Moment Inequality of the Minimum for Nonnegative Negatively Orthant Dependent Random Variables

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Abstract. Let $\{x_n, n \geq 1\}$ be a sequence of positive numbers and $\{\xi_n, n \geq 1\}$ be a sequence of nonnegative negatively orthant dependent (NOD) random variables satisfying certain distribution conditions. An exponential inequality for the minimum $\min_{1 \leq i \leq n} x_i \xi_i$ is given. In addition, the moment inequalities of the minimum $(\mathbb{E} \, k - \min_{1 \leq i \leq n} |x_i \xi_i|^p)^{1/p}$ for nonnegative negatively orthant dependent random variables are established, where $p > 0$ and $k = 1, 2, \cdots, n$. Our results generalize the corresponding ones for independent random variables to the case of negatively orthant dependent random variables.

1. Introduction

For a given sequence of real numbers $a_1, \cdots, a_n$, we denote the $k - th$ smallest one by $k - \min_{1 \leq i \leq n} a_i$; thus, $1 - \min_{1 \leq i \leq n} a_i = \min_{1 \leq i \leq n} a_i$, and $2 - \min_{1 \leq i \leq n} a_i$ is the next smallest, etc. That is to say that $(k - \min_{1 \leq i \leq n} a_i)_{k=1}^n$ is the non-decreasing rearrangement of the sequence $(a_i)_{i=1}^n$. In the same way we denote the $k - th$ biggest number by $k - \max_{1 \leq i \leq n} a_i$.

For decades, many authors have studied the moment inequalities of the maximum. See for example, Gordon et al. [8] considered expressions of the form

$$\mathbb{E} \left(\sum_{k=1}^{m} k - \max_{1 \leq i \leq n} |x_i f_i|^p \right),$$

where $f_1, f_2, \cdots, f_n$ are random variables and $x_1, x_2, \cdots, x_n$ are real numbers. Since the functions $\left(\sum_{k=1}^{m} k - \max_{1 \leq i \leq n} |x_i f_i|^p \right)^{1/p}$ are norms on $\mathbb{R}^n$, such forms appear naturally in the study of various parameters associated with the geometry of Banach spaces. Other applications of these forms can be found in Kwapien and Schütt [15] and Kwapien and Schütt [16].

Recently, Gordon et al. [9] considered expressions of the form $\left(\sum_{k \in I} k - \min_{1 \le i \le n} |x_i|^p\right)^{1/p}$ for subsets $I \subseteq \{1, \cdots, n\}$. These are not norms if I is not an integer interval starting at 1. Hence, for a given sequence of random variables $f_1, \cdots, f_n$, the computation of expressions such as

$$\mathbb{E}\left(k - \min_{1 \le i \le n} |f_i|^p\right) = \mathbb{E}\left((n - k + 1) - \max_{1 \le i \le n} |f_i|^p\right)$$

requires completely different techniques. Such minima, also called order statistics, have been intensively studied during last century. We refer an interested reader to Arnold and Narayanaswamy [4] and David and Nagaraja [6] for basic facts, known results, and references. Most works dealt with the case of independent and identically distributed random variables. Sometimes the condition "to be identically distributed was substituted by the condition" the $f_i's$ have the same first and the same second moments. Gordon et al. [9] dropped these conditions and dealt with sequences of random variables having no restrictions on their moments.

The main results of Gordon et al. [9] are based on the following (α, β)-condition.

Let $\alpha > 0$ and $\beta > 0$ be parameters. We say that a random variable ξ satisfies the (α, β)-condition if

$$\mathbb{P}(|\xi| \le t) \le \alpha t \quad \text{for every } t \ge 0 \tag{1.1}$$

and

$$\mathbb{P}(|\xi| > t) \le e^{-\beta t} \quad \text{for every } t \ge 0. \tag{1.2}$$

It should be noted that many random variables, including $N(0, 1)$ Gaussian variables (with $\alpha = \beta = \sqrt{2/\pi}$) and exponentially distributed variables (with $\alpha = \beta = 1$), satisfy the (α, β)-condition. Gordon et al. [9] provided the following example satisfying the (α, β)-condition.

