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Fixed Point Theorem in Fuzzy Metric Spaces Using Altering Distance

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Abstract. The aim of the presented paper is to study the fixed point theorems in complete and compact fuzzy metric spaces as improvement of some recent results (M.S. Khan, M. Swaleh, S. Sessa, 1984.)[20]. For this purpose, the condition of the maximum type defined by altering distance is used. The research is illustrated by three examples.

1. Introduction

Famous Banach and Edelstein results [2], [10] have fundamental role in many fixed point theorems [1], [3]-[9], [15], [17]-[20], [23]-[25], [27]-[29]. It is well known that the fuzzy metric spaces are a generalization of the metric spaces, based on the theory of fuzzy sets [30]. Kramosil and Michalek [22] introduced a fuzzy metric spaces performing the probabilistic metric spaces approach to the fuzzy settings. Further on, George and Veeramani [13], [14] obtained a Hausdorff topology for specific fuzzy metric spaces with important applications in quantum physics [11], [12]. Accordingly, many authors translated the various contraction mappings from metric to fuzzy metric spaces, using different t-norms [21]. In 1984, M.S. Khan et al. [20] improved the metric fixed point theory by introducing a control function called an altering distance function.

Definition 1.1. *[20] A function $\psi : [0,\infty) \to [0,\infty)$ is an altering distance function if*
(i) ψ is monotone increasing and continuous
(ii) $\psi(t) = 0$ if and only if $t = 0$.

Using the notion of altering distance Khan et al. generalized the Banach contraction principle in metric spaces.

Theorem 1.2. *[20] Let (X, d) be a complete metric space, $\psi : [0,\infty) \to [0,\infty)$ and $f : X \to X$ be a mapping which satisfies the following inequality $\psi(d(fx, fy)) \le a\psi(d(x, y))$ for all $x, y \in X$ and for some $0 \le a < 1$. Then f has a unique fixed point.*

This result was motivation for further studies in metric spaces, as well as in the probabilistic metric spaces [6], [18], [23], [25]. Recently, Shen et al. [28] introduced the notion of altering distance in fuzzy metric spaces and gave a fixed point results in complete and compact fuzzy metric spaces.

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Definition 1.3. *[28] A function $\varphi : [0,1] \to [0,1]$ is an altering distance if:*
(P1) φ is strictly decreasing and left continuous;
(P2) $\varphi(\lambda) = 0$ if and only if $\lambda = 1$.

Obviously, we obtain that $\lim\limits_{\lambda \to 1^-} \varphi(\lambda) = \varphi(1) = 0$.

Theorem 1.4. *[28] Let (X, M, T) be a complete fuzzy metric space and f a self- map on X and suppose that $\varphi : [0,1] \to [0,1]$ satisfies properties (P1) and (P2). Furthermore, let k be a function from $(0,\infty)$ into $(0,1)$. If for any $t > 0$, f satisfies the following condition:*

$$\varphi(M(f(x), f(y), t)) \leq k(t) \cdot \varphi(M(x, y, t)), \tag{1}$$

where $x, y \in X$ and $x \neq y$, then f has a unique fixed point.

Theorem 1.5. *[28] Let (X, M, T) be a compact fuzzy metric space and f a continuous self- map on X and suppose that $\varphi : [0,1] \to [0,1]$ satisfies properties (P1) and (P2). If for any $t > 0$, f satisfies the following condition:*

$$\varphi(M(f(x), f(y), t)) < \varphi(M(x, y, t)), \tag{2}$$

where $x, y \in X$ and $x \neq y$, then f has a unique fixed point.

The main purpose of this paper is to improve mentioned results introducing more general contraction condition.

Now, we give basic definitions.

Definition 1.6. *[27] A mapping $T : [0,1] \times [0,1] \to [0,1]$ is called a triangular norm (a t-norm) if the following conditions are satisfied:*

(a) $T(a, 1) = a$ *for all* $a \in [0,1]$;

(b) $T(a, b) = T(b, a)$ *for all* $a, b \in [0,1]$;

(c) $a \geq b,\ c \geq d \Rightarrow \ T(a, c) \geq T(b, d) \ (a, b, c, d \in [0,1])$;

(d) $T(a, T(b, c)) = T(T(a, b), c) \ (a, b, c \in [0,1])$.

Basic t-norms [21] are $T_M(x, y) = \min(x, y)$, $T_P(x, y) = x \cdot y$, $T_L(x, y) = \max(x + y - 1, 0)$.

