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A Note on Closedness and Connectedness in the Category of Proximity Spaces

Muammer Kula[a], Tuğba Maraşlı[a], Samed Özkan[a]

[a]*Department of Mathematics, Erciyes University, Kayseri, Turkey*

Abstract. In this paper, an explicit characterization of the separation properties T_0 and T_1 at a point p is given in the topological category of proximity spaces. Furthermore, the (strongly) closed and (strongly) open subobjects of an object are characterized in the category of proximity spaces and also the characterization of each of the various notions of the connected objects in this category are given.

1. Introduction

The basic concepts of the theory of proximity spaces were emerged by Frigyes Riesz [30] at the mathematical congress in Roma in 1908. This theory was rediscovered and axiomatized by Efremovich [31], [32] in 1934, but not published until 1951. He characterized the proximity relation "A is close to B" as a binary relation on subsets of a set X. In the meanwhile, in 1941, a study was made by Wallace [33], [34] regarding "separation of sets". This study can be considered as the primordial version of the proximity concept.

A large part of the early work in proximity spaces was done by Smirnov [28] and [29]. He showed which topological spaces admit a proximity relation compatible with the given topology [29]. Smirnov was also the first to discover relationship between proximities and uniformities.

Efremovich [32] defined the closure of a subset A of X to be the collection of all points of X "close" A. Thereby he showed that a topology (completely regular) can be introduced in a proximity space. He also showed that every completely regular space X can be turned into a proximity space by using Urysohn's function. This is to say, $A\bar{\delta}B$ iff there exist a continuous function f mapping X into $[0, 1]$ such that $f(A) = 0$ and $f(B) = 1$.

In later years, some authors such as Leader [25], Lodato [26] and Pervin [27] have worked with weaker axioms than Efremovich's proximity axioms. For instance, Pervin omitted the symmetry condition, Leader and Lodato used a weakened form of one of the Efremovich's axioms. In this way, some generalized proximities were appeared.

Baran [2] defined separation properties at a point p, i.e., locally (see [3], [5], [9] and [13]), then generalized this to point free definitions by using the generic element, [20] p. 39, method of topos theory for an arbitrary topological category over sets. One reason for doing this is that, in general, objects in a topos may not have

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Email addresses: kulam@erciyes.edu.tr (Muammer Kula), tubamaras@hotmail.com (Tuğba Maraşlı), samedozkan@erciyes.edu.tr (Samed Özkan)

points, however they always have a generic point. The other reason is that the notions of "closedness" and "strong closedness" in arbitrary topological categories is defined in terms of T_0 and T_1 at a point, p. 335 [2]. The notions of "closedness" and "strong closedness" in set based topological categories are introduced by Baran [2], [4], [8] and it is shown in [9], [10], [13] that these notions form an appropriate closure operator in the sense of Dikranjan and Giuli [15] in some well-known topological categories.

The main goal of this paper is

1. to give the characterization of the separation properties T_0 and T_1 at a point p in the topological category of proximity spaces,
2. to characterize the (strongly) closed and (strongly) open subspaces of a proximity space,
3. to give the characterization of the notions of connectedness and strong connectedness in the category of proximity spaces.

2. Preliminaries

The following are some basic definitions and notations which we will use throughout the paper.

Let $\mathcal{E}$ and $\mathcal{B}$ be any categories. The functor $\mathcal{U} : \mathcal{E} \longrightarrow \mathcal{B}$ is said to be topological or that $\mathcal{E}$ is a topological category over $\mathcal{B}$ if $\mathcal{U}$ is concrete (i.e., faithful and amnestic), has small (i.e., sets) fibers, and for which every $\mathcal{U}$-source has an initial lift or, equivalently, for which each $\mathcal{U}$-sink has a final lift [1].

Note that a topological functor $\mathcal{U} : \mathcal{E} \longrightarrow \mathcal{B}$ is said to be normalized if constant objects, i.e., subterminals, have a unique structure [1], [5], [12], [16], or [17].

Recall in [1] or [17], that an object $X \in \mathcal{E}$ (where $X \in \mathcal{E}$ stands for $X \in \mathrm{Ob}(\mathcal{E})$), a topological category, is discrete iff every map $\mathcal{U}(X) \to \mathcal{U}(Y)$ lifts to a map $X \to Y$ for each object $Y \in \mathcal{E}$ and an object $X \in \mathcal{E}$ is indiscrete iff every map $\mathcal{U}(Y) \to \mathcal{U}(X)$ lifts to a map $Y \to X$ for each object $Y \in \mathcal{E}$.

Let $\mathcal{E}$ be a topological category and $X \in \mathcal{E}$. A is called a subspace of X if the inclusion map $i : A \to X$ is an initial lift (i.e., an embedding) and we denote it by $A \subset X$.

Definition 2.1. *[22] An (Efremovich) proximity space is a pair (X, δ), where X is a set and δ is a binary relation on the powerset of X such that*

(P1) $A\delta B$ *iff* $B\delta A$;
(P2) $A\delta(B \cup C)$ *iff* $A\delta B$ *or* $A\delta C$;
(P3) $A\delta B$ *implies* $A, B \neq \emptyset$;
(P4) $A \cap B \neq \emptyset$ *implies* $A\delta B$;
(P5) $A\bar{\delta}B$ *implies there is an* $E \subseteq X$ *such that* $A\bar{\delta}E$ *and* $(X - E)\bar{\delta}B$;

where $A\bar{\delta}B$ means it is not true that $A\delta B$.

A function $f : (X, \delta) \to (Y, \delta')$ between two proximity spaces is called a *proximity mapping (or a p-map)* iff $f(A)\delta' f(B)$ whenever $A\delta B$. It can easily be shown that f is a p-map iff $f^{-1}(C)\bar{\delta} f^{-1}(D)$ whenever $C\bar{\delta}' D$.

We denote the category of proximity spaces and proximity mappings by **Prox**.

Definition 2.2. *[36] Let X be a nonempty set. A proximity-base on X is a binary relation $\mathfrak{B}$ on $P(X)$ satisfying the axioms $(B1)$ through $(B5)$ given below:*

(B1) $(\emptyset, X) \notin \mathfrak{B}$;
(B2) *If* $A \cap B \neq \emptyset$ *implies* $(A, B) \in \mathfrak{B}$;
(B3) $(A, B) \in \mathfrak{B}$ *iff* $(B, A) \in \mathfrak{B}$;
(B4) *If* $(A, B) \in \mathfrak{B}$ *and* $A \subseteq A^*$, $B \subseteq B^*$ *then* $(A^*, B^*) \in \mathfrak{B}$;
(B5) *If* $(A, B) \notin \mathfrak{B}$ *then there exists a set* $E \subseteq X$ *such that* $(A, E) \notin \mathfrak{B}$ *and* $(X - E, B) \notin \mathfrak{B}$.

