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Convergence Theorems for Bregman Strongly Nonexpansive Mappings in Reflexive Banach Spaces

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Abstract. In this paper, we study a strong convergence theorem for a common fixed point of a finite family of Bregman strongly nonexpansive mappings in the framework of reflexive real Banach spaces. As a consequence, we prove convergence theorem for a common fixed point of a finite family of Bergman relatively nonexpansive mappings. Furthermore, we apply our method to prove strong convergence theorems of iterative algorithms for finding a common zero of a finite family of Bregman inverse strongly monotone mappings and a solution of a finite family of variational inequality problems.

1. Introduction

In this paper, without specifications, let E be a reflexive real Banach space with the norm $\|.\|$; and E^* as its dual. Let $f : E \to (-\infty, +\infty]$ be a proper, lower semi- continuous and convex function. We denote by $\mathrm{dom} f$, the domain of f, that is, the set $\{x \in E : f(x) < +\infty\}$. Let $x \in \mathrm{int}(\mathrm{dom} f)$. The subdifferential of f at x is the convex set defined by

$$\partial f(x) = \{x^* \in E^* : f(x) + \langle x^*, y - x\rangle \leq f(y), \forall y \in E\}, \tag{1}$$

where the Fenchel conjugate of f is the function $f^* : E^* \to (-\infty, +\infty]$ defined by $f^*(x^*) = \sup\{\langle x^*, x\rangle - f(x) : x \in E\}$. A function f on E is *coercive* [14] if the sublevel set of f is bounded; equivalently, $\lim_{\|x\|\to\infty} f(x) = \infty$. A function f on E is said to be *strongly coercive* [27] if $\lim\limits_{\|x\|\to\infty} \dfrac{f(x)}{\|x\|} = \infty$. For any $x \in \mathrm{int}(\mathrm{dom} f)$ and $y \in E$, the right-hand derivative of f at x in the direction of y is defined by

$$f^0(x, y) := \lim_{t\to 0^+} \frac{f(x + ty) - f(x)}{t}. \tag{2}$$

The function f is said to be Gâteaux differentiable at x if $\lim\limits_{t\to 0^+} \big(f(x + ty) - f(x)\big)/t$ exists for any y. In this case, $f^0(x, y)$ coincides with $\nabla f(x)$, the value of the gradient ∇f of f at x. The function f is said to be Gâteaux

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differentiable if it is Gâteaux differentiable for any $x \in \text{int}(\text{dom } f)$. The function f is said to be Fréchet differentiable at x if this limit is attained uniformly in $\|y\| = 1$. Finally, f is said to be uniformly Fréchet differentiable on a subset C of E if the limit is attained uniformly for $x \in C$ and $\|y\| = 1$.

Let $f : E \to (-\infty, +\infty]$ be a Gâteaux differentiable function. The function $D_f : dom f \times \text{int}(\text{dom } f) \to [0, +\infty)$ defined as follows:

$$D_f(x, y) := f(x) - f(y) - \langle \nabla f(y), x - y \rangle$$

is called the *Bregman distance with respect to* f [12].

The Legendre function $f : E \to (-\infty, +\infty]$ is defined in [2]. It is well known that in reflexive spaces, f is Legendre function if and only if it satisfies the following conditions:

(L1) The interior of the domain of f, int(dom f), is nonempty, f is Gâteaux differentiable on int(dom f) and dom$f$=int(dom$f$);
(L2) The interior of the domain of f^*, int(domf^*), is nonempty, f^* is Gâteaux differentiable on int(domf^*) and domf^*=int(domf^*).

Since E is reflexive, we know that $(\partial f)^{-1} = \partial f^*$ (see [8]). This, with (L1) and (L2), imply the following equalities: $\nabla f = (\nabla f^*)^{-1}$, $ran \nabla f = dom \nabla f^* = \text{int}(\text{dom } f^*)$ and $ran \nabla f^*$=dom(∇f) =int(dom$f$), where $ran \nabla f$ denotes the range of ∇f.

When the subdifferential of f is single-valued, it coincides with the gradient $\partial f = \nabla f$ (see [18]). By Bauschke *et al.* [2] the conditions (L1) and (L2) also yields that the function f and f^* are strictly convex on the interior of their respective domains.

