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On Generalized Spiral-like Analytic Functions

Khalida Inayat Noor[a], Nazar Khan[a], Muhammad Aslam Noor[a]

[a]*Mathematics Department, COMSATS Institute of Information Technology, Park Road, Islamabad, Pakistan*

Abstract. In this paper, we use the concept of bounded Mocanu variation to introduce a new class of analytic functions, defined in the open unit disc, which unifies a number of classes previously studied such as those of functions with bounded radius rotation and bounded Mocanu variation. It also generalizes the concept of β-spiral likeness in some sense. Some interesting properties of this class including inclusion results, arclength problems and a sufficient condition for univalency are studied.

1. Introduction

Let $\mathcal{A}$ denote the class of functions f :

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disk $E = \{z : |z| < 1\}$. Let S, S^*, C and M_α be the subclasses of $\mathcal{A}$ which consist of, respectively, univalent, starlike (with respect to origin), convex and α-starlike functions. Let P be the class of functions p, analytic in E with $p(0) = 1$ and satisfying $Re\{p(z)\} > 0, \quad z \in E$. The class P is called the class of analytic functions with positive real part. It is well-known [3] that a number of important classes of analytic functions ($e.g. C, S^*, M_\alpha$) are related through their derivatives by the functions in P. These functions play a significant role in solving problems from signal theory and construction of quadrature formulas. In the recent years, several interesting subclasses of analytic functions have been introduced and investigated, see [1,7,9,10,12,17,20,21]. Motivated and inspired by the recent research going on, we introduce and investigate a new class of analytic function using the concept of bounded Mocanu variation. This new class of analytic functions unifies a number of classes previously studied. We obtain some new results including inclusion results, radius problem and arclength problems for this new class of analytic functions. A sufficient condition for univalency is investigated. Results obtained in this paper may stimulate further research in this field.

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Email addresses: khalidanoor@hotmail.com (Khalida Inayat Noor), nazarmaths@gmail.com (Nazar Khan), noormaslam@hotmail.com (Muhammad Aslam Noor)

Let P_k be the class of functions p, analytic in E, satisfying the properties
$p(0) = 1, \quad z = re^{i\theta}$ and

$$p(z) = \frac{1}{2}\int_0^{2\pi} \frac{1 + ze^{-it}}{1 - ze^{-it}} d\mu(t), \tag{2}$$

where $\mu(t)$ is a function with bounded variation on $[0, 2\pi]$ such that, for $k \geq 2$,

$$\int_0^{2\pi} d\mu(t) = 2, \quad \text{and} \quad \int_0^{2\pi} \left|d\mu(t)\right| \leq k. \tag{3}$$

This class has been studied in [7]. For $k = 2$, we have the class P.
From (2) and (3), it easily follows that $p \in P_k$, if and only if,

$$p(z) = \left(\frac{k}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2(z), \quad p_1, p_2 \in P. \tag{4}$$

In the following, we list some known classes of analytic functions.

$$(i) \quad S_p^*(\beta) = \left\{ f : f \in \mathcal{A}, \quad e^{i\beta}\frac{zf'}{f} \in P, \quad \beta \quad \text{real}, \quad z \in E, \quad |\beta| < \frac{\pi}{2} \right\}.$$

It is known that $S_p^*(\beta) \subset S$ and $f \in S_p^*(\beta)$ is called β-spiral like function, see [3].

$$(ii) \quad S_p^c(\beta) = \left\{ f : f \in \mathcal{A}, \left[\cos\beta \left(\frac{(zf')'}{f'} \right) + i\sin\beta \frac{zf'}{f} \right] \in P, \quad z \in E, \quad |\beta| < \frac{\pi}{2} \right\}.$$

The functions in this class are called β-spiral convex and it is shown [22] that
$S_p^c(\beta) \subset S_p^*(\beta)$.

$$(iii) \quad M^*(\alpha, \beta) = \left\{ f : f \in \mathcal{A}, \left[\left(e^{i\beta} - \alpha\cos\beta\right) \frac{zf'}{f} + \alpha\cos\beta \left(1 + \frac{zf''}{f'}\right) \right] \in P \right\}$$

where α, β are real $\quad |\beta| < \frac{\pi}{2}, \quad z \in E$.
It is shown in [19] that

$$M^*(\alpha, \beta) \subset S_p^*(\beta).$$

We note that

$$M^*(\alpha, 0) \equiv M_\alpha, \quad M^*(1, \beta) \equiv S_p^c(\beta), \quad \text{and} \quad M^*(0, \beta) \equiv S_p^*(\beta).$$

We now define the following.