2.3 *Let $\mathfrak{B}$ be a proximity-base on a set X and let a binary relation δ on $P(X)$ be defined as follows: $(A, B) \in \delta$ if, given any finite covers $\{A_i : 1 \leq i \leq n\}$ and $\{B_j : 1 \leq j \leq m\}$ of A and B respectively, then there exists a pair (i, j) such that $(A_i, B_j) \in \mathfrak{B}$. δ is a proximity on X finer than the relation $\mathfrak{B}$ [35] or [36].*

2.4 *Let X be a non-empty set, for each $i \in I$, (X_i, δ_i) be a proximity space and $f_i : X \to (X_i, \delta_i)$ be a source in* **Prox**. *Define a binary relation $\mathfrak{B}$ on $P(X)$ as follows: for $A, B \in P(X)$, $A\mathfrak{B}B$ iff $f_i(A)\delta_i f_i(B)$, for all $i \in I$. $\mathfrak{B}$ is a proximity-base on X (Theorem 3.8, [36]). The initial proximity structure δ on X generated by the proximity base $\mathfrak{B}$ is given by for $A, B \in P(X)$, $A\delta B$ iff for any finite covers $\{A_i : 1 \leq i \leq n\}$ and $\{B_j : 1 \leq j \leq m\}$ of A and B respectively, then there exists a pair (i, j) such that $(A_i, B_j) \in \mathfrak{B}$ [36].*

2.5 *Let (X, δ) is a proximity space, Y be a non-empty set and f be a function from a proximity space (X, δ) onto a set Y. The quotient proximity δ^* on Y is given by for every $A, B \subset Y$, $A\delta^* B$ iff for each binary rational s in $[0, 1]$ there is some $C_s \subset Y$ such that $C_0 = A$, $C_1 = B$ and $s < t$ implies $f^{-1}(C_s)\delta f^{-1}(C_t)$ [24] or [37] p.276.*

2.6 *Let X be set and $p \in X$. Let $X \vee_p X$ be the wedge at p ([2] p. 334), i.e., two disjoint copies of X identified at p, i.e., the pushout of $p : 1 \to X$ along itself (where 1 is the terminal object in* **Set***). An epi sink $\{i_1, i_2 : (X, \delta) \longrightarrow (X \vee_p X, \delta')\}(p - maps)$, where i_1, i_2 are the canonical injections, in* **Prox** *is a final lift if and only if the following statement holds. For each pair A, B in the different component of $X \vee_p X$, $A\delta' B$ iff there exist sets C, D in X such that $C\delta\{p\}$ and $\{p\}\delta D$ with $i_k^{-1}(A) = C$ and $i_j^{-1}(B) = D$ for $k, j = 1, 2$ and $k \neq j$. If A and B are in the same component of wedge, then $A\delta' B$ iff there exist sets C, D in X such that $C\delta D$ and $i_k^{-1}(A) = C$ and $i_k^{-1}(B) = D$ for some $k = 1, 2$. Specially, if $i_k(C) = A$ and $i_k(D) = B$, then $(i_k(C), i_k(D)) \in \delta'$ iff $(i_k^{-1}(i_k(C)), i_k^{-1}(i_k(D))) = (C, D) \in \delta$. This is a special case of 2.5.*

2.7 *Let X be a non-empty set. The discrete proximity structure δ on X is given by for $A, B \subset X$, $A\delta B$ iff $A \cap B \neq \emptyset$ [22] p.9.*

2.8 *Let X be a non-empty set. The indiscrete proximity structure δ on X is given by for $A, B \subset X$, $A\delta B$ iff $A \neq \emptyset$ and $B \neq \emptyset$ [23] p.5.*

3. T_0 and T_1 proximity spaces at a point

In this section, we give the characterization of T_0 and T_1 proximity spaces at a point p.

Let B be set and $p \in B$. Let $B \vee_p B$ be the wedge at p. A point x in $B \vee_p B$ will be denoted by $x_1(x_2)$ if x is in the first (resp. second) component of $B \vee_p B$. Note that $p_1 = p_2$.

The principal p-axis map, $A_p : B \vee_p B \to B^2$ is defined by $A_p(x_1) = (x, p)$ and $A_p(x_2) = (p, x)$. The skewed p-axis map, $S_p : B \vee_p B \to B^2$ is defined by $S_p(x_1) = (x, x)$ and $S_p(x_2) = (p, x)$. The fold map at p, $\triangledown_p : B \vee_p B \to B$ is given by $\triangledown_p(x_i) = x$ for $i = 1, 2$ [2], [4].

Note that the maps A_p, S_p and $\triangledown_p$ are the unique maps arising from the above pushout diagram for which $A_p i_1 = (id, f)$, $S_p i_1 = (id, id) : B \to B^2$, $A_p i_2 = S_p i_2 = (f, id) : B \to B^2$, and $\triangledown_p i_j = id$, $j = 1, 2$, respectively, where, $id : B \to B$ is the identity map and $f : B \to B$ is the constant map at p [10].

Remark 3.1. *We define p_1, p_2 by $1 + p$, $p + 1 : B \vee_p B \to B$, respectively where $1 : B \to B$ is the identity map, $f : B \to B$ is constant map at p (i.e., having value p). Note that $\pi_1 A_p = p_1 = \pi_1 S_p$, $\pi_2 A_p = p_2$, $\pi_2 S_p = \nabla_p$, where $\pi_i : B^2 \to B$ is the i-th projection, $i = 1, 2$. When showing A_p and S_p are initial it is sufficient to show that (p_1 and p_2) and (p_1 and ∇_p) are initial lifts, respectively [2], [4].*

Definition 3.2. *(cf. [2], [4]) Let $\mathcal{U} : \mathcal{E} \longrightarrow$ **Set** be a topological functor, X an object in $\mathcal{E}$ with $\mathcal{U}(X) = B$. Let F be a nonempty subset of B. We denote by X/F the final lift of the epi $\mathcal{U}$-sink $q : \mathcal{U}(X) = B \to B/F = (B \backslash F) \cup \{*\}$, where q is the epi map that is the identity on $B \backslash F$ and identifying F with a point $\{*\}$ [2].*

Let p be a point in B.

1. *X is $\overline{T_0}$ at p iff the initial lift of the $\mathcal{U}$-source $\{A_p : B \vee_p B \to \mathcal{U}(X^2) = B^2$ and $\nabla_p : B \vee_p B \to \mathcal{U}\mathcal{D}(B) = B\}$ is discrete, where $\mathcal{D}$ is the discrete functor which is a left adjoint to $\mathcal{U}$.*
2. *X is T_0' at p iff the initial lift of the $\mathcal{U}$ -source $\{id : B \vee_p B \to \mathcal{U}(X \vee_p X) = B \vee_p B$ and $\nabla_p : B \vee_p B \to \mathcal{U}\mathcal{D}(B) = B\}$ is discrete, where $X \vee_p X$ is the wedge in $\mathcal{E}$, i.e., the final lift of the $\mathcal{U}$-sink $\{i_1, i_2 : \mathcal{U}(X) = B \to B \vee_p B\}$ where i_1, i_2 denote the canonical injections.*

3. *X is T_1 at p iff the initial lift of the $\mathcal{U}$-source $\{S_p : B \vee_p B \to \mathcal{U}(X^2) = B^2$ and $\nabla_p : B \vee_p B \to \mathcal{U}\mathcal{D}(B) = B\}$ is discrete.*

Remark 3.3. 1. *Note that for the category **Top** of topological spaces, $\overline{T_0}$ at p, T_0' at p, or T_1 at p reduce to usual T_0 at p or T_1 at p, respectively, where a topological space X is called T_0 at p (resp. T_1 at p) if for each $x \neq p$, there exists a neighborhood of x not containing p or (resp. and) there exists a neighborhood of p not containing x [6].*