If E is a smooth and strictly convex Banach space, then an important and interesting Legendre function is $f(x) := \frac{1}{p}\|x\|^p$ $(1 < p < \infty)$. In this case the gradient ∇f of f coincides with the generalized duality mapping of E, i.e., $\nabla f = J_p$ $(1 < p < \infty)$. In particular, $\nabla f = I$, the identity mapping in Hilbert spaces. From now on we assume that the convex function $f : E \to (-\infty, +\infty]$ is Legendre.

A *Bregman projection* [5] of $x \in \text{int}(\text{dom } f)$ onto the nonempty closed and convex set $C \subset \text{int}(\text{dom} f)$ is the unique vector $P_C^f(x) \in C$ satisfying

$$D_f(P_C^f(x), x) = \inf\{D_f(y, x) : y \in C\}.$$

Remark 1.1. *If E is a smooth and strictly convex Banach space and $f(x) = \|x\|^2$ for all $x \in E$, then we have that $\nabla f(x) = 2Jx$ for all $x \in E$, where J the normalized duality mapping from E into 2^{E^*}, and hence $D_f(x, y)$ reduces to $\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$, for all $x, y \in E$, which is the Lyapunov function introduced by Alber [1] and Bregman projection $P_C^f(x)$ reduces to the generalized projection $\Pi_C(x)$ (see, e.g. [1]) which is defined by*

$$\phi(\Pi_C(x), x) = min_{y \in C} \phi(y, x), \tag{3}$$

If $E = H$, a Hilbert space J is identity mapping and hence Bregman projection $P_C^f(x)$ reduces to the metric projection of H onto C, $P_C(x)$.

Let C be a nonempty, closed and convex subset of int(dom$f$). A mapping $T : C \to C$ is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. T is said to be *quasi-nonexpansive* if $F(T) \neq \emptyset$ and $\|Tx - p\| \leq \|x - p\|$, for all $x \in C$ and $p \in F(T)$, where $F(T)$ stands for the fixed point set of T, that is, $F(T) = \{x \in C : Tx = x\}$. A point $p \in C$ is called an asymptotic fixed point of T (see [19]) if C contains a sequence $\{x_n\}$ which converges

weakly to p such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. We denote by $\hat{F}(T)$ the set of asymptotic fixed points of T.

A mapping $T : C \to \text{int (dom } f)$ with $F(T) \neq \emptyset$ is called:
(i) *quasi-Bregman nonexpansive* [21] with respect to f if,

$$D_f(p, Tx) \leq D_f(p, x), \forall x \in C, p \in F(T);$$

(ii) *Bregman relatively nonexpansive* [21] with respect to f if,

$$D_f(p, Tx) \leq D_f(p, x), \forall x \in C, p \in F(T), \text{ and } \widehat{F}(T) = F(T).$$

(iii) *Bregman strongly nonexpansive* (see [7, 22]) with respect to f and $\widehat{F}(T)$ if,

$$D_f(p, Tx) \leq D_f(p, x), \forall x \in C, p \in \widehat{F}(T) \tag{4}$$

and, if whenever $\{x_n\} \subset C$ is bounded, $p \in \widehat{F}(T)$, and

$$\lim_{n\to\infty} (D_f(p, x_n) - D_f(p, Tx_n)) = 0, \tag{5}$$

it follows that

$$\lim_{n\to\infty} D_f(x_n, Tx_n) = 0. \tag{6}$$

(iv) *Bregman firmly nonexpansive* [23] with respect to f if, for all $x, y \in C$,

$$\langle \nabla f(Tx) - \nabla f(Ty), Tx - Ty \rangle \leq \langle \nabla f(x) - \nabla f(y), Tx - Ty \rangle,$$

or, equivalently,

$$D_f(Tx, Ty) + D_f(Ty, Tx) + D_f(Tx, x) + D_f(Ty, y) \leq D_f(Tx, y) + D_f(Ty, x).$$

Existence and approximation of fixed points of nonexpansive and quasi- nonexpansive mappings have been studied by various authors (see, e.g., [6, 13, 20, 28–31] and the references therein) in Hilbert spaces. But, most of the methods failed to give the same conclusion in Banach spaces more general than Hilbert spaces. One of the reasons is that, nonexpansive mapping in Hilbert spaces may not be nonexpansive in Banach spaces (for example, the resolvent $R_A = (I + A)^{-1}$ of a maximal monotone mapping $A : H \to 2^H$ and the metric projection P_K onto a nonempty, closed and convex subset C of H).