Definition 1.7. *[13] The 3-tuple (X, M, T) is a fuzzy metric space if X is an arbitrary set, T is a continuous t-norm and M is a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions:*

(a) $M(x, y, t) > 0$, *for all* $x, y \in X$, $t > 0$,

(b) $M(x, y, t) = 1$ *for all* $t > 0 \Leftrightarrow x = y$,

(c) $M(x, y, t) = M(y, x, t)$, *for all* $x, y \in X$, $t > 0$,

(d) $T(M(x, y, t), M(y, z, s)) \leq M(x, z, t + s)$, *for all* $x, y, z \in X$, $t, s > 0$,

(e) $M(x, y, \cdot) : (0, \infty) \to [0, 1]$ *is continuous for all* $x, y \in X$.

Lemma 1.8. *[15] $M(x, y, \cdot)$ is non-decreasing for all $x, y \in X$.*

Let (X, M, T) be a fuzzy metric space, and $\tau = \{B(x, r, t) : x \in X, r \in (0,1), t > 0\}$, where $B(x, r, t) = \{y : y \in X, M(x, y, t) > 1 - r\}$. Then τ is a Hausdorff and first countable topology on X induced by the fuzzy metric M [13], [16].

Definition 1.9. *[13], [15] Let (X, M, T) be a fuzzy metric space.*

(a) *A sequence* $\{x_n\}_{n\in\mathbb{N}}$ *in* X *is a Cauchy sequence if for all* $\varepsilon \in (0, 1)$, $t > 0$ *there exists* $n_0 \in \mathbb{N}$ *such that* $M(x_n, x_m, t) > 1 - \varepsilon$ *for all* $n, m \geq n_0$.

(b) *A sequence* $\{x_n\}_{n\in\mathbb{N}}$ *in* X *converges to* x *if for all* $\varepsilon \in (0, 1)$, $t > 0$ *there exists* $n_0 \in \mathbb{N}$ *such that* $M(x_n, x, t) > 1-\varepsilon$ *for all* $n \geq n_0$.

(c) *A fuzzy metric space is complete if every Cauchy sequence is convergent .*

(d) *A fuzzy metric space is compact if every sequence in* X *has a convergent subsequence.*

It is known [13] that in a fuzzy metric space every compact set is closed and bounded.

Proposition 1.10. *[26] Let* (X, M, T) *be a fuzzy metric space. Then* M *is a continuous function on* $X \times X \times (0, \infty)$.

2. Main Results

In the sequel the generalization of results given in Theorems 1.4. and 1.5. with appropriate examples will be presented.

Theorem 2.1. *Let* (X, M, T) *be a complete fuzzy metric spaces and* $f : X \to X$. *Let function* $\varphi : [0, 1] \to [0, 1]$ *satisfies the conditions* $(P1)$ *and* $(P2)$. *If there exist function* $k : (0, \infty) \to (0, 1)$ *such that:*

$$\varphi(M(f(x), f(y), t)) \leq k(t) \cdot \max\{\varphi(M(x, f(x), t)),\ \varphi(M(y, f(y), t)),\ \varphi(M(x, y, t))\}, \tag{3}$$

for any $x, y \in X$, $x \neq y$, $t > 0$, *then* f *has a unique fixed point.*

Proof. Let x_0 be arbitrary point in X and let $x_{n+1} = fx_n$, $\forall n \in \mathbb{N}_0$. Trivially, if $x_{n_0+1} = fx_{n_0} = x_{n_0}$, for some $n_0 \in \mathbb{N}_0$, then x_{n_0} is a fixed point of f. Further, we assume that $x_n \neq x_{n+1}$, $n \in \mathbb{N}_0$. According to (3), with $x = x_{n-1}$ and $y = x_n$, we have that

$$\varphi(M(x_n, x_{n+1}, t)) \leq k(t) \cdot \max\{\varphi(M(x_{n-1}, x_n, t)),\ \varphi(M(x_n, x_{n+1}, t)),\ \varphi(M(x_{n-1}, x_n, t))\},\ \ t > 0. \tag{4}$$

for every $n \in \mathbb{N}$. Suppose that $\max\{\varphi(M(x_{n-1}, x_n, t)),\ \varphi(M(x_n, x_{n+1}, t))\} = \varphi(M(x_n, x_{n+1}, t))$ for some $n \in \mathbb{N}$ and some $t > 0$. Then

$$\varphi(M(x_n, x_{n+1}, t)) \leq k(t) \cdot \varphi(M(x_n, x_{n+1}, t)) < \varphi(M(x_n, x_{n+1}, t)).$$