2. *Let $\mathcal{U} : \mathcal{E} \to$ **Set** be a topological functor, X an object in $\mathcal{E}$ and $p \in \mathcal{U}(X)$ be a retract of X, i.e., the initial lift $h : 1 \to X$ of the $\mathcal{U}$-source $p : 1 \to \mathcal{U}(X)$ is a retract, where 1 is the terminal object in **Set**. Then if X is $\overline{T_0}$ at p (resp. T_1 at p), then X is T_0' at p but the converse of implication is not true, in general [5], Theorem 2.10.*
3. *If $\mathcal{U} : \mathcal{E} \to$ **Set** be a normalized topological functor, then each of $\overline{T_0}$ at p and T_1 at p implies T_0' at p [5], Corollary 2.11.*
4. *A topological space (Semiuniform convergence space) X is T_i, $i = 0, 1$ iff X is T_i, $i = 0, 1$, at p for all points p in X [6] (resp. [14], Remark 3.2). If $\mathcal{U} : \mathcal{E} \to$ **Set** is a topological functor, X an object in $\mathcal{E}$ and $p \in \mathcal{U}(X)$ is a retract of X and X is T_0 (resp. T_1), then X is T_0 at p (resp. T_1 at p) but the reverse implication is not true, in general [5], Theorem 2.6 and [14], Remark 4.2 (3).*
5. *One of the uses of $\overline{T_0}$ at p and T_1 at p is to define the notions of (strong) closedness in set-based topological categories which are introduced in [2], [4]. These notions are used in [2], [7], [10] and [11] to generalize each of the notions of compactness, connectedness, Hausdorffness, and perfectness to arbitrary set-based topological categories. Moreover, it is shown, in [9], [10], and [13] that they form appropriate closure operators in the sense of Dikranjan and Giuli [15] in some well-known topological categories.*

Theorem 3.4. *Let (X, δ) be a proximity space and $p \in X$. (X, δ) is T_1 at p iff for each $x \neq p$, $(\{x\}, \{p\}) \notin \delta$.*

Proof. Suppose (X, δ) is T_1 at p, i.e., by 2.4, 2.7, 3.1 and 3.2, for any sets U, V on the wedge, $p_1 U \delta p_1 V$, $\nabla_p U \delta \nabla_p V$ and $\nabla_p U \delta_d \nabla_p V$ iff $U \cap V \neq \emptyset$ (δ_d is the discrete proximity structure on X). We shall show that the condition holds. Suppose for some $x, p \in X$, $(\{x\}, \{p\}) \in \delta$ with $x \neq p$. Then, by 2.4, 2.7 and 3.1, for $(U, V) \in \delta'$ (δ' is a proximity structure on the wedge) with $U = \{x_1\}$ and $V = \{x_2\}$, $p_1 U \delta p_1 V = p_1(\{x_1\}) \delta p_1(\{x_2\}) = \pi_1 S_p(\{x_1\}) \delta \pi_1 S_p(\{x_2\}) = \pi_1(\{(x, x)\}) \delta \pi_1(\{(p, x)\}) = \{x\} \delta \{p\}$, *i.e.*, $(\{x\}, \{p\}) \in \delta$, $\nabla_p U \delta \nabla_p V = \nabla_p(\{x_1\}) \delta \nabla_p(\{x_2\}) = \pi_2 S_p(\{x_1\}) \delta \pi_2 S_p(\{x_2\}) = \pi_2(\{(x, x)\}) \delta \pi_2(\{(p, x)\}) = \{x\} \delta \{x\}$, *i.e.*, $(\{x\}, \{x\}) \in \delta$, where $\pi_i : X^2 \to X$, $i = 1, 2$, are the projection maps, and $\nabla_p(\{x_1\}) \delta_d \nabla_p(\{x_2\}) = \{x\} \delta_d \{x\}$, i.e., $(\{x\}, \{x\}) \in \delta_d$ (δ_d is the discrete proximity structure on X). But $U \cap V = \emptyset$. This is a contradiction to the fact that (X, δ) is T_1 at p. Hence if $(\{x\}, \{p\}) \in \delta$, then $x = p$.

Conversely, suppose that for each $x \neq p$, $(\{x\}, \{p\}) \notin \delta$. We need to show that (X, δ) is T_1 at p, i.e., by 2.4, 2.7, 3.1 and 3.2, we must show that the proximity structure δ' on $X \vee_p X$ induced by $S_p : X \vee_p X \to \mathcal{U}((X^2, \delta^2)) = X^2$ and $\nabla_p : X \vee_p X \to \mathcal{U}\mathcal{D}((X, \delta_d)) = X$ is discrete, where δ^2 and δ_d are the product proximity structure on X^2 and the discrete proximity structure on X, respectively. Let (U, V) be any set in δ', i.e., $\pi_i S_p(U) \delta \pi_i S_p(V)$ ($i = 1, 2$) and $\nabla_p U \delta_d \nabla_p V$.

Since δ_d is the discrete proximity structure and $\nabla_p U \delta_d \nabla_p V$, then $\nabla_p U \cap \nabla_p V \neq \emptyset$. It follows that there exists $x \in \nabla_p U \cap \nabla_p V$. Hence, there exist $y \in U$ and $z \in V$ such that $\nabla_p y = x = \nabla_p z$. If $x = p$, then $y = p_i = z$, ($i = 1, 2$) and $p_i \in U \cap V$.

If $x \neq p$, then $y = x_i$, $z = x_j$ ($i, j = 1, 2$). We need to show that $U \cap V \neq \emptyset$.

If $p \in \nabla_p U \cap \nabla_p V$, then $p_i \in U \cap V$ ($i = 1, 2$).

Suppose that $p \notin \nabla_p U \cap \nabla_p V$. We show that both U and V are in the first or in the second or in both component of $X \vee_p X$.

If U subset of the first component of $X \vee_p X$ and V subset of the second component of $X \vee_p X$, then $\{x_1\} \in U$ and $\{x_2\} \in V$. It follows that $\pi_1 S_p(\{x_1\}) \delta \pi_1 S_p(\{x_2\}) = \pi_1(\{(x, x)\}) \delta \pi_1(\{(p, x)\}) = \{x\} \delta \{p\}$, *i.e.*, $(\{x\}, \{p\}) \in \delta$. Since $(\{x\}, \{p\}) \notin \delta$ (by assumption), $(\{x_1\}, \{x_2\}) \notin \delta'$ by the condition ($P2$) of 2.1.

The case U subset of the second component of $X \vee_p X$ and V subset of the first component of $X \vee_p X$ can be handled similarly. Hence U and V can not be in different component of $X \vee_p X$.

If U and V are in both component of $X \vee_p X$, then $U \supseteq \{x_1, x_2\}$ and $V \supseteq \{x_1, x_2\}$. Hence $U \cap V \neq \emptyset$.

If U subset of the first component of $X \vee_p X$ and V subset of both component of $X \vee_p X$, then $U \supseteq \{x_1\}$ and $V \supseteq \{x_1, x_2\}$. Hence $U \cap V \neq \emptyset$.

If U subset of both component of $X \vee_p X$ and V subset of the second component of $X \vee_p X$, then $U \supseteq \{x_1, x_2\}$ and $V \supseteq \{x_2\}$. Hence $U \cap V \neq \emptyset$.

