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A Second Order Differential Equation with Generalized Sturm-Liouville Integral Boundary Conditions at Resonance

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Abstract. By using Mawhin coincidence degree theory, we investigate the existence of solutions for a class of second order nonlinear differential equations with generalized Sturm-Liouville integral boundary conditions at resonance. The results extend some known conclusions of integral boundary value problem at resonance for nonlinear differential equations.

1. Introduction

In this paper, we discuss the following nonlinear differential equation with generalized Sturm-Liouville integral boundary conditions at resonance

$$x'' = f(t, x(t), x'(t)) + e(t), t \in (0, 1), \tag{1.1}$$

$$ax(0) - bx'(0) = \int_0^1 g(t)x(t)d\xi(t),\ cx(1) + dx'(1) = \int_0^1 h(t)x(t)d\eta(t), \tag{1.2}$$

where $f : [0, 1] \times R^2 \to R$ is a Carathéodory function, $h(t),\ g(t) \in C[0, 1],\ e(t) \in L^1[0, 1]$. $\xi(t)$ and $\eta(t)$ are bounded functions satisfying $m\xi(t) > 1,\ m\eta(t) > c,\ m = \min_{t\in(0,1)}\{h(t), g(t)\}$, and constants $a, b, c, d \in R$.

It is well known that nonlinear differential equations with integral boundary conditions have been used in description of many phenomena in the applied sciences. For instance, heat conduction, chemical engineering and underground water flow and so on [1-3]. Therefore, boundary value problem (BVP for short) with integral boundary conditions has been studied by many authors [4-8]. For example, Zhang et al. [4] applied Mawhin coincidence degree theorem to considered second order nonlinear boundary value problem of the following type

$$x'' = f(t, x(t), x'(t)), t \in (0, 1), \tag{1.3}$$

$$x'(0) = \int_0^1 h(t)x'(t)dt,\ \ x'(1) = \int_0^1 g(t)x'(t)dt, \tag{1.4}$$

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where $h, g \in C([0,1],[0,+\infty))$ with $\int_0^1 h(t)dt = 1,\ \int_0^1 g(t)dt = 1$, and they obtained some results based on the following assumption

$$\Delta = \begin{vmatrix} 1 - \int_0^1 tg(t)dt & \frac{1}{2}(1 - \int_0^1 t^2 g(t)dt) \\ \int_0^1 th(t)dt & \frac{1}{2}\int_0^1 t^2 h(t)dt \end{vmatrix} \neq 0.$$

But in this paper, we will prove this condition is redundant.

The differential equations of the form (1.1) have also been discussed in [7] but with the boundary conditions as follow

$$x(0) = \int_0^1 \alpha(t)x(t)d\xi(t),\ \ x(1) = \int_0^1 \beta(t)x'(t)d\eta(t), \tag{1.5}$$

where $f : [0,1]\times R^2 \to R$ is a Carathéodory function, $e(t) \in L^1[0,1]$, $\alpha(t),\ \beta(t) \in C[0,1]$. $\xi(t),\ \eta(t)$ are bounded, variation, nondecreasing and monotonous functions. And $M\eta(t) < 1,\ m\xi(t) > 1,\quad m = \min_{t\in(0,1)} \alpha(t),\ M = \max_{t\in(0,1)} \beta(t)$.

However, without the case of resonance, in [8], by using Leray-Schauder continuation theorem, the authors considered second-order boundary value problem with generalized Sturm-Liouville integral boundary conditions

$$x'' = f(t, x(t), x'(t)) + e(t), t \in (0,1),$$

$$\alpha x(0) - \beta x'(0) = \int_0^1 a(t)x(t)dt,\ \ \gamma x(1) + \delta x'(1) = \int_0^1 b(t)x(t)dt.$$

On the basis of above papers, in this paper, we shall use Mawhin coincidence degree theory to investigate the existence of solution for a class of boundary value problem with generalized Sturm-Liouville integral boundary conditions at resonance. We extend some results in refs [4, 7].