To overcome this problem, researchers use the distance function $D_f(.,.)$ introduced by Bregman [5] instead of norm which opened a growing area of research in designing and analyzing iterative techniques for solving variational inequalities, approximating equilibria, computing fixed points of nonlinear mappings (see, e.g., [3–5, 11] and the references therein).

If T is a Bregman firmly nonexpansive mapping and f is a Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E, then it is known in [23] that $F(T) = \widehat{F}(T)$ and $F(T)$ is closed and convex (see [23]). It also follows that every Bregman firmly nonexpansive mapping is Bregman strongly nonexpansive with respect to $F(T) = \widehat{F}(T)$.

Very recently, by using Bregman projection, Reich and Sabach [22] proposed an algorithm for finding a common fixed point of finitely many Bregman strongly nonexpansive mappings $T_i : C \to C$ $(i = 1, 2, ..., N)$ satisfying $\cap_{i=1}^N F(T_i) \neq \emptyset$ in a reflexive Banach space E as follows:

$$\begin{cases} x_0 \in E, \text{ chosen arbitrarily}, \\ y_n^i = T_i(x_n + e_n^i), \\ C_n^i = \{z \in E : D_f(z, y_n^i) \leq D_f(z, x_n + e_n^i)\}, \\ C_n = \cap_{i=1}^N C_n^i, \\ Q_n^i = \{z \in E : \langle \nabla f(x_0) - \nabla f(x_n), z - x_n \rangle \leq 0\}, \\ x_{n+1} = P^f_{C_n \cap Q_n}(x_0), \forall n \geq 0. \end{cases} \tag{7}$$

Under some suitable conditions, they proved that the sequence $\{x_n\}$ generated by (7) converges strongly to a point in $\cap_{i=1}^{N} F(T_i)$, and applied it to approximate a solution of convex feasibility and equilibrium problems.

In [24], Reich and Sabach proposed the following algorithm for finding a common fixed point of finitely many Bregman firmly nonexpansive self mappings T_i $(i = 1, 2, ..., N)$ on E satisfying $\cap_{i=1}^{N} F(T_i) \neq \emptyset$. For $x_1 \in E$ let the sequence $\{x_n\}$ be defined by

$$\begin{cases} Q_0^i = E, \\ y_n^i = T_i(x_n + e_n^i), \\ Q_{n+1}^i = \{z \in Q_n^i : \langle \nabla f(x_n + e_n^i) - \nabla f(y_n^i), z - y_n^i \rangle \leq 0\}, \\ Q_n = \cap_{i=1}^{N} Q_n^i, \\ x_{n+1} = P^f_{Q_{n+1}}(x_0), \forall n \geq 1, \end{cases} \tag{8}$$

They proved that, under some suitable conditions, the sequence $\{x_n\}$ generated by (8) converges strongly to $\cap_{i=1}^{N} F(T_i)$, and applied it to the solution of convex feasibility and equilibrium problems.

Remark 1.2. *But it is worth mentioning that the iteration processes* (7) *and* (8) *seem not easy to use in the sense that at each stage of iteration, the set(s) C_n and (or) Q_n are (is) computed and the next iterate is taken as the Bregman projection of x_0 onto the intersection of C_n and Q_n or Q_n.*

In 2012, Suantai *et al.* [25] used the following Halpern's iterative scheme for Bregman strongly nonexpansive self mapping T on E: For $x_1 \in E$ let $\{x_n\}$ be a sequence defined by

$$x_{n+1} = \nabla f^*(\alpha_n \nabla f(u) + (1 - \alpha_n)\nabla f(Tx_n)), \forall n \geq 1, \tag{9}$$

where $\{\alpha_n\} \subset (0, 1)$ satisfying $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. They proved that the sequence $\{x_n\}$ generated by (9) converges strongly to a fixed point of T.

We remark that the map T in the above theorem remains a self-mapping on E. If, however, the domain of T is a nonempty closed convex subset C of E (and this is the case in several applications) and T is a self-mapping on C, then the iteration processes (9) may fail to be well defined.

In this paper, we study an iteration scheme which converges strongly to a common fixed point of a finite family of Bregman strongly nonexpansive self-mappings on a nonempty, closed and convex subset C of int(dom$f$). As a consequence, we obtain strong convergence theorem for finding a common zero of a finite family of inverse strongly monotone mappings and a common solution of a finite family of variational inequality problems. Our results improve and generalize many known results in the current literature; see, for example, [22, 25, 32].