If U and V are in the first component of $X \vee_p X$, then $U \supseteq \{x_1\}$ and $V \supseteq \{x_1\}$. Similarly if U and V are in the second component of $X \vee_p X$, then $U \supseteq \{x_2\}$ and $V \supseteq \{x_2\}$. Hence $U \cap V \neq \emptyset$.

If $(\{x_i\}, \{x_i\}) \in \delta'$, then $\pi_1 S_p(\{x_1\})\delta\pi_1 S_p(\{x_1\}) = \{x\}\delta\{x\}$, i.e., $(\{x\}, \{x\}) \in \delta$, $\pi_2 S_p(\{x_1\})\delta\pi_2 S_p(\{x_1\}) = \{x\}\delta\{x\}$, i.e., $(\{x\}, \{x\}) \in \delta$ and $\pi_1 S_p(\{x_2\})\delta\pi_1 S_p(\{x_2\}) = \{p\}\delta\{p\}$, i.e., $(\{p\}, \{p\}) \in \delta$, $\pi_2 S_p(\{x_2\})\delta\pi_2 S_p(\{x_2\}) = \{x\}\delta\{x\}$, i.e., $(\{x\}, \{x\}) \in \delta$.

We must have $(U, V) \supseteq (\{x_i\}, \{x_i\})$, $(i = 1, 2)$, i.e., $U \cap V \neq \emptyset$ and consequently, by 2.4, 2.7, 3.1 and 3.2, (X, δ) is T_1 at p. □

Theorem 3.5. *Let (X, δ) be a proximity space and $p \in X$. (X, δ) is $\overline{T_0}$ at p iff for each $x \neq p$, $(\{x\}, \{p\}) \notin \delta$.*

Proof. Suppose (X, δ) is $\overline{T_0}$ at p, i.e., by 2.4, 2.7, 3.1 and 3.2, for any sets U, V on the wedge, $p_1 U\delta p_1 V$, $p_2 U\delta p_2 V$ and $\nabla_p U\delta_d\nabla_p V$ iff $U \cap V \neq \emptyset$ (δ_d is the discrete proximity structure on X). We shall show that the condition holds. Suppose for some $x, p \in X$, $(\{x\}, \{p\}) \in \delta$ with $x \neq p$. Then, by 2.4, 2.7 and 3.1, for $(U, V) \in \delta'$ (δ' is a proximity structure on the wedge) with $U = \{x_1\}$ and $V = \{x_2\}$, $p_1 U\delta p_1 V = p_1(\{x_1\})\delta p_1(\{x_2\}) = \pi_1 A_p(\{x_1\})\delta\pi_1 A_p(\{x_2\}) = \pi_1(\{(x, p)\})\delta\pi_1(\{(p, x)\}) = \{x\}\delta\{p\}, i.e., (\{x\}, \{p\}) \in \delta$, $p_2 U\delta p_2 V = p_2(\{x_1\})\delta p_2(\{x_2\}) = \pi_2 A_p(\{x_1\})\delta\pi_2 A_p(\{x_2\}) = \pi_2(\{(x, p)\})\delta\pi_2(\{(p, x)\}) = \{p\}\delta\{x\}, i.e., (\{p\}, \{x\}) \in \delta$, where $\pi_i : X^2 \to X$, $i = 1, 2$, are the projection maps, and $\nabla_p(\{x_1\})\delta_d\nabla_p(\{x_2\}) = \{x\}\delta_d\{x\}$, i.e., $(\{x\}, \{x\}) \in \delta_d$ (δ_d is the discrete proximity structure on X). But $U \cap V = \emptyset$. This is a contradiction to the fact that (X, δ) is $\overline{T_0}$ at p. Hence if $(\{x\}, \{p\}) \in \delta$, then $x = p$.

Conversely, suppose that for each $x \neq p$, $(\{x\}, \{p\}) \notin \delta$. We need to show that (X, δ) is $\overline{T_0}$ at p, i.e., by 2.4, 2.7, 3.1 and 3.2, we must show that the proximity structure δ' on $X \vee_p X$ induced by $A_p : X \vee_p X \to \mathcal{U}((X^2, \delta^2)) = X^2$ and $\nabla_p : X \vee_p X \to \mathcal{UD}((X, \delta_d)) = X$ is discrete, where δ^2 and δ_d are the product proximity structure on X^2 and the discrete proximity structure on X, respectively. Let (U, V) be any set in δ', i.e., $\pi_i A_p(U)\delta\pi_i A_p(V)$ $(i = 1, 2)$ and $\nabla_p(U)\delta_d\nabla_p(V)$.

Since δ_d is the discrete proximity structure and $\nabla_p U\delta_d\nabla_p V$, then $\nabla_p U \cap \nabla_p V \neq \emptyset$. It follows that there exists $x \in \nabla_p U \cap \nabla_p V$. Hence, there exist $y \in U$ and $z \in V$ such that $\nabla_p y = x = \nabla_p z$. If $x = p$, then $y = p_i = z$, $(i = 1, 2)$ and $p_i \in U \cap V$.

If $x \neq p$, then $y = x_i$, $z = x_j$ $(i, j = 1, 2)$. We need to show that $U \cap V \neq \emptyset$.

If $p \in \nabla_p U \cap \nabla_p V$, then $p_i \in U \cap V$ $(i = 1, 2)$.

Suppose that $p \notin \nabla_p U \cap \nabla_p V$. We show that both U and V are in the first or in the second or in both component of $X \vee_p X$.

If U subset of the first component of $X \vee_p X$ and V subset of the second component of $X \vee_p X$, then $\{x_1\} \in U$ and $\{x_2\} \in V$. It follows that $\pi_1 A_p(\{x_1\})\delta\pi_1 A_p(\{x_2\}) = \pi_1(\{(x, p)\})\delta\pi_1(\{(p, x)\}) = \{x\}\delta\{p\}, i.e., = (\{x\}, \{p\}) \in \delta$. Since $(\{x\}, \{p\}) \notin \delta$ (by assumption), $(\{x_1\}, \{x_2\}) \notin \delta'$ by the condition $(P2)$ of 2.1.

The case U subset of the second component of $X \vee_p X$ and V subset of the first component of $X \vee_p X$ can be handled similarly. Hence U and V can not be in different component of $X \vee_p X$.

If U and V are in both component of $X \vee_p X$, then $U \supseteq \{x_1, x_2\}$ and $V \supseteq \{x_1, x_2\}$. Hence $U \cap V \neq \emptyset$.

If U subset of the first component of $X \vee_p X$ and V subset of both component of $X \vee_p X$, then $U \supseteq \{x_1\}$ and $V \supseteq \{x_1, x_2\}$. Hence $U \cap V \neq \emptyset$.

If U subset of both component of $X \vee_p X$ and V subset of the second component of $X \vee_p X$, then $U \supseteq \{x_1, x_2\}$ and $V \supseteq \{x_2\}$. Hence $U \cap V \neq \emptyset$.

If U and V are in the first component of $X \vee_p X$, then $U \supseteq \{x_1\}$ and $V \supseteq \{x_1\}$. Similarly if U and V are in the second component of $X \vee_p X$, then $U \supseteq \{x_2\}$ and $V \supseteq \{x_2\}$. Hence $U \cap V \neq \emptyset$.