Therefore,

$$\varphi(M(x_n, x_{n+1}, t)) \leq k(t) \cdot \varphi(M(x_{n-1}, x_n, t)),\ n \in \mathbb{N},\ t > 0. \tag{5}$$

Repeating the same process we conclude that

$$\varphi(M(x_n, x_{n+1}, t)) \leq (k(t))^n \cdot \varphi(M(x_0, x_1, t)),\ n \in \mathbb{N}. \tag{6}$$

Letting $n \to \infty$ we have

$$\lim_{n\to\infty} \varphi(M(x_n, x_{n+1}, t)) = 0, \tag{7}$$

and since the function φ left-continuous, by $(P2)$, we get

$$\lim_{n\to\infty} M(x_n, x_{n+1}, t) = 1. \tag{8}$$

Lets prove, by contradiction, that $\{x_n\}$ is a Cauchy sequence in (X, M, T). Suppose that there exist $0 < \varepsilon < 1$ and two sequences of integers $\{p(n)\}$ and $\{q(n)\}$ such that

$$p(n) > q(n) > n, \forall n \in \mathbb{N}_0, \tag{9}$$

$$M(x_{p(n)}, x_{q(n)}, t) \leq 1-\varepsilon \text{ and } M(x_{p(n)-1}, x_{q(n)}, t) > 1-\varepsilon, \quad n \in \mathbb{N}_0, \ t > 0. \tag{10}$$

By (8) follows that for any ε_1 and ε_2 ($0 < \varepsilon_2 < \varepsilon_1 < \varepsilon$) there exist a positive integers n_1 and n_2 such that

$$M(x_{p(n)}, x_{p(n)-1}, t) \geq 1-\varepsilon_1, \ M(x_{q(n)}, x_{q(n)-1}, t) \geq 1-\varepsilon_1, \quad n > n_1, \ t > 0. \tag{11}$$

$$M(x_{q(n)}, x_{q(n)-1}, \frac{t}{2})) \geq 1-\varepsilon_2, \quad n > n_2, \ t > 0. \tag{12}$$

Also, by Definition 1.7,

$$M(x_{p(n)-1}, x_{q(n)-1}, t) \geq T(M(x_{p(n)-1}, x_{q(n)}, \frac{t}{2}), M(x_{q(n)}, x_{q(n)-1}, \frac{t}{2})), \quad n > n_1, \ t > 0. \tag{13}$$

Now, by (10), (12), (13) and Definition 1.6. follows that

$$M(x_{p(n)-1}, x_{q(n)-1}, t) \geq T(1-\varepsilon, 1-\varepsilon_2), \tag{14}$$

for all $n > \max\{n_1, n_2\}$. Since ε_2 is arbitrary and T is continuous we have

$$M(x_{p(n)-1}, x_{q(n)-1}, t) \geq T(1-\varepsilon, 1) = 1-\varepsilon, \quad \text{as } n \to \infty. \tag{15}$$

Applying (3), (10), (11) and (15) we get contradiction:

$$\begin{aligned} \varphi(1-\varepsilon) &< \varphi(M(x_{p(n)}, x_{q(n)}, t) \\ &\leq k(t) \cdot \max\{\varphi(M(x_{p(n)-1}, x_{p(n)}, t)), \ \varphi(M(x_{q(n)-1}, x_{q(n)}, t)), \ \varphi(M(x_{p(n)-1}, x_{q(n)-1}, t))\} \\ &\leq k(t) \cdot \max\{\varphi(1-\varepsilon), \ \varphi(1-\varepsilon), \ \varphi(1-\varepsilon)\} < \varphi(1-\varepsilon), \quad \text{as } n \to \infty. \end{aligned} \tag{16}$$

If it is not possible to find sequences $\{p(n)\}$ and $\{q(n)\}$ with properties (9) and (10) then there exists $n_0 \in \mathbb{N}_0$ such that $M(x_{n_0+s+2}, x_{n_0+s+1}, t) \leq 1-\varepsilon$, for any $s \in \mathbb{N}$. Moreover, $\{M(x_{n_0+s+2}, x_{n_0+s+1}, t)\}$ is a monotone and bounded sequence, for any $t > 0$, i.e. $\lim_{s \to 0} M(x_{n_0+s+2}, x_{n_0+s+1}, t) = \alpha$, for some $\alpha \in (0, 1-\varepsilon]$. Then,

$$\begin{aligned} &\varphi(M(x_{n_0+s+2}, x_{n_0+s+1}, t)) \\ &\quad \leq k(t) \cdot \max\{\varphi(M(x_{n_0+s+2}, x_{n_0+s+1}, t)), \varphi(M(x_{n_0+s+1}, x_{n_0+s}, t)), \varphi(M(x_{n_0+s+1}, x_{n_0+s}, t))\}, \quad t > 0. \end{aligned} \tag{17}$$