Example 1.1. *Let $q \ge 1$ and ξ be a nonnegative random variable with the probability density function $p(x) = c_q \exp(-x^q)$, where $c_q = 1/\Gamma(1 + 1/q)$. Then ξ satisfies (1.1) and (1.2) with parameters $\alpha = \beta = c_q$.*

An important case is the case $q = 2$ which corresponds to the Gaussian random variable. Example 1.1 implies that $N(0, 1)$ Gaussian random variables satisfy the (α, β)-condition with $\alpha = \beta = \sqrt{2/\pi}$. We would like also to note that if $q = 1$, then we have an exponentially distributed random variable. In this case $\alpha = \beta = 1$.

Using the (α, β)-condition, Gordon et al. [9] obtained the moment inequalities of the minimum for independent random variables as follows:

Theorem A. *Let $\alpha > 0$, $\beta > 0$ and $p > 0$. Let $(x_i)_{i=1}^n$ be a sequence of real numbers and $\xi_1, \xi_2, \cdots, \xi_n$ be random variables satisfying the (α, β)-condition. Then*

$$\frac{1}{1+p} \alpha^{-p} \left(\sum_{i=1}^n \frac{1}{|x_i|}\right)^{-p} \le \mathbb{E}\left(\min_{1 \le i \le n} |x_i \xi_i|^p\right). \tag{1.3}$$

Moreover, if $\xi_1, \xi_2, \cdots, \xi_n$ are independent, then

$$\mathbb{E}\left(\min_{1 \le i \le n} |x_i \xi_i|^p\right) \le \beta^{-p} \Gamma(1 + p) \left(\sum_{i=1}^n \frac{1}{|x_i|}\right)^{-p}, \tag{1.4}$$

where $\Gamma(\cdot)$ is the Gamma-function.

Theorem B. *Let $\alpha > 0$, $\beta > 0$. Let $p > 0$ and $2 \le k \le n$. Let $0 < x_1 \le x_2 \le \cdots \le x_n$ and $\xi_1, \xi_2, \cdots, \xi_n$ be independent random variables satisfying the (α, β)-condition. Then*

$$c(p, \alpha) \max_{1 \le j \le k} \frac{k + 1 - j}{\sum_{i=j}^n 1/x_i} \le \left(\mathbb{E} \, k - \min_{1 \le i \le n} |x_i \xi_i|^p\right)^{1/p} \le \beta^{-1} C(p, k) \max_{1 \le j \le k} \frac{k + 1 - j}{\sum_{i=j}^n 1/x_i}, \tag{1.5}$$

where $c(p, \alpha) = \frac{1}{2e\alpha} \left(1 - \frac{1}{4\sqrt{\pi}}\right)^{1/p}$ and $C(p, k) = 4\sqrt{2}\max\{p, \ln(1 + k)\}$.

The main purpose of the paper is to generalize the results of Theorem A and Theorem B for independent random variables to the case of negatively orthant dependent random variables.

A finite collection of random variables $X_1, X_2, \cdots, X_n$ is said to be negatively upper orthant dependent (NUOD, in short) if for all real numbers $x_1, x_2, \cdots, x_n$,

$$P(X_i > x_i, i = 1, 2, \cdots, n) \leq \prod_{i=1}^{n} P(X_i > x_i), \qquad (1.6)$$

and negatively lower orthant dependent (NLOD, in short) if for all real numbers $x_1, x_2, \cdots, x_n$,

$$P(X_i \leq x_i, i = 1, 2, \cdots, n) \leq \prod_{i=1}^{n} P(X_i \leq x_i). \qquad (1.7)$$

A finite collection of random variables $X_1, X_2, \cdots, X_n$ is said to be negatively orthant dependent (NOD, in short) if they are both NUOD and NLOD. An infinite sequence $\{X_n, n \geq 1\}$ is said to be NOD if every finite subcollection is NOD.