Definition 1.1. Let $f \in \mathcal{A}$. Then, for β real and $|\beta| < \frac{\pi}{2}, \quad f \in R_k^*(\beta)$ if and only if

$$\left\{ e^{i\beta}\frac{zf'}{f} \right\} \in P_k, \quad z \in E, k \geq 2.$$

We note that $R_k^*(0) \equiv R_k$, the class of functions of bounded radius rotation, see [3] and $\quad R_2^*(\beta) \equiv S_p^*(\beta) \quad$ and $\quad R_2^*(0) \equiv S^*$.

Definition 1.2. Let $f \in \mathcal{A}$ and let, for $\quad \frac{f(z)f'(z)}{z} \neq 0$ in E,

$$J(\alpha, \beta, f(z)) = \left(e^{i\beta} - \alpha\cos\beta\right) \frac{zf'(z)}{f(z)} + \alpha\cos\beta \left(1 + \frac{zf''(z)}{f'(z)}\right). \tag{5}$$

Then

$$f \in M_k^*(\alpha, \beta) \quad \text{if and only if} \quad J_k(\alpha, \beta, f) \in P_k, \quad \text{for} \quad z \in E, \ \alpha, \beta \quad \text{real and} \quad |\beta| < \frac{\pi}{2}.$$

We note. as a special case, that the class $M_p^*(\alpha, 0)$ coincides with the class of functions with bounded Mocanu variation, see [2].

Definition 1.3. Let $f \in \mathcal{A}$ with $\frac{f(z)f'(z)}{z} \neq 0$ in E, and let

$$\tilde{J}(\alpha,\beta,\gamma,f(z)) = \left\{\left(e^{i\beta} - \alpha\cos\beta\right)\frac{zf'(z)}{f(z)} + \frac{\alpha\cos\beta}{1-\gamma}\left[1-\gamma+\frac{zf''(z)}{f'(z)}\right]\right\},$$

for real α, β, $|\beta| < \frac{\pi}{2}$ and $\frac{-1}{2} \leq \gamma < 1$. Then

$$f \in B_k(\alpha,\beta,\gamma) \iff \tilde{J}(\alpha,\beta,\gamma,f) \in P_k \quad \text{for} \quad z \in E, \quad k \geq 2.$$

For any real α, $\frac{-1}{2} \leq \gamma < 1$, $\beta = 0$, we note that the identity function belongs to $B_k(\alpha, 0, \gamma)$ so that $B_k(\alpha,\beta,\gamma)$ is not empty in general.

Special Cases.

(i) For $\beta = 0$, we have the class $B_k(\alpha, \gamma)$ introduced and studied in [8].

(ii) With $k = 2$, $0 \leq \alpha \leq 1$, $B_2(\alpha, 0, 0)$ is a subclass of $\mathcal{A}$ introduced by Mocanu [6].

(iii) The class $B_2(\alpha, 0, \gamma)$ consists entirely of univalent functions, see [18].

(iv) $B_2(\alpha,\beta,0) \equiv M_p^*(\alpha,\beta)$.

(v) $B_k(0, 0, \gamma) \equiv R_k$, where R_k denotes the class of bounded radius rotation, see [4].

(vi) $B_2(0,\beta,0) \equiv S_p^*(\beta) \subset S$.

(vii) $B_2(1,\beta,0) \equiv S_p^c(\beta)$.

2. Preliminary Results

In this Section, we recall some known results which we shall need later.

Lemma 2.1 [15]. Let $p \in P$ for $z \in E$. Then, for $s > 0$, $\mu \neq -1$(complex),

$$Re\left\{p(z) + \frac{szp'(z)}{p(z)+\mu}\right\} > 0,$$

for

$$|z| < \frac{|\mu+1|}{\sqrt{A + \sqrt{A^2 - |\mu^2-1|^2}}}, \quad A = 2(s+1)^2 + |\mu|^2 - 1. \tag{6}$$

This bound is best possible.