Letting $s \to \infty$ in (17), we get

$$\varphi(\alpha) \leq k(t) \cdot \max\{\varphi(\alpha), \varphi(\alpha), \varphi(\alpha)\} < \varphi(\alpha), \tag{18}$$

which is a contradiction. Hence, $\{x_n\}$ is a Cauchy sequence, and since (X, M, T) is complete space there exists $x \in X$ such that $\lim_{n \to \infty} x_n = x$. By (3), with $x = x_{n-1}$ and $y = x$, we have

$$\varphi(M(x_n, f(x), t)) \leq k(t) \cdot \max\{\varphi(M(x_{n-1}, x_n, t)), \ \varphi(M(x, f(x), t)), \ \varphi(M(x_{n-1}, x, t))\}, \quad n \in \mathbb{N}, \ t > 0. \tag{19}$$

If we take $n \to \infty$ in (19) we have

$$\varphi(M(x, f(x), t)) \leq k(t) \cdot \max\{\varphi(1), \ \varphi(M(x, f(x), t)), \ \varphi(1)\} = k(t) \cdot \varphi(M(x, f(x), t)), \quad t > 0, \tag{20}$$

i.e.

$$(1-k(t)) \cdot \varphi(M(x, f(x), t)) \leq 0, \quad t > 0. \tag{21}$$

We conclude that $\varphi(M(x, f(x), t)) = 0$. Now, by Definition 1.7 and ($P2$) follows that $M(x, f(x), t) = 1$, i.e. $x = f(x)$. We will show that x is a unique fixed point. Assume that there exist another fixed point v such that $v \neq x$. We use (3) to obtain contradiction:

$$\begin{aligned} &\varphi(M(x, v, t)) = \varphi(M(f(x), f(v), t)) \\ &\quad \leq k(t) \cdot \max\{\varphi(M(x, f(x), t)), \ \varphi(M(v, f(v), t)), \ \varphi(M(x, v, t)))\} < \varphi(M(x, v, t)), \quad t > 0. \end{aligned} \tag{22}$$

□

Remark 2.2. (*i*) *Obviously, in comparison to the (1) condition (3) has advantage when* $M(f(x), f(y), t) = M(x, y, t)$, *for some* $x, y \in X, x \neq y,\ t > 0$.

(*ii*) *With appropriate changes in the proof of Theorem 2.1. condition (3) could be replaced by another one:*

$$\begin{aligned}&\varphi(M(f(x), f(y), t)) \qquad (23)\\ &\leq k_1(t) \cdot \min\{\varphi(M(f(x), y, t)), \varphi(M(x, f(x), t)), \varphi(M(x, f(y), t)), \varphi(M(y, f(y), t))\} + k_2(t) \cdot \varphi(M(x, y, t)),\end{aligned}$$

$x,\ y \in X,\ x \neq y,\ t > 0$, *where* $k_1 : (0, \infty) \to [0, 1), k_2 : (0, \infty) \to (0, 1), k_1(t) + k_2(t) < 1$. *Trivially, if we take* $k_1(t) = 0,\ t > 0$ *we have generalization of condition (1).*

Example 2.3. *Let* $X = \{A, B, C, D, E\}$ *be subset of* $\mathbb{R}^2$, *where* $A(0, 0), B(1, 0), C(0, 1), D(2, 0), E(0, -2)$. *Let* $f : X \to X$ *is defined by*

$$f(A) = f(C) = f(D) = A,\ f(B) = C,\ f(E) = D.$$

Let $\varphi(\tau) = 1 - \sqrt{\tau},\ \tau \in [0, 1]$ *and* $M(x, y, t) = e^{-\frac{2d(x,y)}{t}},\ t > 0$, *where by* $d(x, y)$ *is denoted Euclidean distance in* $\mathbb{R}^2$. *Note that, by* (X, M, T) *is given a complete fuzzy metric space with respect to the* $t-$ *norm* $T_p(x, y) = x \cdot y$. *Also,* φ *satisfies conditions* $(P1)$ *and* $(P2)$ *and function* $k : (0, \infty) \to (0, 1)$ *defined by*

$$k(t) = \begin{cases} 1 - e^{-\frac{5}{t}}, & t \in (0,\ 2] \\ \dfrac{t}{t + \frac{1}{5}}, & t \in (2, \infty) \end{cases}$$

satisfies condition (3). So, by Theorem 2.1. follows that f *has a unique fixed point. On the other hand, if we take points A and B condition (1) is not satisfied, i.e.*

$$1 - e^{-\frac{1}{t}} > k(t)\,(1 - e^{-\frac{1}{t}}),\ \ t \in (0, 2].$$

The same holds for pairs $(A, E), (B, D)$ *and* (B, E).