The notion of NOD random variables was introduced by Lehmann [17] and developed by Joag-Dev and Proschan [11]. Obviously, independent random variables are NOD. Joag-Dev and Proschan [11] pointed out that negatively associated (NA, in short) random variables are NOD, but neither NUOD nor NLOD implies NA. They also presented an example in which $X = (X_1, X_2, X_3, X_4)$ possesses NOD, but does not possess NA. So we can see that NOD is weaker than NA. A number of limit theorems for NOD random variables have been established by many authors. We refer to Volodin [25] for the Kolmogorov exponential inequality, Asadian et al. [5] and Gan et al. [7] for the Rosenthals type inequality, Kim [12] for Hájek–Rényi type inequality, Amini et al. [1, 3], Ko and Kim [10], Klesov et al. [13], Wu and Zhu [33], Wu [28], Shen [19, 21] and Wu et al. [31] for almost sure convergence, Wu and Jiang [32] for the strong consistency of M estimator in a linear model, Kuczmaszewska [14], Taylor et al. [24], Wang et al. [27] and Sung [22] for exponential inequalities, Amini and Bozorgnia [2], Wu [29, 30], Qiu et al. [18], Zarei and Jabbari [34], Sung [23], Wang et al. [26] and Shen [20] for complete convergence, and so forth.

The paper is organized as follows: the notation and preliminaries are given in Section 2. Our main results are presented in Section 3 and the proofs of the main results are provided in Section 4.

2. Notation and Preliminaries

In this section, we will give some notation and a useful lemma, which will be used to prove the main results of the paper.

We say that $(A_j)_{j=1}^{k}$ is a partition of $\{1, 2, \cdots, n\}$ if $\emptyset \neq A_j \subseteq \{1, 2, \cdots, n\}$, $j \leq k$, $\bigcup_{j=1}^{k} A_j = \{1, 2, \cdots, n\}$, and $A_i \bigcap A_j = \emptyset$ for $i \neq j$. By $1/t$ we mean ∞ if $t = 0$ and 0 if $t = \infty$.

We will use the following simple property of $k - \min$ which holds for every sequence $(a_i)_{i=1}^{n}$. For every partition $(A_j)_{j=1}^{k}$ of $\{1, 2, \cdots, n\}$,

$$k - \min_{1 \leq i \leq n} a_i \leq \max_{1 \leq j \leq k} \left\{ \min_{i \in A_j} a_i \right\}. \qquad (2.1)$$

Let x be a positive number. The Gamma-function is defined by

$$\Gamma(x) = \int_{0}^{\infty} t^{x-1} e^{-t} dt.$$

By Stirling's formula, we can get that for every $x \geq 1$,

$$\sqrt{2\pi x} \left(\frac{x}{e}\right)^x < \Gamma(x + 1) < \sqrt{2\pi x} \left(\frac{x}{e}\right)^x e^{\frac{1}{12x}}. \qquad (2.2)$$

The following lemma is useful.

Lemma 2.1 *(Gordon et al. [9], Lemma 4).* *Let $1 \leq k \leq n$. Let $(a_i)_{i=1}^n$ be a nonincreasing sequence of positive numbers. Then there exists a partition $(A_l)_{l=1}^k$ of $\{1, 2, \cdots, n\}$ such that*

$$\min_{1 \leq l \leq k} \sum_{i \in A_l} a_i \geq \frac{1}{2} \min_{1 \leq j \leq k} \frac{1}{k+1-j} \sum_{i=j}^n a_i. \tag{2.3}$$

3. Main Results

Firstly, we will present an exponential inequality for the minimum of nonnegative NOD random variables, which will be used to prove the moment inequality of the minimum for nonnegative NOD random variables.

Theorem 3.1. *Let $\beta > 0$ and $\{x_n, n \geq 1\}$ be a sequence of positive numbers and $\{\xi_n, n \geq 1\}$ be a sequence of nonnegative NOD random variables satisfying (1.2). For fixed $n \geq 1$, denote $a = \sum_{i=1}^n 1/x_i$. Then for every $t > 0$,*

$$\mathbb{P}\left\{\omega : \min_{1 \leq i \leq n} x_i \xi_i(\omega) > t\right\} \leq e^{-\beta a t}. \tag{3.1}$$

By using the exponential inequality for the minimum of nonnegative NOD random variables above, we can get the moment inequalities of the minimum as follows:

Theorem 3.2. *Let $\beta > 0$ and $p > 0$. Let $\{x_n, n \geq 1\}$ be a sequence of real numbers and $\{\xi_n, n \geq 1\}$ be a sequence of nonnegative NOD random variables satisfying (1.2). Then for each $n \geq 1$,*

$$\mathbb{E}\left(\min_{1 \leq i \leq n} |x_i \xi_i|^p\right) \leq \beta^{-p} \Gamma(1+p) \left(\sum_{i=1}^n \frac{1}{|x_i|}\right)^{-p}. \tag{3.2}$$

An immediate consequence of this theorem is the following corollary.