Lemma 2.2 [16]. Let $f \in \mathcal{A}$ with $\frac{f(z)f'(z)}{z} \neq 0$ in E. Then f belongs to the class of Bazilevic (univalent) functions if and only if, for $0 \leq \theta_1 < \theta_2 \leq 2\pi$ and $0 < r < 1$, we have

$$\int_{\theta_1}^{\theta_2}\left[Re\left\{1 + \frac{zf''(z)}{f'(z)} + (\rho-1)\frac{zf'(z)}{f(z)}\right\} - \alpha_1 Im\frac{zf'(z)}{f(z)}\right]d\theta \geq -\pi,$$

where $z = re^{i\theta}$, $\rho > 0$ and α_1 real.

Lemma 2.3 [5]. Let $u = u_1 + iu_2, \quad v = v_1 + iv_2$ and let $\Psi(u,v)$ be a complex -valued function satisfying the conditions:
(i). $\Psi(u,v)$ is continuous in a domain $\mathcal{D} \subset \mathcal{C}^2$.
(ii). $(1,0) \in \mathcal{D}$ and $\Psi(1,0) > 0$.
(iii). $Re\{\Psi(iu_2, v_1) \leq 0$ whenever $(iu_2, v_1) \in \mathcal{D}$ and $v_1 \leq \frac{-1}{2}(1+u_2^2)$.

If $h(z) = 1+c_1z+c_2z^2+\ldots$, is a function analytic in E, such that $(h(z), zh'(z)) \in \mathcal{D}$ and $Re\{\Psi(h(z), zh'(z))\} > 0$ for $z \in E$, then $Reh(z) > 0$ in E.

Lemma 2.4 [13]. Let $f \in R_2^*(\beta) \equiv S_p^*(\beta)$. Then for each $\beta, \quad |\beta| < \frac{\pi}{2}$, the following sharp inequality holds.

$$Re\frac{zf'(z)}{f(z)} \geq \frac{1-2(\cos\beta)r+(\cos 2\beta)r^2}{1-r^2}.$$

3. Main Results

Theorem 3.1. Let $\alpha > 0, \quad |\beta| < \frac{\pi}{2}, \quad k \geq 2$. Then $M_k^*(\alpha,\beta) \subset R_k^*(\beta)$.

Proof. Let $f \in M_k^*(\alpha,\beta)$ and let

$$e^{i\beta}\frac{zf'(z)}{f(z)} = (\cos\beta)\,p(z) + i\sin\beta,$$

where $p(z)$ is analytic in E with $p(0) = 1$.

After simple computation, we have

$$\begin{aligned} J(\alpha,\beta,f(z)) &= \left(e^{i\beta} - \alpha\cos\beta\right)\frac{zf'(z)}{f(z)} + \alpha\cos\beta\left(1 + \frac{zf''(z)}{f'(z)}\right) \\ &= (\cos\beta)p(z) + i\sin\beta + \alpha\cos^2\beta\frac{zp'(z)}{(\cos\beta)p(z)+i\sin\beta}, \end{aligned} \tag{7}$$

and $J(\alpha,\beta,f) \in P_k$ in E.

Define

$$\Phi_{\alpha,\beta}(z) = \frac{1}{1+i\tan\beta}\frac{z}{(1-z)^\alpha} + \frac{i\tan\beta}{1+i\tan\beta}\frac{z}{(1-z)^{\alpha+1}}.$$

Then, using similar convolution technique, see [8], we have

$$\left(p(z) \star \frac{\Phi_{\alpha,\beta}(z)}{z}\right) = p(z) + \frac{\alpha zp'(z)}{p(z)+i\tan\beta}. \tag{8}$$

Now, from (4), (7) and (8), we have

$$(\cos\beta)Re\left[p_i(z) + \frac{\alpha zp_j'(z)}{p_j(z)+i\tan\beta}\right] > 0, \quad \text{for} \quad j = 1,2 \quad \text{and} \quad z \in E.$$