If $(\{x_i\}, \{x_i\}) \in \delta'$, then $\pi_1 A_p(\{x_1\})\delta\pi_1 A_p(\{x_1\}) = \{x\}\delta\{x\}$, i.e., $(\{x\}, \{x\}) \in \delta$, $\pi_2 A_p(\{x_1\})\delta\pi_2 A_p(\{x_1\}) = \{p\}\delta\{p\}$, i.e., $(\{p\}, \{p\}) \in \delta$ and $\pi_1 A_p(\{x_2\})\delta\pi_1 A_p(\{x_2\}) = \{p\}\delta\{p\}$, i.e., $(\{p\}, \{p\}) \in \delta$, $\pi_2 A_p(\{x_2\})\delta\pi_2 A_p(\{x_2\}) = \{x\}\delta\{x\}$, i.e., $(\{x\}, \{x\}) \in \delta$.

We must have $(U, V) \supseteq (\{x_i\}, \{x_i\})$, $(i = 1, 2)$, i.e., $U \cap V \neq \emptyset$ and consequently, by 2.4, 2.7, 3.1 and 3.2, (X, δ) is $\overline{T_0}$ at p. □

Remark 3.6. *Let (X, δ) be a proximity space and $p \in X$. It follows from 3.4, 3.5 that (X, δ) is $\overline{T_0}$ at p if and only if (X, δ) is T_1 at p if and only if for each $x \neq p$, $(\{x\}, \{p\}) \notin \delta$.*

Theorem 3.7. *All proximity spaces are T_0' at p.*

Proof. Suppose that (X, δ) is a proximity space and $p \in X$. By 2.4, 2.6, 2.7, 3.1 and 3.2 we will show that for any $(i_k(U), i_k(V)) \in \delta'$ (δ' is a proximity structure on $X \vee_p X$), if $i_k(U, V) = (i_k(U), i_k(V)) \in \delta'$ $(k = 1, 2)$ for some $(U, V) \in \delta$ $(U, V \subset X)$ and $(\nabla_p(i_k(U)), \nabla_p(i_k(V))) \in \delta_d$ (δ_d is the discrete proximity structure on X), then we will show that $(i_k(U), i_k(V)) \supseteq (\{p_k\}, \{p_k\})$ or $(i_k(U), i_k(V)) \supseteq (\{x_k\}, \{x_k\})$, $(k = 1, 2)$, i.e., $i_k(U) \cap i_k(V) \neq \emptyset$.

Since δ_d is the discrete proximity structure and $\nabla_p(i_k(U))\delta_d\nabla_p(i_k(V))$, then $\nabla_p(i_k(U)) \cap \nabla_p(i_k(V)) \neq \emptyset$. It follows that there exists $x \in \nabla_p(i_k(U)) \cap \nabla_p(i_k(V))$. Hence, there exist $y \in i_k(U)$ and $z \in i_k(V)$ such that $\nabla_p y = x = \nabla_p z$. If $x = p$, then $y = p_k = z$, $(k = 1, 2)$ and $p_k \in i_k(U) \cap i_k(V)$.

If $x \neq p$, then $y = x_k$, $z = x_n$ $(k, n = 1, 2)$. We need to show that $i_k(U) \cap i_k(V) \neq \emptyset$.

If $p \in \nabla_p(i_k(U)) \cap \nabla_p(i_k(V))$, then $p_k \in i_k(U) \cap i_k(V)$ $(k = 1, 2)$.

Suppose that $p \notin \nabla_p(i_k(U)) \cap \nabla_p(i_k(V))$. We show that both $i_k(U)$ and $i_k(V)$ are in the first or in the second or in both component of $X \vee_p X$.

If $i_k(U)$ subset of the first component of $X \vee_p X$ and $i_k(V)$ subset of the second component of $X \vee_p X$, then $\{x_1\} \in i_k(U)$ and $\{x_2\} \in i_k(V)$. But, if $(i_k(U), i_k(V)) \supseteq (\{x_1\}, \{x_2\}) \in \delta'$ for some $(U, V) \in \delta$ and $k = 1$ (resp. $k = 2$), then $(\{x_1\}, \{x_2\}) \in (i_1(U), i_1(V))$ which shows that x_2 (resp. x_1) must be in the first (resp. second) component of $X \vee_p X$, a contradiction since $x \neq p$.

Similarly, if $i_k(U)$ subset of the second component of $X \vee_p X$ and $i_k(V)$ subset of the first component of $X \vee_p X$, then $\{x_2\} \in i_k(U)$ and $\{x_1\} \in i_k(V)$. But, if $(i_k(U), i_k(V)) \supseteq (\{x_2\}, \{x_1\}) \in \delta'$ for some $(U, V) \in \delta$ and $k = 1$ (resp. $k = 2$), then $(\{x_2\}, \{x_1\}) \in (i_1(U), i_1(V))$ which shows that x_2 (resp. x_1) must be in the first (resp. second) component of $X \vee_p X$, a contradiction since $x \neq p$. Hence $i_k(U)$ and $i_k(V)$ can not be in different component of $X \vee_p X$.

If $i_k(U)$ and $i_k(V)$ are in both component of $X \vee_p X$, then $i_k(U) \supseteq \{x_1, x_2\}$ and $i_k(V) \supseteq \{x_1, x_2\}$. Hence $i_k(U) \cap i_k(V) \neq \emptyset$.

If $i_k(U)$ subset of the first component of $X \vee_p X$ and $i_k(V)$ subset of both component of $X \vee_p X$, then $i_k(U) \supseteq \{x_1\}$ and $i_k(V) \supseteq \{x_1, x_2\}$. Hence $i_k(U) \cap i_k(V) \neq \emptyset$.

If $i_k(U)$ subset of both component of $X \vee_p X$ and $i_k(V)$ subset of the second component of $X \vee_p X$, then $i_k(U) \supseteq \{x_1, x_2\}$ and $i_k(V) \supseteq \{x_2\}$. Hence $i_k(U) \cap i_k(V) \neq \emptyset$.

If $i_k(U)$ and $i_k(V)$ are in the first component of $X \vee_p X$, then $i_k(U) \supseteq \{x_1\}$ and $i_k(V) \supseteq \{x_1\}$. Similarly if $i_k(U)$ and $i_k(V)$ are in the second component of $X \vee_p X$, then $i_k(U) \supseteq \{x_2\}$ and $i_k(V) \supseteq \{x_2\}$. Hence $i_k(U) \cap i_k(V) \neq \emptyset$.

We must have $(i_k(U), i_k(V)) \supseteq (\{x_i\}, \{x_i\})$, $(i = 1, 2)$, i.e., $i_k(U) \cap i_k(V) \neq \emptyset$ and consequently, by 2.4, 2.6, 2.7, 3.1 and 3.2, (X, δ) is T_0' at p. □

Remark 3.8. *If a proximity space (X, δ) is $\overline{T_0}$ at $p \in X$ or T_1 at $p \in X$, then it is T_0' at p. However, the converse is not true generally. For example, let $X = \{a, b\}$ and $\delta = \{(X, X), (\{a\}, \{a\}), (\{b\}, \{b\}), (X, \{a\}), (\{a\}, X), (X, \{b\}), (\{b\}, X), (\{a\}, \{b\}), (\{b\}, \{a\})\}$. Then (X, δ) is T_0' at $p = a$ but it is not $\overline{T_0}$ at $p = a$ or T_1 at p since $(\{a\}, \{b\}) \in \delta$ but $a \neq b$.*

4. Closedness and Connectedness

In this section, the (strongly) closed and (strongly) open subobjects of an object are characterized in the category of proximity spaces. Furthermore, the characterization of each of the various notions of the connected objects in this category are given.