2. Preliminaries

Let $f : E \to (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. The *modulus of total convexity* of f at $x \in$ dom f is the function $\nu_f(x, .) : [0; +\infty) \to [0; +\infty]$ defined by

$$\nu_f(x, t) := \inf\{D_f(y, x) : y \in dom f, \|y - x\| = t\}.$$

The function f is called *totally convex at x* if $\nu_f(x, t) > 0$ whenever $t > 0$. The function f is called *totally convex* if it is totally convex at any point $x \in int(dom\ f)$ and is said to be *totally convex on bounded sets* if $\nu_f(B, t) > 0$ for any nonempty bounded subset B of E and $t > 0$, where the modulus of total convexity of the function f on the set B is the function $\nu_f : int\ (dom f) \times [0, +\infty) \to [0, +\infty]$ defined by

$$\nu_f(B, t) := \inf\{V_f(x, t) : x \in B \cap dom\ f\}.$$

We know that f is totally convex on bounded sets if and only if f is uniformly convex on bounded sets (see [11], Theorem 2.10). The next Lemma will be useful in the proof of our main results.

Lemma 2.1. *[17] Let $f : E \to (-\infty, +\infty]$ be a bounded, uniformly Frêchet differentiable and totaly convex function on bounded subsets of E. Assume that ∇f^* is bounded on bounded subsets of dom $f^* = E^*$ and let C be a nonempty subset of int(dom f). Let $\{T_i : i = 1..., N\}$ be N Bregman strongly nonexpansive mappings from K into itself satisfying $\cap_{i=1}^N \widehat{F}(T_i) \neq \emptyset$. Let $T = T_N \circ T_{N-1} \circ ... \circ T_1$. Then T is Bregman strongly nonexpansive mapping and $\widehat{F}(T) = \cap_{i=1}^N \widehat{F}(T_i)$.*

Lemma 2.2. *[23] Let C be a nonempty closed and convex subset of int (dom f) and $T : C \to C$ be a quasi-Bregman nonexpansive mapping with respect to f. Then $F(T)$ is closed and convex.*

Lemma 2.3. *[9] The function $f : E \to (-\infty, +\infty)$ is totally convex on bounded subsets of E if and only if for any two sequences $\{x_n\}$ and $\{y_n\}$ in int(dom f) and dom f, respectively, such that the first one is bounded,*

$$\lim_{n\to\infty} D_f(y_n, x_n) = 0 \implies \lim_{n\to\infty} \|y_n - x_n\| = 0.$$

Lemma 2.4. *[11] Let C be a nonempty, closed and convex subset of E. Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function and let $x \in E$. then*
(i) $z = P_C^f(x)$ if and only if $\langle \nabla f(x) - \nabla f(z), y - z\rangle \leq 0, \forall y \in C$.
(ii) $D_f(y, P_C^f(x)) + D_f(P_C^f(x), x) \leq D_f(y, x), \forall y \in C$.

Lemma 2.5. *[18] Let $f : E \to (-\infty, +\infty]$ be a proper, lower semi-continuous and convex function, then $f^* : E^* \to (-\infty, +\infty]$ is a proper, weak* lower semi-continuous and convex function. Thus, for all $z \in E$, we have*

$$D_f(z, \nabla f^*(\sum_{i=1}^N t_i \nabla f(x_i))) \leq \sum_{i=1}^N t_i D_f(z, x_i) \tag{10}$$

Lemma 2.6. *[17] Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable on int(dom f) such that ∇f^* is bounded on bounded subsets of dom f^*. Let $x^* \in E$ and $\{x_n\} \subset int(E)$. If $\{D_f(x, x_n)\}$ is bounded, so is the sequence $\{x_n\}$.*

Let $f : E \to \mathbb{R}$ be a Legendre and Gâteaux differentiable function. Following [1] and [12], we make use of the function $V_f : E \times E^* \to [0, +\infty)$ associated with f, which is defined by

$$V_f(x, x^*) = f(x) - \langle x, x^*\rangle + f^*(x^*), \forall x \in E, x^* \in E^*. \tag{11}$$

Then V_f is nonnegative and

$$V_f(x, x^*) = D_f(x, \nabla f^*(x^*)), \text{ for all } x \in E \text{ and } x^* \in E^*. \tag{12}$$

Moreover, by the subdifferential inequality,

$$V_f(x, x^*) + \langle y^*, \nabla f^*(x^*) - x\rangle \leq V_f(x, x^* + y^*), \tag{13}$$

$\forall x \in E$ and $x^*, y^* \in E^*$ (see [15]).