We formulate the functional $\Psi(u,v)$ by choosing $u = u_1 + iu_2 = p_j(z)$ and $v = v_1 + iv_2 = zp_j'(z)$ as

$$\Psi(u,v) = u + \frac{\alpha v}{u + i\tan\beta}.$$

We note that the first two conditions of Lemma 2.3 are clearly satisfied. We verify the third condition as follows.

$$\begin{aligned} Re\Psi(iu_2, v_1) &= Re\left\{\frac{\alpha v_1}{i(u_2+\tan\beta)}\right\} \\ &\leq -Re\left\{\frac{\alpha(1+u_2^2)}{i2(u_2+\tan\beta)}\right\} \\ &= Re\left\{\frac{i\alpha(1+u_2^2)}{2(u_2+\tan\beta)}\right\} = 0. \end{aligned}$$

This shows that all the conditions of Lemma 2.3 are satisfied and therefore $p_j \in P, \quad z \in E, \quad j = 1,2.$ Consequently $p \in P_k$ in E and this completes the proof. □

Theorem 3.2. For $0 < \alpha_1 \leq \alpha_2 < 1,$, $M_k^*(\alpha_2, \beta) \subset M_k^*(\alpha_1, \beta)$.

Proof. Let $f \in M_k^*(\alpha_2, \beta)$. Then

$$\begin{aligned} J(\alpha_1, \beta, f(z)) &= \left(1-\frac{\alpha_1}{\alpha_2}\right)e^{i\beta}\frac{zf'(z)}{f(z)} \\ &\quad +\frac{\alpha_1}{\alpha_2}\left[\left(e^{i\beta}-\alpha_2\cos\beta\right)\frac{zf'(z)}{f(z)}+\alpha_2\cos\beta\left(1+\frac{zf''(z)}{f'(z)}\right)\right] \\ &= \left(1-\frac{\alpha_1}{\alpha_2}\right)p(z)+\frac{\alpha_1}{\alpha_2}h(z) = H(z), \end{aligned}$$

where

$$\begin{aligned} p(z) &= e^{i\beta}\frac{zf'(z)}{f(z)} \in P_k, \quad \text{by Theorem 3.1.} \\ h(z) &= J(\alpha_2, \beta, f(z)) \in P_k, \quad \text{since} \quad f \in M_k^*(\alpha_2, \beta). \end{aligned}$$

The class P_k is known [3] to be a convex set, and hence it follows that $H \in P_k$. This implies that $f \in M_k^*(\alpha_1, \beta)$. □

We now deal with the converse case of Theorem 3.2 as follows.

Theorem 3.3. Let $f \in R_k^*(\beta)$. Then, for $\alpha > 0$, $f \in M_k^*(\alpha, \beta)$ for $|z| < r_0$, where

$$r_0 = \frac{\sec\beta}{\sqrt{A+\sqrt{A^2-\sec^4\beta}}}, \quad A = 2(\alpha+1)^2+\tan^2\beta-1. \tag{9}$$

This result is sharp.

Proof. Let

$$e^{i\beta}\frac{zf'(z)}{f(z)} = (\cos\beta)p(z)+i\sin\beta, \quad p \in P_k$$

where $p(z)$ is given by (4).

Proceeding as in Theorem 3.1, we have

$$\begin{aligned} J(\alpha, \beta, f(z)) &= \left(\frac{k}{4}+\frac{1}{2}\right)\left[(\cos\beta)p_1(z)+i\sin\beta+(\alpha\cos^2\beta)\frac{zp_1'(z)}{(\cos\beta)p_1(z)+i\sin\beta}\right] \\ &\quad -\left(\frac{k}{4}-\frac{1}{2}\right)\left[(\cos\beta)p_2(z)+i\sin\beta+(\alpha\cos^2\beta)\frac{zp_2'(z)}{(\cos\beta)p_2(z)+i\sin\beta}\right]. \end{aligned} \tag{10}$$

Now, for $j = 1, 2$

$$Re\left[(\cos\beta)p_j(z) + i\sin\beta + (\alpha\cos^2\beta)\frac{zp_j'(z)}{(\cos\beta)p_j(z) + i\sin\beta}\right]$$
$$= \cos\beta Re\left[p_j(z) + \frac{\alpha zp_j'(z)}{p_j(z) + i\tan\beta}\right].$$