Example 2.4. *Let* $X \subseteq \mathbb{R}^2$ *and* $f : X \to X$ *is defined the same as in Example 2.3. Let* $\varphi(\tau) = 1 - \tau,\ \tau \in [0, 1]$ *and* $M(x, y, t) = \dfrac{t}{t + d(x, y)},\ t > 0$. *By* (X, M, T) *is given a complete fuzzy metric space with respect to the* $t-$ *norm* $T_p(x, y) = x \cdot y$ *and* φ *satisfies conditions* $(P1)$ *and* $(P2)$. *Function* $k : (0, \infty) \to (0, 1)$ *defined by*

$$k(t) = \begin{cases} \dfrac{e^4}{t + e^4}, & t \in (0,\ 1] \\ \dfrac{t}{t + \frac{1}{e^4}}, & t \in (1, \infty) \end{cases}$$

satisfies condition (3) and by Theorem 2.1 follows that f *has a unique fixed point. Again, condition (1) is not satisfied for pairs* $(A, B), (A, E)$ *and* (B, D).

Theorem 2.5. *Let* (X, M, T) *be a compact fuzzy metric spaces and* $f : X \to X$ *be a continuous function. Let function* $\varphi : [0, 1] \to [0, 1]$ *satisfies the conditions* $(P1)$ *and* $(P2)$. *If*

$$\varphi(M(f(x), f(y), t)) < \max\{\varphi(M(x, f(x), t)),\ \varphi(M(y, f(y), t)),\ \varphi(M(x, y, t))\},\ x, y \in X,\ x \neq y,\ t > 0, \qquad (24)$$

then f *has a unique fixed point.*

Proof. Let $x_0 \in X$. We define sequence $x_{n+1} = f(x_n), n \in \mathbb{N}_0$ and suppose that $x_{n+1} \neq x_n, n \in \mathbb{N}_0$. Since (X, M, T) is compact, sequence $\{x_n\}_{n\in\mathbb{N}_0}$ has a subsequence $\{x_{k(n)}\}_{n\in\mathbb{N}_0}$ such that $\lim_{n\to\infty} x_{k(n)} = x,\ x \in X$. Then, there is a sequence of natural numbers $\{p_n\}_{n\in\mathbb{N}_0}$ such that $x_{k(n)+p_n} = x_{k(n+1)},\ n \in \mathbb{N}_0$. We have the following relations:

$$\lim_{n\to\infty} x_{k(n)} = \lim_{n\to\infty} x_{k(n+1)} = \lim_{n\to\infty} x_{k(n)+p_n} = x,$$

$$\lim_{n\to\infty} f(x_{k(n)}) = \lim_{n\to\infty} x_{k(n)+1} = f(x),\ \lim_{n\to\infty} f(x_{k(n)+1}) = f^2(x).$$

Suppose that $f(x) \neq x$. By (24), for every $n \in \mathbb{N}_0$, we have

$$\varphi(M(x_{k(n+1)}, f(x_{k(n+1)}), t)) = \varphi(M(x_{k(n)+p_n}, x_{k(n)+p_n+1}, t)) \tag{25}$$
$$< \max\{\varphi(M(x_{k(n)+p_n-1}, x_{k(n)+p_n}, t)),\ \varphi(M(x_{k(n)+p_n}, x_{k(n)+p_n+1}, t)),\ \varphi(M(x_{k(n)+p_n-1}, x_{k(n)+p_n}, t))\}.$$

If $\varphi(M(x_{k(n)+p_n}, x_{k(n)+p_n+1}, t)) = \max\{\varphi(M(x_{k(n)+p_n-1}, x_{k(n)+p_n}, t)),\ \varphi(M(x_{k(n)+p_n}, x_{k(n)+p_n+1}, t))\}$ we get a contradiction. So,

$$\varphi(M(x_{k(n)+p_n}, x_{k(n)+p_n+1}, t)) < \varphi(M(x_{k(n)+p_n-1}, x_{k(n)+p_n}, t)).$$