Corollary 3.1. *Let $\alpha > 0$, $\beta > 0$ and $p > 0$. Let $\{x_n, n \geq 1\}$ be a sequence of real numbers and $\{f_n, n \geq 1\}$, $\{\xi_n, n \geq 1\}$ be sequences of random variables satisfying the (α, β)-condition. Assume that $\{\xi_n, n \geq 1\}$ is a sequence of nonnegative NOD random variables. Then for each $n \geq 1$,*

$$\mathbb{E}\left(\min_{1 \leq i \leq n} |x_i \xi_i|^p\right) \leq \Gamma(2+p) \alpha^p \beta^{-p} \mathbb{E}\left(\min_{1 \leq i \leq n} |x_i f_i|^p\right). \tag{3.3}$$

In particular, if $\{f_n, n \geq 1\}$ and $\{\xi_n, n \geq 1\}$ are sequences of $N(0, 1)$ Gaussian random variables, then for each $n \geq 1$,

$$\mathbb{E}\left(\min_{1 \leq i \leq n} |x_i \xi_i|^p\right) \leq \Gamma(2+p) \mathbb{E}\left(\min_{1 \leq i \leq n} |x_i f_i|^p\right). \tag{3.4}$$

The following Theorem 3.3 generalizes the estimates for the expectation of the minimum to the case of the $k - th$ minimum.

Theorem 3.3. *Let $\beta > 0$ and $p > 0$. Let $\{x_n, n \geq 1\}$ be a nondecreasing sequence of positive numbers and $\{\xi_n, n \geq 1\}$ be a sequence of nonnegative NOD random variables satisfying (1.2). Then for each $n \geq 2$ and $2 \leq k \leq n$,*

$$\left(\mathbb{E} \, k - \min_{1 \leq i \leq n} |x_i \xi_i|^p\right)^{1/p} \leq \beta^{-1} C(p, k) \max_{1 \leq j \leq k} \frac{k+1-j}{\sum_{i=j}^n 1/x_i}, \tag{3.5}$$

where $C(p, k) = 4\sqrt{2} \max\{p, \ln(1+k)\}$.

4. Proof of the Main Results

Proof of Theorem 3.1. Denote

$$A_i(t) = \{\omega : x_i \xi_i(\omega) > t\} = \{\omega : \xi_i(\omega) > t/x_i\}, \; i = 1, 2, \cdots, n$$

and

$$A(t) = \left\{\omega : \min_{1 \le i \le n} x_i \xi_i(\omega) > t\right\} = \bigcap_{i=1}^{n} A_i(t).$$

We have by (1.2) that

$$\mathbb{P}(A_i(t)) \le \exp(-\beta t/x_i), \; i = 1, 2, \cdots, n.$$

It follows from (1.6) and the inequality above that

$$\mathbb{P}(A(t)) \le \prod_{i=1}^{n} \mathbb{P}(A_i(t)) \le \exp\left(-\beta t \sum_{i=1}^{n} 1/x_i\right) = e^{-\beta at},$$

which completes the proof of the theorem. □

Proof of Theorem 3.2. It is easily seen that if $x_i = 0$ for some i, then the expectation in (3.2) is 0 and (3.2) is trivial. That is to say, Theorem 3.2 holds true. Therefore, without loss of generality, we assume that $x_i > 0$ for each i.

