$

Then, we have

$$\int_T^{\sigma(t)} H(\sigma(t),\sigma(s))\rho^\sigma(s)\Phi(s)\Delta s \le H(\sigma(t),T)w(T)$$

$$+\Big[\int_T^{\sigma(t)} H^{\Delta s}(\sigma(t),s) + H(\sigma(t),\sigma(s))\frac{\rho^\Delta(s)}{\rho(s)}\Big]w(s)\Delta s - \int_T^{\sigma(t)} H(\sigma(t),\sigma(s))\gamma\rho^\sigma(s)\beta(s)R(s,T_1)\frac{w^\lambda(s)}{\rho^\lambda(s)}\Delta s.$$

It then follows from (23) that

$$\int_T^{\sigma(t)} H(\sigma(t),\sigma(s))\rho^\sigma(s)\Phi(s)\Delta s \quad\le\quad H(\sigma(t),T)w(T)$$

$$+ \int_T^{\sigma(t)} \Big[-\frac{h(t,s)}{\rho(s)}H(\sigma(t),\sigma(s))^{\frac{\gamma}{\gamma+1}}\Big]w(s)\Delta s - \int_T^{\sigma(t)} H(\sigma(t),\sigma(s))\gamma\rho^\sigma(s)\beta(s)R(s,T_1)\frac{w^\lambda(s)}{\rho^\lambda(s)}\Delta s.$$

Then

$$\int_T^{\sigma(t)} H(\sigma(t),\sigma(s))\rho^\sigma(s)\Phi(s)\Delta s \quad\le\quad H(\sigma(t),T)w(T) + \int_T^{\sigma(t)} \Big[\frac{h_-(t,s)}{\rho(s)}H(\sigma(t),\sigma(s))^{\frac{\gamma}{\gamma+1}}\Big]w(s)\Delta s$$

$$- \quad \int_T^{\sigma(t)} H(\sigma(t),\sigma(s))\gamma\rho^\sigma(s)\beta(s)R(s,T_1)\frac{w^\lambda(s)}{\rho^\lambda(s)}\Delta s. \tag{26}$$

Therefore, as in Theorem 3.1. , by letting

$$A^\lambda := H(\sigma(t),\sigma(s))\gamma\rho^\sigma(t)\beta(t)R(t,T_1)\frac{w^\lambda(t)}{\rho^\lambda(t)},$$

$$B^{\lambda-1} := \frac{h_-(t,s)}{\lambda(\gamma\rho^\sigma(t)\beta(t)R(t,T_1))^{\frac{1}{\lambda}}}.$$

Then using the inequality [18]

$$\lambda AB^{\lambda-1} - A^\lambda \le (\lambda-1)B^\lambda,$$

we have

$$\int_T^{\sigma(t)} \Big[\frac{h_-(t,s)}{\rho(s)}H(\sigma(t),\sigma(s))^{\frac{\gamma}{\gamma+1}}\Big]w(s)\Delta s - \int_T^{\sigma(t)} H(\sigma(t),\sigma(s))\gamma\rho^\sigma(s)\beta(s)R(s,T_1)\frac{w^\lambda(s)}{\rho^\lambda(s)}\Delta s$$

$$\le \int_T^{\sigma(t)} \frac{(h_-(t,s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}(\beta(s)\rho^\sigma(s)R(t,T_1))^\gamma}\Delta s$$

From this last inequality and (26), we find

$$\int_T^{\sigma(t)} H(\sigma(t),\sigma(s))\rho^\sigma(s)\Phi(s)\Delta s \le H(\sigma(t),T)w(T) + \int_T^{\sigma(t)} \frac{(h_-(t,s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}(\beta(s)\rho^\sigma(s)R(t,T_1))^\gamma}\Delta s.$$

Then for $T > T_1$ we have

$$\int_T^{\sigma(t)} \left[H(\sigma(t),\sigma(s))\rho^\sigma(s)\Phi(s) - \frac{(h_-(t,s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}(\beta(s)\rho^\sigma(s)R(s,T_1))^\gamma}\right]\Delta s \le H(\sigma(t),T)w(T),$$

and this implies that

$$\frac{1}{H(\sigma(t),T)}\int_T^{\sigma(t)} \left[H(\sigma(t),\sigma(s))\rho^\sigma(s)\Phi(s) - \frac{(h_-(t,s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}(\beta(s)\rho^\sigma(s)R(s,T_1))^\gamma}\right]\Delta s < w(T),$$

for all large T, which contradicts (24). This completes the proof of Theorem 3.2. □

Remark 3.3. *From Theorem 3.1, we can obtain different conditions for oscillation of equation (1.1) with different choices of* $\rho(t)$.