Using Lemma 2.1, with $s = \alpha > 0$, $\mu = i\tan\beta$, it follows that

$$Re\left[p_j(z) + \frac{\alpha zp_j'(z)}{p_j(z) + i\tan\beta}\right] > 0, \quad \text{for} \quad z < r_0,$$

where r_0 is given by (9). Consequently, form (10), it follows that $J(\alpha, \beta, f) \in P_k$ for $|z| < r_0$ and the proof is complete. □

Theorem 3.4. Let $f \in R_k^*(\beta)$. Then $f \in R_k$ for $|z| < r_\beta$, where

$$r_\beta = \frac{1}{\cos\beta + \sin\beta}. \tag{11}$$

This result is sharp.

Proof. Let

$$\frac{zf'(z)}{f(z)} = p(z) = \left(\frac{k}{4} + \frac{1}{2}\right)p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right)p_2(z),$$

where p is analytic in E with $p(0) = 1$. Then

$$e^{i\beta}\frac{zf'(z)}{f(z)} = \left(\frac{k}{4} + \frac{1}{2}\right)\left(e^{i\beta}p_1(z)\right) - \left(\frac{k}{4} - \frac{1}{2}\right)\left(e^{i\beta}p_2(z)\right).$$

Since $f \in R_k^*(\beta)$, it follows that $e^{i\beta}p_j(z) \in P_k$, $j = 1, 2$. Using Lemma 2.4, we have

$$Re\{p_j(z)\} \geq \frac{1 - 2(\cos\beta)r + (\cos 2\beta)r^2}{1 - r^2}$$

and thus it follows that $p_j \in P$ for $|z| < r_\beta$, where r_β is given by (11) The function

$$p(z) = \left(\frac{k}{4} + \frac{1}{2}\right)\frac{1+z}{1-z} - \left(\frac{k}{4} - \frac{1}{2}\right)\frac{1-z}{1+z}$$

gives us the sharpness. □

Theorem 3.5. Let $f \in \mathcal{A}$. Then $f \in B_k(\alpha, \beta, \gamma)$, $\alpha \neq 0$, if and only if, there exists a function $g \in B_k(0, \beta, \gamma) \equiv R_k^*(\beta)$ such that

$$f(z) = \left[m\int_0^z t^{m-1}\left(\frac{g(t)}{t}\right)^{\frac{(1-\gamma)e^{i\beta}}{\alpha\cos\beta}} dt\right]^{\frac{1}{m}}, \tag{12}$$

where

$$m = 1 + \frac{(1-\gamma)\left(e^{i\beta} - \alpha\cos\beta\right)}{\alpha\cos\beta}. \tag{13}$$

Proof. From (12), we have after some computation,

$$\begin{aligned} e^{i\beta}\frac{zg'(z)}{g(z)} &= \left(e^{i\beta}-\alpha\cos\beta\right)\frac{zf'(z)}{f(z)}+\frac{\alpha\cos\beta}{1-\gamma}\left[1-\gamma+\frac{zf''(z)}{f'(z)}\right] \\ &= \tilde{J}(\alpha,\beta,\gamma,f(z)). \end{aligned}$$

If the right hand side belongs to P_k, so does the left hand side and conversely. □

Theorem 3.6. Let $f\in B_k(\alpha,\beta,\gamma)$. Then the function $g\in R_k^*(\beta)$, where

$$\left(\frac{g(z)}{z}\right)^{e^{i\beta}}=\left(\frac{f(z)}{z}\right)^{e^{i\beta}-\alpha\cos\beta}\left(f'(z)\right)^{\frac{\alpha\cos\beta}{1-\gamma}}. \tag{14}$$

Proof. Logarithmic differentiation of (14) and simple calculations yield

$$e^{i\beta}\frac{zg'(z)}{g(z)}=\left(e^{i\beta}-\alpha\cos\beta\right)\frac{zf'(z)}{f(z)}+\frac{\alpha\cos\beta}{1-\gamma}\left[1-\gamma+\frac{zf''(z)}{f'(z)}\right],$$

and since $f\in B_k(\alpha,\beta,\gamma)$, we immediately obtain the required result. □