2. Preliminaries

Now, some notations and an abstract existence result [9] are introduced.

Let Y, Z be real Banach spaces and let $L : \mathrm{dom}L \subset Y \to Z$ be a linear operator which is a Fredholm map of index zero and $P : Y \to Y, Q : Z \to Z$ be continuous projectors such that $\mathrm{Im}P = \mathrm{Ker}L$, $\mathrm{Ker}Q = \mathrm{Im}L$ and $Y = \mathrm{Ker}L \oplus \mathrm{Ker}P$, $Z = \mathrm{Im}L \oplus \mathrm{Im}Q$. It follows that $L|_{\mathrm{dom}L\cap\mathrm{Ker}P} : \mathrm{dom}L \cap \mathrm{Ker}P \to \mathrm{Im}L$ is invertible, we denote the inverse of that map by K_P. Let Ω be an open bounded subset of Y such that $\mathrm{dom}L \cap \Omega \neq \emptyset$, the map $N : Y \to Z$ is said to be $L-$compact on $\bar{\Omega}$ if the map $QN(\bar{\Omega})$ is bounded and $K_P(I - Q)N : \bar{\Omega} \to Y$ is compact.

Lemma 2.1. ([9, Theorem IV])*Let L be a Fredholm map of index zero and let N be $L-$compact on $\bar{\Omega}$. Assume that the following conditions are satisfied:*
(i) $Lx \neq \lambda Nx$ for every $(x, \lambda) \in ((domL\backslash KerL) \cap \partial\Omega) \times [0,1]$;
(ii) $Nx \notin ImL$ for every $x \in KerL \cap \partial\Omega$;
(iii) $deg(QN|_{KerL}, \Omega \cap KerL, 0) \neq 0$, where $Q : Z \to Z$ is a continuous projector as above with $ImL = KerQ$.
Then the abstract equation $Lx = Nx$ has at least one solution in $domL \cap \bar{\Omega}$.

For $x \in C^1[0,1]$, we use the norm $\|x\|_\infty = \max_{t\in(0,1)} |x(t)|,\ \|x\| = \max\{\|x\|_\infty, \|x'\|_\infty\}$, denote the norm in $L^1[0,1]$ by $\|\cdot\|_1$, and the Sobolev space $W^{2,1}(0,1)$ as

$$W^{2,1}(0,1) = \{x : [0,1] \to R : x, x' \text{are absolutely continuous on } [0,1] \text{ with } x'' \in L^1[0,1]\}.$$

Let $Y = C^1[0,1],\ Z = L^1[0,1]$, and define the linear operator $L : \mathrm{dom}L \subset Y \to Z$ as $Lx = x'',\ x \in \mathrm{dom}L$, where

$$\mathrm{dom}L = \{x \in W^{2,1}(0,1) : x \text{ satisfies boundary conditions (1.2)}\}.$$

Define $N : Y \to Z$ as

$$Nx = f(t, x(t), x'(t)) + e(t), \quad t \in (0, 1),$$

then BVP (1.1), (1.2) can be written as $Lx = Nx$.

The resonance conditions of BVP (1.1), (1.2) are as follow

$$\int_0^1 g(t)d\xi(t) = a, \int_0^1 g(t)td\xi(t) = -b, \int_0^1 h(t)d\eta(t) = c, \int_0^1 h(t)td\eta(t) = c + d. \tag{C}$$