Remark 3.4. *The conclusion of Theorem 3.1 remains intact if assumption (13) is replaced by the two conditions*

$$\limsup_{t\to\infty} \int_T^t \rho^\sigma(s)\Phi(s)\Delta s = \infty,$$

$$\limsup_{t\to\infty} \int_T^t \frac{((\rho^\Delta(s))_+)^{\gamma+1}}{(\gamma+1)^{\gamma+1}(\beta(s)\rho^\sigma(s)R(s,t_*))^\gamma}\Delta s < \infty.$$

Remark 3.5. *The conclusion of Theorem 3.2 remains intact if assumption (24) is replaced by the two conditions2*

$$\limsup_{t\to\infty} \frac{1}{H(\sigma(t),T)}\int_T^{\sigma(t)} H(\sigma(t),\sigma(s))\rho^\sigma(s)\Phi(s)\Delta s = \infty,$$

$$\liminf_{t\to\infty} \frac{1}{H(\sigma(t),T)}\int_T^{\sigma(t)} \frac{(h_-(t,s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}(\beta(s)\rho^\sigma(s)R(s,T_1))^\gamma}\Delta s < \infty.$$

Example 3.6. *Consider the third order quasilinear neutral delay dynamic equations on time scales*

$$\left(x(t) + \frac{1}{3}x(\tau_0(t))\right)^{\Delta\Delta\Delta} + \frac{1}{t}x^{\frac{1}{3}}(\frac{t}{2}) + \frac{1}{t}x^{\frac{5}{3}}(\frac{t}{2}) = 0, \tag{27}$$

where $r(t) = 1,\ \alpha = \frac{1}{3},\ \gamma = 1,\ \beta = \frac{5}{3},\ q_1(t) = q_2(t) = \frac{1}{t},\ \ \mu$ *is a positive constant.*

The condition (2), (5) and (10) hold. By Theorem 3.1, pick $\rho(t) = t$*, we have*

$$\limsup_{t\to\infty} \int_T^t \left[\rho^\sigma(s)\Phi(s) - \frac{((\rho^\Delta(s))_+)^{\gamma+1}}{(\gamma+1)^{\gamma+1}(\beta(s)\rho^\sigma(s)R(s,t_*))^\gamma}\right]\Delta s$$

$$\limsup_{t\to\infty} \int_T^t \left[\mu - \frac{1}{s(s-t_*)}\right]\Delta s = \infty.$$

Hence, every solution of eq. (27) is oscillatory or tends to zero if $\mu > 0$.

Example 3.7. *Consider the third order quasilinear neutral delay dynamic equations on time scales*

$$\left(\frac{1}{t^2}(x(t)+\frac{1}{3}x(\tau_0(t)))\right)^{\Delta\Delta\Delta}+\frac{\sigma(t)}{\tau(t)}x^{\frac{1}{3}}(\tau_1(t))+\frac{\sigma(t)}{\tau(t)}x^{\frac{5}{3}}(\tau_2(t))=0, \tag{28}$$

where $r(t)=\frac{1}{t^2}$, $\alpha=\frac{1}{3}$, $\gamma=1$, $\beta=\frac{5}{3}$, $q_1(t)=q_2(t)=\frac{\sigma(t)}{\tau(t)}$, μ *is a positive constant.*

The condition (2), (5) and (10) hold. By Theorem 3.1, pick $\rho(t)=1$, *we have*

$$\limsup_{t\to\infty}\int_T^t\left[\rho^\sigma(s)\Phi(s)-\frac{((\rho^\Delta(s))_+)^{\gamma+1}}{(\gamma+1)^{\gamma+1}(\beta(s)\rho^\sigma(s)R(s,t_*))^\gamma}\right]\Delta s$$

$$\limsup_{t\to\infty}\int_T^t\mu\Delta s=\infty.$$

Hence, every solution of eq. (28) is oscillatory or tends to zero if $\mu>0$.