Theorem 3.7. $B_k(\alpha,\beta,\gamma)\subset B_k(\alpha_1,\beta,\gamma),\quad 0\le\alpha_1<\alpha.$

Proof. Let $f\in B_k(\alpha,\beta,\gamma)$. Now

$$\begin{aligned} &\frac{1}{1-\gamma}\left[\left(e^{i\beta}-\alpha_1\cos\beta\right)(1-\gamma)\frac{zf'(z)}{f(z)}+\alpha_1\cos\beta\left(1-\gamma+\frac{zf''(z)}{f'(z)}\right)\right] \\ &=\frac{\alpha_1}{\alpha}\left[\left(e^{i\beta}-\alpha\cos\beta\right)\frac{zf'(z)}{f(z)}+\frac{\alpha\cos\beta}{1-\gamma}\left(1-\gamma+\frac{zf''(z)}{f'(z)}\right)\right] \\ &-\left(\frac{\alpha_1-\alpha}{\alpha}\right)e^{i\beta}\frac{zf'(z)}{f(z)} \\ &=\frac{\alpha_1}{\alpha}H_1(z)+\left(1-\frac{\alpha_1}{\alpha}\right)H_2(z)=H(z), \end{aligned}$$

where

$$\begin{aligned} H_1 &= \left(e^{i\beta}-\alpha\cos\beta\right)\frac{zf'}{f}+\frac{\alpha\cos\beta}{1-\gamma}\left(1-\gamma+\frac{zf''}{f'}\right)\in P_k \\ H_2 &= e^{i\beta}\frac{zf'}{f}\in P_k,\quad \text{(by Theorem 3.1.).} \end{aligned}$$

Since P_k is a convex set [11], $H\in P_k$ and this completes the proof. □

Theorem 3.8. Let $f\in B_k(\alpha,\beta,\gamma),\quad \alpha>0$. Then f is univalent in E for

$$k\le\frac{2[\alpha\cos\beta(1+2\gamma)+1-\gamma]}{1-\gamma}.$$

Proof. Let $f\in B_k(\alpha,\beta,\gamma)$. Then

$$H(z)=\left\{\left(e^{i\beta}-\alpha\cos\beta\right)\frac{zf'(z)}{f(z)}+\frac{\alpha\cos\beta}{1-\gamma}\left(1-\gamma+\frac{zf''(z)}{f'(z)}\right)\right\}\in P_k,\quad z\in E.$$

That is

$$\begin{aligned} \left[\frac{(1-\gamma)\cos\beta+\alpha\gamma\cos\beta}{\alpha\cos\beta}-1\right]\frac{zf'(z)}{f(z)} &+ \frac{(zf'(z))'}{f'(z)}+i\left[\frac{(1-\gamma)\sin\beta}{\alpha\cos\beta}\right]\frac{zf'(z)}{f(z)} \\ &= H(z)+\gamma. \end{aligned}$$

Therefore, for $0 \le \theta_1 < \theta_2 \le 2\pi, \quad z = re^{i\theta}$, we have

$$\int_{\theta_1}^{\theta_2} \left\{ Re\left[1 + \frac{zf''(z)}{f'(z)} + (\beta_1 - 1)\frac{zf'(z)}{f(z)}\right] - \alpha_1 Im \frac{zf'(z)}{f(z)} \right\} d\theta$$
$$> -\left[\left(\frac{k}{2} - 1\right)\left(\frac{1-\gamma}{\alpha\cos\beta}\right)\pi - 2\gamma\pi\right],$$

where

$$\beta_1 = \frac{(1-\gamma)\cos\beta + \alpha\gamma\cos\beta}{\alpha\cos\beta} = \frac{1-\gamma+\alpha\gamma}{\alpha}$$
$$\alpha_1 = \frac{1-\gamma}{\alpha}\tan\beta.$$