Define some signs as follow

$$M = \max_{t\in(0,1)} h(t),$$

$$\Lambda(p, q) = \left| \begin{array}{cc} \frac{1}{p(p+1)} \int_0^1 g(t)t^{p+1}d\xi(t) & \frac{1}{p(p+1)} \left[\int_0^1 h(t)t^{p+1}d\eta(t) - c - d(p+1) \right] \\ \frac{1}{q(q+1)} \int_0^1 g(t)t^{q+1}d\xi(t) & \frac{1}{q(q+1)} \left[\int_0^1 h(t)t^{q+1}d\eta(t) - c - d(q+1) \right] \end{array} \right|,$$

$$m_{11} = \frac{1}{p(p+1)} \int_0^1 g(t)t^{p+1}d\xi(t), \; m_{12} = -\frac{1}{p(p+1)}[\int_0^1 h(t)t^{p+1}d\eta(t) - c - d(p+1)],$$

$$m_{21} = -\frac{1}{q(q+1)} \int_0^1 g(t)t^{q+1}d\xi(t), \; m_{22} = \frac{1}{q(q+1)}[\int_0^1 h(t)t^{q+1}d\eta(t) - c - d(q+1)].$$

Lemma 2.2. Assume conditions (*C*) hold, then there exists $p \in Z^+$, $q \in Z^+$, $q \geq p + 1$, such that $\Lambda(p, q) \neq 0$.

Proof. It is obvious that there exists $q \in Z^+$, such that $\int_0^1 h(t)t^{q+1}d\eta(t) \neq c + d(q + 1)$. If else, we have $\int_0^1 h(t)t^{q+1}d\eta(t) = c + d(q + 1)$.

Because of

$$\int_0^1 h(t)t^{q+1}d\eta(t) \leq M \int_0^1 t^{q+1}d\eta(t) = M\eta(1) - M \int_0^1 \eta(t)dt^{q+1},$$

then

$$\lim_{q\to+\infty} \int_0^1 h(t)t^{q+1}d\eta(t) < \lim_{q\to+\infty}[M\eta(1) - M \int_0^1 \eta(t)dt^{q+1}] = M\eta(1).$$

Since $\eta(t)$ is a bounded function, so there exists $M_1 \in R$ such that $\lim_{q\to+\infty} \int_0^1 h(t)t^{q+1}d\eta(t) \leq M_1$. But when $d \neq 0$, we have $\lim_{q\to+\infty}[c + d(q + 1)] = \infty$ which is a contradiction.

On the other hand, when $d = 0$, since $\lim_{q\to+\infty} \int_0^1 h(t)t^{q+1}d\eta(t) \geq m\eta(1) > c$, but $\lim_{q\to+\infty}[c + d(q + 1)] = c$. Hence $\int_0^1 h(t)t^{q+1}d\eta(t) \neq c + d(q + 1)$.

Similarly, due to $\lim_{q\to+\infty} \int_0^1 g(t)t^{q+1}d\xi(t) > m\xi(1) > 1$. Then for each $l \in Z$, there exists $k_l \in \{ln+1, \ldots, (l+1)n\}$ such that $\int_0^1 g(t)t^{k_l+1}d\xi(t) \neq 0$.

Set

$$S = \{k_l \in Z : \int_0^1 h(t)t^{p+1}d\eta(t) - c - d(p + 1) = \frac{\int_0^1 g(t)t^{p+1}d\xi(t)[\int_0^1 h(t)t^{k_l+1}d\eta(t) - c - d(k_l + 1)]}{\int_0^1 g(t)t^{k_l+1}d\xi(t)}\},$$

then S is a finite set. If else, there have a monotone sequence $\{k_{l_s}\},\ s \in Z^+,\ k_{l_s} \le k_{l_{s+1}}$, such that

$$\begin{aligned}\int_0^1 h(t)t^{p+1}d\eta(t) - c - d(p+1) &= \frac{\int_0^1 g(t)t^{p+1}d\xi(t)[\int_0^1 h(t)t^{k_{l_s}+1}d\eta(t) - c - d(k_{l_s}+1)]}{\int_0^1 g(t)t^{k_{l_s}+1}d\xi(t)}\\ &= \lim_{k_{l_s}\to\infty} \frac{\int_0^1 g(t)t^{p+1}d\xi(t)[\int_0^1 h(t)t^{k_{l_s}+1}d\eta(t) - c - d(k_{l_s}+1)]}{\int_0^1 g(t)t^{k_{l_s}+1}d\xi(t)}\\ &= \infty,\end{aligned}$$

which contradicts to the definition of S. Hence $\Lambda(p,q) \neq 0$. The proof is completed.