Let B be set and $p \in B$. The infinite wedge product $\vee_p^\infty B$ is formed by taking countably many disjoint copies of B and identifying them at the point p. Let $B^\infty = B \times B \times ...$ be the countable cartesian product of B. Define $A_p^\infty : \vee_p^\infty B \to B^\infty$ by $A_p^\infty(x_i) = (p, p, ..., p, x, p, ...)$, where x_i is in the i-th component of the infinite wedge and x is in the i-th place in $(p, p, ..., p, x, p, ...)$ (infinite principal p-axis map), and $\nabla_p^\infty : \vee_p^\infty B \longrightarrow B$ by $\nabla_p^\infty(x_i) = x$ for all $i \in I$ (infinite fold map), [2], [4].

Note, also, that A_p^∞ is the unique map arising from the multiple pushout of $p : 1 \to B$ for which $A_p^\infty i_j = (p, p, ..., p, id, p, ...) : B \to B^\infty$, where the identity map, id, is in the j-th place [10].

Definition 4.1. *(cf. [2], [4]) Let $\mathcal{U} : \mathcal{E} \longrightarrow$ **Set** be a topological functor, X an object in $\mathcal{E}$ with $\mathcal{U}(X) = B$. Let F be a nonempty subset of B. We denote by X/F the final lift of the epi $\mathcal{U}$-sink $q : \mathcal{U}(X) = B \to B/F = (B\backslash F) \cup \{*\}$, where q is the epi map that is the identity on $B\backslash F$ and identifying F with a point $\{*\}$ [2].*

Let p be a point in B.

1. *p is closed iff the initial lift of the $\mathcal{U}$-source $\{A_p^\infty : \vee_p^\infty B \to \mathcal{U}(X^\infty) = B^\infty$ and $\nabla_p^\infty : \vee_p^\infty B \to \mathcal{U}\mathcal{D}(B) = B\}$ is discrete.*
2. *$F \subset X$ is closed iff $\{*\}$, the image of F, is closed in X/F or $F = \emptyset$.*
3. *$F \subset X$ is strongly closed iff X/F is T_1 at $\{*\}$ or $F = \emptyset$.*
4. *If $B = F = \emptyset$, then we define F to be both closed and strongly closed.*

Recall that a prebornological space is a pair $(B, \mathfrak{F})$, where $\mathfrak{F}$ is a family of subsets of B that is closed under nonempty finite union and contains all finite nonempty subsets of B. A morphism $(B, \mathfrak{F}) \to (B_1, \mathfrak{F}_1)$ of such spaces is a function $f : B \to B_1$ such that $f(C) \in \mathfrak{F}_1$ if $C \in \mathfrak{F}$. We denote by ***PBorn***, the category thus obtained. This category is topological category over ***Set*** [8].

The category ***Prord*** of preordered spaces has as objects the pairs (B, R), where B is a set and R is a reflexive and transitive relation on B and has as morphism $(B, R) \to (B_1, R_1)$ those functions $f : B \to B_1$ such that if aRb, then $f(a)R_1 f(b)$ for all $a, b \in B$. This category is topological category over ***Set*** [13].

Lemma 4.2. *([13], Theorem 3.6) Let (B, R) be a preordered set (i.e., R is a reflexive and transitive relation on B), and $\emptyset \neq F \subset B$. Then,*

(i) F is a closed subset of B iff for any $x \in B$, if there exists $a, b \in F$ such that xRa and bRx, then $x \in F$.
(ii) F is a strongly closed subset of B iff for each $x \in B$, if there exists $a \in F$ such that xRa or aRx, then $x \in F$.

Lemma 4.3. *([4], Theorem 3.9 and 3.10) Let $(B, \mathfrak{F})$ be a prebornological space. Then,*

(i) A subset $F \subset B$ is closed iff $B = F$ or $F = \emptyset$.
(ii) All subsets of B are strongly closed.

Remark 4.4.
1. *In **Top**, the notion of closedness coincides with the usual one [2] and F is strongly closed iff F is closed and for each $x \notin F$ there exists a neighbourhood of F missing x [2]. If a topological space is T_1, then the notions of closedness and strong closedness coincide [2].*
2. *In general, for an arbitrary topological category, the notions of closedness and strong closedness are independent of each other. To see this, let $B = \{-1, 1\}$, $R = \{(-1, 1), (-1, -1), (1, 1)\}$ and $F = \{1\}$. Then (B, R) is a preordered set and by 4.2, F is closed, but F is not strongly closed. On the other hand, let $B = \mathbb{R}$, the set of real numbers, and $\mathfrak{F} = P(\mathbb{R}) - \{\emptyset\}$, the set of all nonempty subsets of $\mathbb{R}$. Note, [8] Remark 3.2, that $(B, \mathfrak{F})$ is a prebornological space and by 4.3, Q, the set of rational numbers, is strongly closed, but Q is not closed.*

Theorem 4.5. *Let (X, δ) be a proximity space and $p \in X$. $\{p\}$ is closed in X iff for any $B \subset X$, if $\{p\}\delta B$, then $p \in B$.*

Proof. Suppose $\{p\}$ is closed in X, i.e., by 4.1, the proximity structure δ' on $\vee_p^\infty X$ induced by $A_p^\infty : \vee_p^\infty X \to \mathcal{U}(X^\infty) = X^\infty$ and $\nabla_p^\infty : \vee_p^\infty X \to \mathcal{U}\mathcal{D}(X) = X$ is discrete. We will show that for any $B \subset X$, if $\{p\}\delta B$, then $p \in B$. Suppose that $\{p\}\delta B$ while $p \notin B$, for some $B \subset X$. Then for some $x \neq p$ and $x \in B$, we get $(\{x\}, \{p\}) \in \delta$ by the condition $(P2)$ of 2.1. Note that $(\{x_1\}, \{x_2\}) \in \delta'$, since $\pi_1 A_p^\infty\{x_1\}\delta\pi_1 A_p^\infty\{x_2\} = \pi_1\{(x, p, p, ...)\}\delta\pi_1\{(p, x, p, ...)\} = \{x\}\delta\{p\}$, i.e., $(\{x\}, \{p\}) \in \delta$ (by assumption), $\pi_2 A_p^\infty\{x_1\}\delta\pi_2 A_p^\infty\{x_2\} = \pi_2\{(x, p, p, ...)\}\delta\pi_2\{(p, x, p, ...)\} = \{p\}\delta\{x\}$, i.e., $(\{p\}, \{x\}) \in \delta$ (by assumption and by the condition $(P1)$ of 2.1), for $i \geq 3$, $\pi_i A_p^\infty\{x_1\}\delta\pi_i A_p^\infty\{x_2\} = \pi_i\{(x, p, p, ...)\}\delta\pi_i\{(p, x, p, ...)\} = \{p\}\delta\{p\}$, i.e., $(\{p\}, \{p\}) \in \delta$ (by the condition $(P4)$ of 2.1) where $\pi_i : X^\infty \to X$ projection function and $\nabla_p^\infty(\{x_1\})\delta_d\nabla_p^\infty(\{x_2\}) = \{x\}\delta_d\{x\}$, i.e., $(\{x\}, \{x\}) \in \delta_d$ (δ_d is the discrete proximity structure on X). But $\{x_1\} \cap \{x_2\} = \emptyset$. This is a contradiction to the fact that δ' is discrete. Hence if $p \notin B$, then $\{p\}\bar{\delta}B$.