Using lemma 2.2, it follows that f is univalent if

$$\left[\left(\frac{k}{2} - 1\right)\left(\frac{1-\gamma}{\alpha\cos\beta}\right) - 2\gamma\right] \le 1$$

and this proves the result. □

Theorem 3.9. Let $f \in B_k(\alpha, \beta, \gamma), \quad \alpha > 0$ and let $L_r(f)$ denote the length of the curve C,

$$C = f(re^{i\theta}), \quad 0 \le \theta \le 2\pi, \quad \text{and} \quad M(r) = \max_{0\le\theta\le 2\pi} \left|f(re^{i\theta})\right|.$$

Then, for $0 < r < 1$,

$$L_r(f) \le \frac{\pi M(r)}{\alpha\cos\beta} \begin{cases} \left[k\left(1 + \left|\alpha\cos\beta - e^{i\beta}\right|\right) + \dfrac{2\alpha\gamma\cos\beta}{1-\gamma}\right], & 0 < \alpha < 2 \\ k\left(1 + \sqrt{\alpha(\alpha-2)\cos^2\beta + 1}\right) + \dfrac{2\alpha\gamma\cos\beta}{1-\gamma}, & \alpha \ge 2 \end{cases} \tag{15}$$

Proof. With $z = re^{i\theta}$, and integration by parts, we have

$$\begin{aligned} L_r(f) &= \int_0^{2\pi} \left|zf'(z)\right| d\theta = \int_0^{2\pi} zf'(z)e^{-i\arg(zf'(z))} d\theta \\ &= \int_0^{2\pi} f(z)e^{-i\arg(zf'(z))} Re\left\{\frac{(zf'(z))'}{f'(z)}\right\} d\theta \\ &\le \frac{M(r)}{\alpha\cos\beta} \int_0^{2\pi} \left| Re\left\{\tilde{J}(\alpha,\beta,\gamma,f(z)) + \left(\alpha\cos\beta - e^{i\beta}\right)\frac{zf'(z)}{f(z)} + \frac{\alpha\gamma\cos\beta}{1-\gamma}\right\}\right| d\theta \\ &\le \frac{M(r)}{\alpha\cos\beta} \int_0^{2\pi} \left|Re\tilde{J}(\alpha,\beta,\gamma,f(z))\right| d\theta \\ &\quad + \frac{M(r)}{\alpha\cos\beta}\left[\left|\alpha\cos\beta - e^{i\beta}\right| \int_0^{2\pi} \left|Re\frac{zf'(z)}{f(z)}\right| d\theta + \int_0^{2\pi} \left|Re\frac{\alpha\gamma\cos\beta}{1-\gamma}\right| d\theta\right] \\ &\le \frac{M(r)}{\alpha\cos\beta}\left[k\pi + \left|\alpha\cos\beta - e^{i\beta}\right|(k\pi) + \frac{2\alpha\gamma\cos\beta}{1-\gamma}\pi\right], \end{aligned}$$

and this gives us the required result. □

Remark 3.1. For $\alpha > 0$ and $f \in B_k(\alpha, \beta, \gamma)$, we can write (3.9) as

$$L_r(f) \le \frac{\pi M(r)}{\alpha\cos\beta}\left\{k(2 + \alpha\cos\beta) + \frac{2\alpha\gamma\cos\beta}{1-\gamma}\right\}.$$

Theorem 3.10. Let $f \in B_k(\alpha, \beta, \gamma)$, $\alpha > 0$ and be given by (1). Then, for $n \geq 2$,

$$n|a_n| = O(1)M\left(\frac{n-1}{n}\right),$$

where $O(1)$ is a constant depending on α, β, γ and k only.

Proof. The result follows immediately from Theorem 3.9, since

$$n|a_n| \leq \frac{1}{2\pi r^n} \int_0^{2\pi} \left|zf'(z)\right| d\theta = \frac{1}{2\pi r^n} L_r(f).$$

□

Theorem 3.11. The class $B_k(0, \beta, \gamma)$ is preserved under the integral operator $I_c(f)$ defined as follows.

$$I_c(f) = F(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t)dt, \quad c > 0. \tag{16}$$