Lemma 2.3. If conditions (C) hold, then $L : \mathrm{dom}L \subset Y \to Z$ is a Fredholm operator of index zero. Furthermore, the continuous projector operator $Q : Z \to Z$ can be written by

$$Qy = (T_1y)t^{p-1} + (T_2y)t^{q-1},$$

where

$$T_1y = \frac{1}{\Lambda(p,q)}[m_{22}Q_1y + m_{21}Q_2y],\ T_2y = \frac{1}{\Lambda(p,q)}[m_{12}Q_1y + m_{11}Q_2y],$$

$$Q_1y = \int_0^1 g(t)(\int_0^t (t-s)y(s)ds)d\xi(t),\ Q_2y = \int_0^1 h(t)(\int_0^t (t-s)y(s)ds)d\xi(t) - c\int_0^1 (1-s)y(s)ds - d\int_0^1 y(s)ds.$$

The linear operator $K_p : \mathrm{Im}L \to \mathrm{dom}L \cap \mathrm{Ker}P$ can be defined by

$$K_py(t) = \frac{1}{2}\int_0^1 (t-s)y(s)ds,\ y \in \mathrm{Im}L.$$

As well, we have $\|K_py\| \le \|y\|_1, y \in \mathrm{Im}L$.
Proof. It is easy to get that $\mathrm{Ker}L = \{x \in \mathrm{dom}L : x = c_1 + c_2t,\ c_1, c_2 \in R\}$. Now we proof that

$$\mathrm{Im}L = \{y \in Z : Q_1y = Q_2y = 0\}. \tag{2.1}$$

Since the equation

$$x'' = y, \tag{2.2}$$

has a solution $x(t)$ such that boundary conditions (1.2) , if and only if

$$Q_1y = Q_2y = 0. \tag{2.3}$$

In fact, if (2.2) has a solution $x(t)$ satisfies the boundary conditions (1.2), then we have

$$x(t) = x(0) + x'(0)t + \int_0^1 (t-s)y(s)ds.$$

According to conditions (C), we have $Q_1y = Q_2y = 0$.

On the other hand, we let $x(t) = c_1 + c_2t + \int_0^t (t-s)y(s)ds$, where c_1, c_2 are constants. If (2.3) holds, then $x(t)$ is a solution of (2.2) with boundary condition (1.2). Hence (2.1) holds.

From Lemma 2.2, there exists $p \in Z^+,\ q \in Z^+,\ q \ge p+1$, such that $\Lambda(p,q) \neq 0$. Define

$$T_1y = \frac{1}{\Lambda(p,q)}[m_{22}Q_1y + m_{21}Q_2y],\ T_2y = \frac{1}{\Lambda(p,q)}[m_{12}Q_1y + m_{11}Q_2y],$$

and $Qy = (T_1y)t^{p-1} + (T_2y)t^{q-1}$, then dim Im$Q = 2$.

So one has

$$
\begin{aligned}
T_1((T_1y)t^{p-1}) &= \frac{1}{\Lambda(p,q)}[m_{22}Q_1((T_1y)t^{p-1}) + m_{21}Q_2((T_1y)t^{p-1})] \\
&= \frac{1}{\Lambda(p,q)}[m_{22}Q_1(t^{p-1}) + m_{21}Q_2(t^{p-1})]T_1y \\
&= \frac{1}{\Lambda(p,q)}[m_{22}m_{11} - m_{21}m_{12}]T_1y \\
&= T_1y,
\end{aligned}
$$

$$
\begin{aligned}
T_1((T_2y)t^{q-1}) &= \frac{1}{\Lambda(p,q)}[m_{22}Q_1((T_1y)t^{q-1}) + m_{21}Q_2((T_1y)t^{q-1})] \\
&= \frac{1}{\Lambda(p,q)}[m_{22}Q_1(t^{q-1}) + m_{21}Q_2(t^{q-1})]T_1y \\
&= \frac{1}{\Lambda(p,q)}[-m_{22}m_{21} + m_{21}m_{22}]T_1y \\
&= 0.
\end{aligned}
$$