Conversely, suppose that the condition holds. We need to show that $\{p\}$ is closed in X. By 4.1, we must show that the proximity structure δ' on $\vee_p^\infty X$ induced by $A_p^\infty : \vee_p^\infty X \to \mathcal{U}(X^\infty) = X^\infty$ and $\nabla_p^\infty : \vee_p^\infty X \to \mathcal{U}\mathcal{D}(X) = X$ is discrete. Since δ_d is the discrete proximity structure, then $\nabla_p^\infty U \cap \nabla_p^\infty V \neq \emptyset$ for all $U, V \subset \vee_p^\infty X$. If $\nabla_p^\infty U \cap \nabla_p^\infty V \neq \emptyset$, then there exists $x \in \nabla_p^\infty U \cap \nabla_p^\infty V$, i.e., $x \in \nabla_p^\infty U$ and $x \in \nabla_p^\infty V$. Hence, there exists $y \in U$ and there exists $z \in V$ such that $\nabla_p^\infty y = x = \nabla_p^\infty z$. If $x = p$, then $y = p_i = z$ for all $i \in I$, and $p_i \in U \cap V$.

If $x \neq p$, then $y = x_i$, $z = x_j$ $(i \neq j)$ for all $i, j \in I$. We need to show that $U \cap V \neq \emptyset$.

If $p \in \nabla_p^\infty U \cap \nabla_p^\infty V$, then $p_i \in U \cap V$ for all $i \in I$.

Suppose that $p \notin \nabla_p^\infty U \cap \nabla_p^\infty V$. We show that both U and V are in the i. component or in the j. component or in the i. component and in the j. component of $\vee_p^\infty X$.

If U subset of the i. component of $\vee_p^\infty X$ and V subset of the j. component of $\vee_p^\infty X$, then $\{x_i\} \in U$ and $\{x_j\} \in V$. It follows that $\pi_i A_p^\infty(\{x_i\})\delta\pi_i A_p^\infty(\{x_j\}) = \{x\}\delta\{p\}$, i.e., $(\{x\}, \{p\}) \in \delta$ and $\pi_j A_p^\infty(\{x_i\})\delta\pi_j A_p^\infty(\{x_j\}) = \{p\}\delta\{x\}$, i.e., $(\{p\}, \{x\}) \in \delta$. Since $(\{p\}, \{x\}) \notin \delta$ (by assumption), $(\{x_i\}, \{x_j\}) \notin \delta'$ by the condition $(P2)$ of 2.1.

The case U subset of the j. component of $\vee_p^\infty X$ and V subset of the i. component of $\vee_p^\infty X$ can be handled similarly. Hence U and V can not be in different component of $\vee_p^\infty X$.

If U and V are in the i. component of $\vee_p^\infty X$ and in the j. component of $\vee_p^\infty X$, then $U \supseteq \{x_i, x_j\}$, $V \supseteq \{x_i, x_j\}$. Hence $U \cap V \neq \emptyset$.

If U subset of the i. component and the j. component of $\vee_p^\infty X$ and V subset of the j. component of $\vee_p^\infty X$, then $U \supseteq \{x_i, x_j\}$ and $V \supseteq \{x_j\}$. Hence $U \cap V \neq \emptyset$.

If U subset of the i. component of $\vee_p^\infty X$ and V subset of the i. component of $\vee_p^\infty X$ and of the j. component of $\vee_p^\infty X$, then $U \supseteq \{x_i\}$ and $V \supseteq \{x_i, x_j\}$. Hence $U \cap V \neq \emptyset$.

If U and V are in the i. component of $\vee_p^\infty X$, then $U \supseteq \{x_i\}$ and $V \supseteq \{x_i\}$. Similarly if U and V are in the j. component of $\vee_p^\infty X$, then $U \supseteq \{x_j\}$ and $V \supseteq \{x_j\}$. Hence $U \cap V \neq \emptyset$.

If $(\{x_i\}, \{x_i\}) \in \delta'$ for all $i \in I$, then $\pi_i A_p^\infty(\{x_i\})\delta\pi_i A_p^\infty(\{x_i\}) = \{x\}\delta\{x\}$, i.e., $(\{x\}, \{x\}) \in \delta$ and $\nabla_p^\infty(\{x_i\})\delta_d\nabla_p^\infty(\{x_i\}) = \{x\}\delta_d\{x\}$, i.e., $(\{x\}, \{x\}) \in \delta_d$ for any point x in B.

Consequently if $(U, V) \in \delta'$, then $(U, V) \supseteq (\{x_i\}, \{x_i\})$ for all $i \in I$ or $(U, V) \supseteq (\{p_i\}, \{p_i\})$ for all $i \in I$. Hence by 2.7, δ' is discrete on $\vee_p^\infty X$, i.e., $\{p\}$ is closed in X.

□

Theorem 4.6. *Let (X, δ) be a proximity space. Then $\emptyset \neq F \subset X$ is closed iff $x \in F$ whenever $\{x\}\delta F$ for all $x \in X$.*

Proof. Suppose that F is closed and $x \notin F$ whenever $\{x\}\delta F$ for some $x \in X$. By Definition 4.1, F is closed iff $\{*\}$ is closed in X/F and by 4.5, $\{*\}$ is closed in X/F iff for any $B \subset X/F$, $\{*\}\delta' B$ then $* \in B$, where δ' is the quotient proximity structure on X/F that is induced by the map $q : X \to X/F$ (defined in the Definition 4.1). Since $\{x\}\delta F$ and q is a p-map, we have $q(\{x\})\delta' q(F)=\{x\}\delta'\{*\}$, i.e., $(\{x\}, \{*\}) \in \delta'$, a contradiction since $* \notin \{x\} \subset X/F$.

Conversely, suppose that $x \in F$ whenever $\{x\}\delta F$ for any $x \in X$. We shall show that F is closed, i.e., $\{*\}\delta' B$ then $* \in B$, for any $B \subset X/F$. Suppose that $* \notin B$, for some $B \subset X/F$. Since $\{*\}\delta' B$, then for each binary rational s in $[0, 1]$ there is some $C_s \subset X/F$ such that $C_0 = \{*\}$, $C_1 = B$ and $s < t$ implies $q^{-1}(C_s)\delta q^{-1}(C_t)$. $q^{-1}(\{*\})\delta q^{-1}(B) = F\delta B$, by definition of q-map and 2.5. Since $F\delta B$, then there exists $y \in B$ such that $F\delta\{y\}$ by the condition $(P2)$ of 2.1. But for all $y \in B$, $y \notin F$ since $* \notin B$. This is a contradiction. □

Theorem 4.7. *Let (X, δ) be a proximity space. Then $\emptyset \neq F \subset X$ is strongly closed iff $x \in F$ whenever $\{x\}\delta F$ for all $x \in X$.*

Proof. Suppose that F is strongly closed and $x \notin F$ whenever $\{x\}\delta F$ for some $x \in X$. By Definition 4.1, F is strongly closed iff X/F is T_1 at $\{*\}$ and by 3.4, X/F is T_1 at $\{*\}$ iff for any $B \subset X/F$, $\{*\}\delta' B$ then $* \in B$, where δ' is the quotient proximity structure on X/F that is induced by the map $q : X \to X/F$ (defined in the Definition 4.1). Since $\{x\}\delta F$ and q is a p-map, we have $q(\{x\})\delta' q(F)=\{x\}\delta'\{*\}$, i.e., $(\{x\}, \{*\}) \in \delta'$, a contradiction since $* \notin \{x\} \subset X/F$.