References

[1] S. Hilger, Analysis on measure chains A unified approach to continuous and discrete calculus, Results Math. 18 (1990) 18–56.
[2] R.P. Agarwal, M. Bohner, D. O'Regan and A. Peterson, Dynamic equations on time scales: A survey, J. Comput. Appl. Math. 141 (2002) 1–26.
[3] M. Bohner, A. Peterson, Dynamic Equations on Time Scales: An Introduction with Applications, Birkhuser, Boston, 2001.
[4] R. P. Agarwal, D. ORegan, and S. H. Saker, Oscillation criteria for second-order nonlinear neutral delay dynamic equations, Journal of Mathematical Analysis and Applications, 300 (2004) 203-217.
[5] Y. Şahiner, Oscillation of second-order neutral delay and mixed-type dynamic equations on time scales, Advances in Difference Equations, 2006 (2006) Article ID 65626, 9 pages.
[6] H.-W.Wu, R.-K. Zhuang, and R. M. Mathsen, Oscillation criteria for second-order nonlinear neutral variable delay dynamic equations, Applied Mathematics and Computation, 178 (2006) 321-331.
[7] S. H. Saker, Oscillation of second-order nonlinear neutral delay dynamic equations on time scales, Journal of Computational and Applied Mathematics, 187 (2006) 123-141.
[8] Zhang, Wang, Oscillation of second-order nonlinear neutral dynamic equations on time scales, Applied Mathematics and Computation, 216 (2010) 2837–2848.
[9] M. T. Şenel, Kamenev-type oscillation criteria for the second-order nonlinear dynamic equations with damping on time scales, Abstr. Appl. Anal. 2012 (2012) 1–18.
[10] M. T. Şenel, Oscillation theorems for dynamic equation on time scales, Bull. Math. Anal. Appl. 3 (2011) 101–105.
[11] T. Li, R. P. Agarwal, and M. Bohner, Some oscillation results for second-order neutral dynamic equations, Hacet. J. Math. Stat. 41 (2012) 715–721.
[12] T. Candan, Oscillation of second order nonlinear neutral dynamic equations on time scales with distributed deviating arguments, Comput. Math. Appl. 62 (2011) 4118–4125.
[13] Y. Sun, Z. Han, T. Li, G. Zhang, Oscillation criteria for second-order quasilinear neutral delay dynamic equations on time scales, Advances in Difference Equations, 2010 (2010), Article ID 512437, 14 pages.
[14] Z. Han, T. Li, S. Sun, C. Zhang, Oscillation Behavior of Third Order Neutral Emden-Fowler Delay Dynamic Equations On Time Scales, Advances in Difference Equations, 2010 (2010) Article ID 586312, 23 pages.
[15] S. R. Grace, J. R. Graef, M. A. El-Beltagy, On the oscillation of third order delay dynamic equations on time scales, Appl. Math. Comput. 63 (2012) 775–782.
[16] T. Li, Z. Han, S. Sun, Y. Zhao, Oscillation Results for Third Order Nonlinear Delay Dynamic Equations on Time Scales, Bull. Malays. Math. Sci. Soc. 34 (2011) 639–648.
[17] L. Erbe, T. S. Hassan, A. Peterson, Oscillation of Third Order Nonlinear Functional Dynamic Equations on Time Scales, Diff. Equ. Dynam. Sys. 18 (2010) 199–227.
[18] G.H. Hardy, J. E. Littlewood and G. Pólya, Inequalities, reprint of the 1952 edition, Cambridge Mathematical Library, Cambridge Univ. Press, Cambridge, 1988.
[19] M. T. Şenel, Behavior of solutions of a third-order dynamic equation on time scales, J. Inequal. Appl. 2013 (2013) 1–7.
[20] S. H. Saker, J. R. Graef, Oscillation of third-order nonlinear neutral functional dynamic equations on time scales, Dynam. Syst. Appl. 21 (2012) 583–606.
[21] C. Zhang, S.H. Saker, T. Li, Oscillation of third-order neutral dynamic equtions on time scales, Dyn. Contin. Discrete Impuls. Syst., Ser. B, Appl. Algorithms 2013 (in press).
[22] T. Li, Z. Han, C. Zhang, Y. Sun, Oscillation criteria for third-order nonlinear delay dynamic equations on time scales, Bull. Math. Anal. Appl. 3 (2011) 52–60.