Similarly, we have

$$T_2((T_1y)t^{p-1}) = 0,\ T_2((T_2y)t^{q-1}) = T_2y.$$

Then we can get

$$
\begin{aligned}
Q^2y &= Q(T_1y(t))t^{p-1} + (T_2y(t))t^{q-1}) \\
&= T_1((T_1y)t^{p-1}) + T_1((T_2y)t^{q-1}) + T_2((T_1y)t^{p-1}) + T_2((T_2y)t^{q-1}) \\
&= Qy.
\end{aligned}
$$

Hence Q is a operator which we need.

Next we show that $\mathrm{Ker}Q = \mathrm{Im}L$. If $y \in \mathrm{Ker}Q$, then $Qy = 0$. By the definition of Qy we obtain

$$
\begin{cases}
\dfrac{m_{22}}{\Lambda(p,q)}Q_1y + \dfrac{m_{21}}{\Lambda(p,q)}Q_2y = 0, \\[2mm]
\dfrac{m_{12}}{\Lambda(p,q)}Q_1y + \dfrac{m_{11}}{\Lambda(p,q)}Q_2y = 0.
\end{cases}
$$

According to

$$
\begin{vmatrix}
\dfrac{m_{22}}{\Lambda(p,q)} & \dfrac{m_{21}}{\Lambda(p,q)} \\[2mm]
\dfrac{m_{12}}{\Lambda(p,q)} & \dfrac{m_{11}}{\Lambda(p,q)}
\end{vmatrix}
= \frac{1}{\Lambda(p,q)} \neq 0,
$$

one has $Q_1y = Q_2y = 0$, i.e. $y \in \mathrm{Im}L$.

If $y \in \mathrm{Im}L$, then $Q_1y = Q_2y = 0$, i.e. $Qy = 0$. Hence $y \in \mathrm{Ker}Q$, and $\mathrm{Ker}Q = \mathrm{Im}L$.

For $y \in Z$, let $y = (y - Qy) + Qy$, then $Q(y - Qy) = Qy - Q^2y = 0$, we have $y - Qy \in \mathrm{Ker}Q = \mathrm{Im}L$. And we know that $Qy \in \mathrm{Im}Q$, so we can get $Z = \mathrm{Im}Q + \mathrm{Im}L$.

On the other hand, for $\forall\ y \in \mathrm{Im}Q \cap \mathrm{Im}L$. Since $y \in \mathrm{Im}Q$, there exist $a, b \in R$ such that $y = at^{p-1} + bt^{q-1}$. From $y \in \mathrm{Im}L$, we have

$$
\begin{cases}
m_{11}a + m_{21}b = 0, \\
m_{12}a + m_{22}b = 0.
\end{cases}
$$

In view of

$$
\begin{vmatrix}
m_{11} & m_{21} \\
m_{12} & m_{22}
\end{vmatrix}
= \Lambda(p,q) \neq 0,
$$

so $a = b = 0$ i.e. $y = 0$. Hence $Z = \mathrm{Im}Q \oplus \mathrm{Im}L$. Since $\dim\mathrm{Ker}L = \dim\mathrm{Im}Q = \mathrm{codim}\mathrm{Im}L = 2$, thus L is a Fredholm map of index zero.