After $p_n - 1$ iteration we have

$$\varphi(M(x_{k(n+1)}, f(x_{k(n+1)}), t)) < \varphi(M(x_{k(n)+1}, f(x_{k(n)+1}), t)),\quad n \in \mathbb{N}_0,\ t > 0. \tag{26}$$

Letting $n \to \infty$ in (26) and using (24) we get

$$\begin{aligned}\varphi(M(x, f(x), t)) &\leq \varphi(M(f(x), f^2(x), t))\\ &< \max\{\varphi(M(x, f(x), t)),\ \varphi(M(f(x), f^2(x), t)),\ \varphi(M(x, f(x), t))\},\ t > 0,\end{aligned}$$

which leads to a contradiction. Hence, we get the existence of a fixed point of mapping f and uniqueness can be proved analogous as in Theorem 2.1.

□

Remark 2.6. (i) *As an alternative for given proof it could be used the fact that continuous function has minimum and maximum on compact set. For fixed $t > 0$ we define continuous function $h(x) = \varphi(M(x, f(x), t))$, $h : X \to [0, 1]$. Then, there is $x_0 \in X$ such that*

$$\inf_{x\in X} \varphi(M(x, f(x), t)) = \varphi(M(x_0, f(x_0), t)). \tag{27}$$

If we suppose that $x_0 \neq f(x_0)$ then by condition (24) follows

$$\varphi(M(f(x_0), f^2(x_0), t)) < \max\{\varphi(M(x_0, f(x_0), t)),\ \varphi(M(f(x_0), f^2(x_0), t)),\ \varphi(M(x_0, f(x_0), t))\}$$

and we get a contradiction because of (27). So, x_0 is a fixed point.

(ii) *Note that, in Theorem 1.5 it is not necessary to suppose that f is continuous. In fact, continuity of f is used only to show that*

$$\lim_{n\to\infty} x_{k(n)+1} = f(x) \text{ and } \lim_{n\to\infty} x_{k(n)+2} = f^2(x). \tag{28}$$

By condition (2) we have that

$$\varphi(M(f(x_{k(n)}), f(x), t)) < \varphi(M(x_{k(n)}, x, t)),\ t > 0,\ n \in \mathbb{N}.$$

Since φ is left-continuous and M is continuous we have

$$\varphi(M(\lim_{n\to\infty} x_{k(n)+1}, f(x), t)) \leq \varphi(M(\lim_{n\to\infty} x_{k(n)}, x, t)) = 0,\ t > 0.$$

So, the first relation in (28) is proved and in the similar way one could show the second one.

Example 2.7. *Let (X, M, T) be a compact fuzzy metric space, $X = [0, 1]$, $T_p(x, y) = x \cdot y$, $M(x, y, t) = e^{-\frac{d(x,y)}{t}}$, $x, y \in X$, $t > 0$, where by $d(x, y)$ is denoted Euclidean distance in $\mathbb{R}$. Altering distance is defined by $\varphi(\tau) = 1 - \tau$, $\tau \in [0, 1]$. We observe continuous function*

$$f(x) = \begin{cases} 1, & x \in \left[0, \frac{7}{10}\right] \\ \frac{11 - 10x}{4} & x \in \left(\frac{7}{10}, \frac{3}{4}\right) \\ \frac{x+1}{2}, & x \in \left[\frac{3}{4}, 1\right] \end{cases}.$$

If we take points $x = \frac{3}{4} - \varepsilon$, $y = \frac{3}{4} + \varepsilon$, $0 < \varepsilon < 0.05$, *then* $|f(x) - f(y)| = |x - y|$ *and condition (2) does not hold. On the other hand,*

$$|f(x) - x| > |f(x) - f(y)|, \quad x \in \left[0, \frac{3}{4}\right), \; y \in [0,1], \; x \neq y$$

and

$$|f(x) - f(y)| = \frac{1}{2}|x - y|, \quad x, y \in \left[\frac{3}{4}, 1\right], \; x \neq y.$$

Hence, condition (24) is satisfied for every $x, y \in [0,1]$, $x \neq y$, $t > 0$, *and by Theorem 2.5 follows that* f *has a unique fixed point.*

3. Conclusion

In this paper the fixed point theorems in complete and compact fuzzy metric spaces using altering distance are studied. It is shown that condition of maximum type is convenient for that purpose. The advantages of given condition are justified by three examples.

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