Denote $B = (\beta \sum_{i=1}^{n} 1/x_i)^{-p}$. Note that

$$\mathbb{E}\left(\min_{1 \le i \le n} |x_i \xi_i|^p\right) = \int_0^\infty \mathbb{P}\left(\min_{1 \le i \le n} |x_i \xi_i|^p > t\right) dt = \int_0^\infty \mathbb{P}\left(\min_{1 \le i \le n} |x_i \xi_i| > t^{1/p}\right) dt. \tag{4.1}$$

By Theorem 3.1, we can see that

$$\mathbb{P}\left(\min_{1 \le i \le n} |x_i \xi_i| > t^{1/p}\right) \le \exp\left(-t^{1/p} B^{-1/p}\right). \tag{4.2}$$

Therefore, it follows from (4.1) and (4.2) that

$$\begin{aligned}
\mathbb{E}\left(\min_{1 \le i \le n} |x_i \xi_i|^p\right) &\le \int_0^\infty \exp\left(-t^{1/p} B^{-1/p}\right) dt \\
&= Bp \int_0^\infty s^{p-1} e^{-s} ds \\
&= Bp\Gamma(p) = B\Gamma(p+1).
\end{aligned}$$

This completes the proof of the theorem. □

Proof of Corollary 3.1. By Theorem 3.2 and Theorem A (1.3), we can get that

$$\begin{aligned}
\mathbb{E}\left(\min_{1 \le i \le n} |x_i \xi_i|^p\right) &\le \beta^{-p} \Gamma(1+p) \left(\sum_{i=1}^{n} \frac{1}{|x_i|}\right)^{-p} \\
&\le \Gamma(2+p) \alpha^p \beta^{-p} \mathbb{E}\left(\min_{1 \le i \le n} |x_i f_i|^p\right),
\end{aligned}$$

which yields the desired result (3.3).

If $\{f_n, n \ge 1\}$ and $\{\xi_n, n \ge 1\}$ are sequences of $N(0,1)$ Gaussian random variables, then $\alpha = \beta = \sqrt{2/\pi}$ follows from Example 1.1. Thus, (3.4) follows from (3.3) immediately. The proof is completed. □

Proof of Theorem 3.3. For fixed $n \geq 2$ and $2 \leq k \leq n$, let $(A_j)_{j=1}^k$ be the partition given by Lemma 2.1 for the sequence $a_i = 1/x_i$, $i = 1, 2, \cdots, n$. For every $q \geq 1$, we have by (2.1) and Theorem 3.2 that

$$
\begin{aligned}
\mathbb{E}\, k - \min_{1 \leq i \leq n} |x_i \xi_i|^p &\leq \mathbb{E} \max_{1 \leq j \leq k} \left\{ \min_{i \in A_j} |x_i \xi_i|^p \right\} \\
&\leq \mathbb{E} \left[\sum_{j=1}^k \left(\min_{i \in A_j} |x_i \xi_i|^p \right)^q \right]^{1/q} \\
&\leq \left(\mathbb{E} \sum_{j=1}^k \min_{i \in A_j} |x_i \xi_i|^{pq} \right)^{1/q} \\
&\leq \left[\Gamma(1 + pq) \beta^{-pq} \sum_{j=1}^k \left(\sum_{i \in A_j} 1/x_i \right)^{-pq} \right]^{1/q} \\
&\leq \beta^{-p} \left(k\Gamma(1 + pq) \right)^{1/q} \max_{1 \leq j \leq k} \left(\sum_{i \in A_j} 1/x_i \right)^{-p}.
\end{aligned}
$$

Therefore, by the inequality above and Lemma 2.1, we can get that

$$
\begin{aligned}
\left(\mathbb{E}\, k - \min_{1 \leq i \leq n} |x_i \xi_i|^p \right)^{1/p} &\leq \beta^{-1} \left(k\Gamma(1 + pq) \right)^{1/(pq)} \max_{1 \leq j \leq k} \left(\sum_{i \in A_j} 1/x_i \right)^{-1} \\
&= \beta^{-1} \left(k\Gamma(1 + pq) \right)^{1/(pq)} \left(\min_{1 \leq j \leq k} \sum_{i \in A_j} 1/x_i \right)^{-1} \\
&\leq 2\beta^{-1} \left(k\Gamma(1 + pq) \right)^{1/(pq)} \max_{1 \leq j \leq k} \frac{k + 1 - j}{\sum_{i=j}^n 1/x_i}.
\end{aligned}
$$

Choosing $q = \frac{\ln(1+k)}{p}$ if $p \leq \ln(1 + k)$ and $q = 1$ if $p > \ln(1 + k)$, we can get the desired result (3.5) from the inequality above and (2.2). This completes the proof of the theorem.

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