Let $P : Y \to Y$ as follow $P(x(t)) = x(0) + x'(0)t,\ t \in (0, 1)$. Then the generalized inverse of L which is K_p : $\mathrm{Im}L \to \mathrm{dom}L \cap \mathrm{Ker}P$ with $K_p(y(t)) = \int_0^1 (t-s)y(s)ds,\ y \in \mathrm{Im}L$. In fact, for $y \in \mathrm{Im}L$, then $L(K_p y) = (K_p)'' = y(t)$. Besides, if $x(t) \in \mathrm{dom}L \cap \mathrm{Ker}P$, one has

$$(K_pL)x(t) = (K_p)x''(t) = \int_0^t (t-s)x''(s)ds = x(t) - x(0) - x'(0)t = x(t) - Px(t).$$

As a result of $x \in \mathrm{dom}L \cap \mathrm{Ker}P$, then $Px(t) = 0$. Thus $(K_pL)x(t) = x(t)$.

Clearly, $\|K_p y\| \le \|y\|_1$. The proof of Lemma 2.3 is completed.

3. Main Results

Theorem 3.1. Suppose the conditions (C) hold, and assume that
(H_1) There exists $\alpha(t),\ \beta(t),\ \gamma(t) \in L^1[0, 1]$. For $\forall(x_1, x_2) \in R^2,\ t \in (0, 1)$, we have

$$|f(t, x_1, x_2)| \le \alpha(t)|x_1| + \beta(t)|x_2| + \gamma(t).$$

(H_2) There exists a constant $A > 0$ such that $x(t) \in \mathrm{dom}L$, if $|x(t)| > A,\ |x'(t)| > A$. For each $t \in (0, 1)$, then either $Q_1N(x(t)) \ne 0$ or $Q_2N(x(t)) \ne 0$.
(H_3) There exists a constant $B > 0$. For $a, b \in R$, when $|a| > B,\ |b| > B$, we have either

$$Q_1N(a + bt) + Q_2N(a + bt) > 0, \tag{3.1}$$

or

$$Q_1N(a + bt) + Q_2N(a + bt) < 0. \tag{3.2}$$

Then BVP (1.1), (1.2) have at least one solution in $C^1[0, 1]$, and we can get that $\|\alpha\|_1 + \|\beta\|_1 < 1$.
Proof. We will divide the proof into following four steps.
Step 1: Let $\Omega_1 = \{x \in \mathrm{dom}L\backslash\mathrm{Ker}L : Lx = \lambda Nx,\ \lambda \in [0, 1]\}$. Then the set Ω_1 is bounded.

Assume that $x \in \Omega_1$, we get $Lx = \lambda Nx$. Then when $\lambda \ne 0$, we have $Nx \in \mathrm{Im}L$, and $Q_1N(x) = Q_2N(x) = 0$. For (H_2), there exists $t_1,\ t_2 \in (0, 1)$, such that $|x(t_1)| \le A,\ |x'(t_2)| \le A$. Since $x,\ x'$ are absolutely continuous and

$$x(t) = x(t_1) + \int_{t_1}^t x'(s)ds,\ x'(t) = x'(t_2) + \int_{t_2}^t x''(s)ds.$$

So we have $\|x\|_\infty \le 2A + \|x''\|_1,\ \|x'\|_\infty \le A + \|x''\|_1$.

From (H_1), we can get

$$\begin{aligned}\|x''\|_1 = \|Lx\|_1 &\le \|Nx\|_1\\ &\le \|\alpha\|_1\|x\|_\infty + \|\beta\|_1\|x'\|_\infty + \|\gamma\|_1 + \|e\|_1\\ &\le (\|\alpha\|_1 + \|\beta\|_1)\|x''\|_1 + (2\|\alpha\|_1 + \|\beta\|_1)A + \|\gamma\|_1 + \|e\|_1.\end{aligned}$$

Hence $\|x''\| \le \dfrac{1}{1 - (\|\alpha\|_1 + \|\beta\|_1)}[(2\|\alpha\|_1 + \|\beta\|_1)A + \|\gamma\|_1 + \|e\|_1]$.