Proof. Set

$$\begin{aligned} e^{i\beta} \frac{zF'(z)}{F(z)} &= (\cos\beta)p(z) + i\sin\beta \\ &= \left(\frac{k}{4} + \frac{1}{2}\right)[(\cos\beta)p_1(z) + i\sin\beta] \\ &\quad -\left(\frac{k}{4} - \frac{1}{2}\right)[(\cos\beta)p_2(z) + i\sin\beta], \end{aligned} \tag{17}$$

where $p(z)$ is analytic in E with $p(0) = 1$. Then, from (16), we have

$$e^{i\beta} \frac{zf'(z)}{f(z)} = \cos\beta \left[p(z) + i\sin\beta + \frac{zp'(z)}{p(z) + c\sec\beta + i\tan\beta}\right]. \tag{18}$$

Define

$$\Phi_{c,\beta}(z) = \left(\frac{1}{1 + c\sec\beta + i\tan\beta}\right)\frac{z}{1-z} + \left(\frac{c\sec\beta + i\tan\beta}{1 + c\sec\beta + i\tan\beta}\right)\frac{z}{(1-z)^2}.$$

Then

$$\left(p(z) \star \frac{\Phi_{c,\beta}(z)}{z}\right) = \left[p(z) + \frac{zp'(z)}{p(z) + c\sec\beta + i\tan\beta}\right]. \tag{19}$$

From (17), (18) and (19), we have

$$\begin{aligned} e^{i\beta} \frac{zf'(z)}{f(z)} &= \left(\frac{k}{4} + \frac{1}{2}\right)\left((\cos\beta)p_1(z) + i\sin\beta + \frac{(\cos^2\beta)zp_1'(z)}{(\cos\beta)p_1(z) + c + i\sin\beta}\right] \\ &\quad -\left(\frac{k}{4} - \frac{1}{2}\right)\left((\cos\beta)p_2(z) + i\sin\beta + \frac{(\cos^2\beta)zp_2'(z)}{(\cos\beta)p_2(z) + c + i\sin\beta}\right]. \end{aligned}$$

Since

$$e^{i\beta} \frac{zf'(z)}{f(z)} \in P_k, \quad \text{for} \quad z \in E,$$

it follows that

$$Re\left[(\cos\beta)p_j(z) + i\sin\beta + \frac{(\cos^2\beta)zp_j'(z)}{(\cos\beta)p_j(z) + c + i\sin\beta}\right] > 0, \quad z \in E, \quad j = 1, 2.$$

We want to show that $Re\{p_j(z)\} > 0$ in E, which will imply that $p \in P_k$ in E. We proceed by performing the functional $\Psi(u, v)$ with $u = u_1 + iu_2 = p_j(z)$ and $v = v_1 + iv_2 = zp'_j(z)$. Thus

$$\Psi(u, v) = u + i\tan\beta + \frac{v}{u + c\sec\beta + i\tan\beta}.$$

The first two conditions of Lemma 2.3 are clearly satisfied. We verify the condition (iii) as follows.

$$\begin{aligned} Re\Psi(iu_2, v_1) &= Re\left[\frac{v_1}{c\sec\beta + i(u_2 + \tan\beta)}\right] \\ &= \frac{c(\sec\beta)v_1}{c^2\sec^2\beta + (u_2 + \tan\beta)^2} \\ &\leq \frac{-1}{2}\frac{c\sec\beta(1 + u_2^2)}{c^2\sec^2\beta + (u_2 + \tan\beta)^2} \leq 0. \end{aligned}$$

This proves $Re\{p_j(z)\} > 0$, $j = 1, 2$ and the proof is complete. □

Using the similar technique of Theorem 3.3, we can easily prove the converse case of Theorem 3.11 as follows.

Theorem 3.12. Let $F \in B_k(0, \beta, \gamma)$ and be defined by (16). Then $f \in B_k(0, \beta, \gamma)$ for $|z| < r_1$, where the value of r_1 is exact and is given by (6) in Lemma 2.1 with $s = 1$ and $\mu = (c\sec\beta + i\tan\beta)$.

Conclusion. In this paper, we have used the concept of bounded Mocanu variation to introduce some new classes of analytic functions in the unit disk. The main results in the paper deal with containment properties between such classes and some distortion estimates. We have also discussed several specials cases of our main results. The ideas and techniques of this work may motivate and inspire the others to explore this interesting field further.

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