Conversely, suppose that $x \in F$ whenever $\{x\}\delta F$ for any $x \in X$. We shall show that F is strongly closed, i.e., by 3.4, if $\{*\}\delta' B$, then $* \in B$, for any $B \subset X/F$. Suppose that $* \notin B$, for some $B \subset X/F$. Since $\{*\}\delta' B$, then for each binary rational s in $[0, 1]$ there is some $C_s \subset X/F$ such that $C_0 = \{*\}$, $C_1 = B$ and $s < t$ implies $q^{-1}(C_s)\delta q^{-1}(C_t)$. $q^{-1}(\{*\})\delta q^{-1}(B) = F\delta B$, by definition of q-map and 2.5. Since $F\delta B$, then there exists $y \in B$ such that $F\delta\{y\}$ by the condition $(P2)$ of 2.1. But for all $y \in B$, $y \notin F$ since $* \notin B$. This is a contradiction. □

Definition 4.8. *Let $\mathcal{E}$ be a topological category over **Set**, X an object in $\mathcal{E}$ and F be a nonempty subset of X.*

1. *$F \subset X$ is open iff F^c, the complement of F, is closed in X.*
2. *$F \subset X$ is strongly open iff F^c, the complement of F, is strongly closed in X [11].*

Note that in ***Top*** the notion of openness coincides with the usual one [11]. If a topological space is T_1, then the notions of openness and strong openness coincide [11].

Theorem 4.9. *Let (X, δ) be a proximity space. $\emptyset \neq F \subset X$ is (strongly) open iff $x \in F^c$ whenever $\{x\}\delta F^c$ for all $x \in X$.*

Proof. It follows from 4.6, 4.7 and 4.8. □

Definition 4.10. *([37], p. 268) Let (X, δ) be a proximity space and $A \subset X$. Define $\bar{A} = \{x | x\delta A\}$ and if $\bar{A} = A$, then A is said to be closed.*

Remark 4.11.

1. *Let (X, δ) be a proximity space and $A \subset X$. A is closed (in the usual above sense) iff for each $x \in X$, if $x\delta A$, then $x \in A$.*
2. *Let (X, δ) be a proximity space. It follows from 4.6, 4.7 and 4.10 that the notions of closedness (in our sense) and strong closedness coincide with the notion of closedness in the usual sense.*

We now give the characterization the various notions of connectedness in the category of proximity spaces.

Definition 4.12. *Let $\mathcal{E}$ be a topological category over **Set** and X be an object in $\mathcal{E}$.*

1. *X is connected iff the only subsets of X both strongly open and strongly closed are X and $\emptyset$ [11].*
2. *X is strongly connected iff the only subsets of X both open and closed are X and $\emptyset$ [11].*
3. *X is D-connected iff any morphism from X to any discrete object is constant (cf. [11], [17], [18], [19], [21]).*

Note that for the category ***Top*** of topological spaces, the notion of strongly connectedness and D-connectedness coincides with the usual notion of connectedness. If a topological space X is T_1, then, by 4.12, the notions of connectedness and strong connectedness coincide [11].

Theorem 4.13. *A proximity space (X, δ) is (strongly) connected iff for any non-empty proper subset F of X, either the condition* (1) *or* (2) *holds.*

1. *$x \notin F$ whenever $\{x\}\delta F$ for some $x \in X$.*
2. *$x \notin F^c$ whenever $\{x\}\delta F^c$ for some $x \in X$.*

Proof. Suppose that (X, δ) is (strongly) connected but conditions (1) and (2) do not hold for some non-empty proper subset F of X. Since the condition (1) does not hold, we get $x \in F$ whenever $\{x\}\delta F$ for all $x \in X$ which means that subset F is (strongly) closed by 4.6 or 4.7. Since the condition (2) does not hold, we get $x \in F^c$ for all $x \in X$, whenever $\{x\}\delta F^c$. This means that F^c is (strongly) closed. So F is (strongly) open by 4.9. Hence F is open and closed. But this is a contradiction since (X, δ) is (strongly) connected.

Conversely, suppose that the condition (1) holds. Then $x \notin F$ whenever $\{x\}\delta F$ for some $x \in X$ and F is not (strongly) closed 4.6 or 4.7. Suppose that the condition (2) holds. Then for some $x \in X$, $x \notin F^c$ whenever $\{x\}\delta F^c$. This means that F^c is not (strongly) closed. So F is not (strongly) open by 4.9. Hence the only subsets of X both (strongly) open and (strongly) closed are X and $\emptyset$. From here (X, δ) is (strongly) connected. □

Theorem 4.14. *Suppose (X, δ) is any proximity space. (X, δ) is D-connected iff $\{x\}\delta\{y\}$ for all $x, y \in X$ with $x \neq y$.*

Proof. Suppose that (X, δ) is D-connected and the condition does not hold, i.e., $(\{x\}, \{y\}) \notin \delta$ for some $x, y \in X$ with $x \neq y$. Let (Y, δ') be a discrete proximity space with $Card Y > 1$. Define $f : (X, \delta) \to (Y, \delta')$ by $f(t) = a$, if $t = x$ and $f(t) = b$, if $t \in \{x\}^c$. We show that f is a p-map. Since (Y, δ') is a discrete proximity space and $\{a\} \cap \{b\} = \emptyset$, $(\{a\}, \{b\}) \notin \delta'$. Since $(\{x\}, \{y\}) \notin \delta$ and $f^{-1}(\{a\}) = \{x\}$, $f^{-1}(\{b\}) = \{x\}^c$, then $(f^{-1}(\{a\}), f^{-1}(\{b\})) \notin \delta$, i.e., $(\{x\}, \{x\}^c) \notin \delta$. Indeed, if $(\{x\}, \{x\}^c) \in \delta$, then $(f(\{x\}), f(\{x\}^c)) = (\{a\}, \{b\}) \in \delta'$. But this is a contradiction since (Y, δ') is the discrete proximity space. Consequently, f is a p-map but it is not a constant map. This is a contradiction since (X, δ) is a D-connected. Hence $\{x\}\delta\{y\}$ for all $x, y \in X$ with $x \neq y$.

Conversely, suppose that the condition holds. Let $f : (X, \delta) \to (Y, \delta')$ be a p-map with (Y, δ') is a discrete proximity space. If $Card Y = 1$, then f is constant and (X, δ) is D-connected. Suppose that $Card Y > 1$ and f is not constant. There exist $x, y \in X$ with $x \neq y$ such that $f(x) \neq f(y)$. Hence $\{f(x)\} \cap \{f(y)\} = \emptyset$ and $(\{f(x)\}, \{f(y)\}) \notin \delta'$. Since f is p-map, $(f^{-1}(\{f(x)\}), f^{-1}(\{f(y)\})) \notin \delta$, i.e., $(\{x\}, \{y\}) \notin \delta$. But this is a contradiction by assumption. Hence f must be constant and this means that (X, δ) is D-connected. □

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