Thus, there exists a constant $N_1 > 0$ such that $\|x\| \le N_1$, and $\|\alpha\|_1 + \|\beta\|_1 < 1$. So Ω_1 is bounded.
Step 2: Let $\Omega_2 = \{x \in \mathrm{Ker}L : Nx \in \mathrm{Im}L\}$. Then the set Ω_2 is bounded.

For $x \in \Omega_2$, we have $x \in \mathrm{Ker}L$ and $Nx \in \mathrm{Im}L = \mathrm{Ker}Q$. So x can be defined by $x = c_1 + c_2t,\ c_1, c_2 \in R$. Since $QNx = 0$, thus $Q_1N(c_1 + c_2t) = Q_2N(c_1 + c_2t) = 0$. From (H_3), we can get $\|x\| \le |c_1| + |c_2| \le 2B$. Therefore Ω_2 is bounded.
Step 3: Let $\Omega_3 = \{x \in \mathrm{Ker}L : \lambda Jx + (1 - \lambda)QNx = 0,\ \lambda \in [0, 1]\}$. Then the set Ω_3 is bounded.

The linear isomorphism $J : \mathrm{Ker}L \to \mathrm{Im}Q$ define by

$$J(c_1 + c_2 t) = \frac{1}{\Lambda(p,q)}(a_1 t^{p-1} + a_2 t^{q-1}),$$

where

$$a_1 = m_{22}|c_1| + m_{21}|c_2|,\ a_2 = m_{12}|c_1| + m_{11}|c_2|.$$

For any $x(t) = c_1 + c_2 t \in \Omega_3$ and $\lambda Jx + (1-\lambda)QNx = 0$, we obtain

$$\begin{cases} \dfrac{m_{22}}{\Lambda(p,q)}[\lambda|c_1| + (1-\lambda)Q_1N(c_1+c_2t)] + \dfrac{m_{21}}{\Lambda(p,q)}[\lambda|c_2| + (1-\lambda)Q_2N(c_1+c_2t)] = 0, \\ \dfrac{m_{12}}{\Lambda(p,q)}[\lambda|c_1| + (1-\lambda)Q_1N(c_1+c_2t)] + \dfrac{m_{11}}{\Lambda(p,q)}[\lambda|c_2| + (1-\lambda)Q_2N(c_1+c_2t)] = 0. \end{cases}$$

In view of

$$\begin{vmatrix} \dfrac{m_{22}}{\Lambda(p,q)} & \dfrac{m_{21}}{\Lambda(p,q)} \\ \dfrac{m_{12}}{\Lambda(p,q)} & \dfrac{m_{11}}{\Lambda(p,q)} \end{vmatrix} = \frac{1}{\Lambda(p,q)} \neq 0,$$

we have

$$\begin{cases} \lambda|c_1| + (1-\lambda)Q_1N(c_1+c_2t) = 0, \\ \lambda|c_2| + (1-\lambda)Q_2N(c_1+c_2t) = 0. \end{cases}$$

If $\lambda = 1$, then $c_1 = c_2 = 0$. If $\lambda \neq 1$ and $|c_1| > B, |c_2| > B$, in the view of the equality and (3.1), we have $\lambda(|c_1| + |c_2|) = -(1-\lambda)[Q_1N(c_1+c_2t) + Q_2N(c_1+c_2t)] < 0$, which contradicts $\lambda(|c_1|+|c_2|) > 0$. Thus $\|x\| \leq |c_1| + |c_2| \leq 2B$, i.e. Ω_3 is bounded.

If (3.2) holds, then set $\Omega_3 = \{x \in \mathrm{Ker}L : -\lambda Jx + (1-\lambda)QNx = 0,\ \lambda \in [0,1]\}$. Similar to the above, we can show Ω_3 is bounded too.

Step 4: Set Ω is an open bounded subset of Y such that $\bigcup_{i=1}^{3} \overline{\Omega}_i \subset \Omega$. By using Ascoli-Arzela theorem, $K_P(I-Q)N : \overline{\Omega} \to Y$ is compact, thus N is $L-$compact on $\overline{\Omega}$. Then from the above discuss, we have
(i) $Lx \neq \lambda Nx$ for every $(x,\lambda) \in ((\mathrm{dom}L \backslash \mathrm{Ker}L) \cap \partial\Omega) \times [0,1]$.
(ii) $Nx \notin \mathrm{Im}L$ for every $x \in \mathrm{Ker}L \cap \partial\Omega$.

Now we will prove (iii) of Lemma 2.1 is satisfied. Let $H(x,\lambda) = \pm\lambda Jx + (1-\lambda)QNx$. By the step 3, we have $H(x,\lambda) \neq 0,\ \forall x \in \partial\Omega \cap \mathrm{Ker}L$. Hence, by using the homotopy property of degree,

$$\begin{aligned} \deg(QN|_{\mathrm{Ker}L}, \Omega \cap \mathrm{Ker}L, 0) &= \deg(H(\cdot,0), \Omega \cap \mathrm{Ker}L, 0) \\ &= \deg(H(\cdot,1), \Omega \cap \mathrm{Ker}L, 0) \\ &\neq 0. \end{aligned}$$

By Lemma 2.1, $Lx = Nx$ has at least one solution in $\mathrm{dom}L \cap \overline{\Omega}$. So BVP (1.1), (1.2) has at least one solution in $C^1[0,1]$. The proof of Theorem 3.1 is completed.

Remark 3.2. If $e(t) = 0,\ a = c = 0,\ b = -1,\ d = 1,\ \xi(t) = \eta(t) = t$, and there exists $g(t),\ h(t) \in C[0,1]$ such that $g(t)x(t) = \alpha(t)x'(t),\ h(t)x(t) = \beta(t)x'(t)$. Then BVP (1.1), (1.2) is the problem discussed in [4].

Remark 3.3. Taking $a = d = 1,\ c = b = 0$, and there exists $h(t) \in C[0,1]$ such that $h(t)x(t) = \beta(t)x'(t)$, then BVP (1.1), (1.5) discussed in [7] is special case of BVP (1.1), (1.2). And in this paper, we only require $\xi(t),\ \eta(t)$ are bounded.

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References

[1] R. E. Ewing, T. Lin, A class of parameter estimation techniques for fluid flow in porous media, Adv. Water Resour. 14 (1991) 89–97.
[2] G. L. Karakostsa, P. Ch. Tsamatos, Multiple positive solution of some Fredholm integral equations arisen from nonlocal boundary-value problems, Electron. J. Differential Equations, 30 (2002) 1-17.
[3] J. M. Gallardo, Second order differential operators with integral boundary conditions and generation of semigroups, Rocky Mountain J. Math. 30 (2000) 1265-1292.
[4] X. Zhang, M. Feng, W. Ge, Existence result of second-order differential equations with integral boundary conditions at resonance, J. Math. Anal. Appl. 353 (2009) 311-319.
[5] B. Ahmad, A. Alsaedi, B. S. Alghamdi, Analytic approximation of solutions of forced Duffing equation with integral boundary conditions, Nonlinear Anal. 9 (2008) 1727-1740.
[6] Y. Wang, W. Ge, Existence of solutions for a third differential equation with integral boundary conditions, Comput. Math. Appl. 53 (2007) 144-154.
[7] T. Wang, X. Han, Existence of solutions for integral boundary value problem at resonance, Pure Appl. Math. 26 (6) (2010) 1032-1039.
[8] X. Tang, Z. Wang, Solvability of a second order boundary value problem with generalized Sturm-Liouville integral boundary condition, J. Jinggangshan University (Sicence and Technology), 30(2009) 32-34 (in Chinese).
[9] J. Mawhin, Topological degree methods in nonlinear boundary value problems, NSFCBMS Regional Conference Series in Mathematics, American Mathematical Society, Providence, RI, 1979.