Lemma 2.7. *[26] Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n\delta_n, n \geq n_0,$$

where $\{\alpha_n\} \subset (0, 1)$ and $\{\delta_n\} \subset R$ satisfying the following conditions: $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$, and $\limsup_{n\to\infty} \delta_n \leq 0$. Then, $\lim_{n\to\infty} a_n = 0$.

Lemma 2.8. *[16] Let $\{a_n\}$ be sequences of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} < a_{n_i+1}$ for all $i \in \mathbb{N}$. Then there exists an increasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$:*

$$a_{m_k} \leq a_{m_k+1} \text{ and } a_k \leq a_{m_k+1}.$$

In fact, m_k is the largest number n in the set $\{1, 2, ..., k\}$ such that the condition $a_n \leq a_{n+1}$ holds.

3. Main Results

Theorem 3.1. *Let $f : E \to \mathbb{R}$ be a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E. Let C be a nonempty, closed and convex subset of int(dom f) and $T_i : C \to C$, for $i = 1, 2, ..., N$, be a finite family of Bregman strongly nonexpansive mappings with respect to f such that $F(T_i) = \widehat{F}(T_i)$, for each $i \in \{1, 2, ..., N\}$. Assume that $\mathcal{F} := \cap_{i=1}^{N} F(T_i)$ is nonempty. For $u, x_1 \in C$, let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = P_C^f \nabla f^*(\alpha_n \nabla f(u) + (1 - \alpha_n)\nabla f(Tx_n)), n = 1, 2, ..., \tag{14}$$

where $T = T_N \circ T_{N-1} \circ ... \circ T_1$, $\{\alpha_n\} \subset (0, 1)$ satisfying $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then, $\{x_n\}$ converges strongly to $p = P_{\mathcal{F}}^f(u)$.

Proof. We note from Lemma 2.2 that $F(T_i)$, for each $i \in \{1, 2, ..., N\}$, is closed and convex and hence $\mathcal{F}$ is closed and convex. Moreover, by Lemma 2.1 we have that $\mathcal{F} = \cap_{i=1}^{N} F(T_i) = F(T)$. Let $p := P_{\mathcal{F}}^f(u) \in \mathcal{F}$. Then, using (14), Lemma 2.4, 2.5 and property of T_i, for each $i = 1, 2, ..., N$, we get that

$$\begin{aligned}
D_f(p, x_{n+1}) &= D_f(p, P_C^f \nabla f^*(\alpha_n \nabla f(u) + (1 - \alpha_n)\nabla f(Tx_n))) \\
&\leq D_f(p, \nabla f^*(\alpha_n \nabla f(u) + (1 - \alpha_n)\nabla f(Tx_n))) \\
&\leq \alpha_n D_f(p, u) + (1 - \alpha_n) D_f(p, Tx_n) \qquad (15) \\
&\leq \alpha_n D_f(p, u) + (1 - \alpha_n) D_f(p, x_n). \qquad (16)
\end{aligned}$$

Thus, by induction we obtain that

$$D_f(p, x_{n+1}) \leq \max\{D_f(p, u), D_f(p, x_1)\} \ \forall n \geq 1,$$

which implies that $\{D_f(p, x_n)\}$ and hence $D_f(p, Tx_n)$ are bounded. Moreover, by Lemma 2.6 we get that the sequence $\{x_n\}$ and $\{Tx_n\}$ are bounded. Let $y_n := \nabla f^*(\alpha_n \nabla f(u) + (1 - \alpha_n)\nabla f(Tx_n))$. Then, from the fact $\alpha_n \to 0$ as $n \to \infty$, we get that

$$\begin{aligned}
D_f(Tx_n, y_n) &\leq \alpha_n D_f(Tx_n, u) + (1 - \alpha_n) D_f(Tx_n, Tx_n) \\
&\leq \alpha_n D_f(Tx_n, u) \to 0, \text{ as } n \to \infty,
\end{aligned}$$

and hence by Lemma 2.3 we have

$$Tx_n - y_n \to 0, \text{ as } n \to \infty. \tag{17}$$

Furthermore, from (14), Lemma 2.4, (12) and (13) we get that

$$\begin{aligned}
D_f(p, x_{n+1}) &= D_f(p, P_C^f \nabla f^*(\alpha_n \nabla f(u) + (1 - \alpha_n)\nabla f(Tx_n))) \\
&\leq D_f(p, \nabla f^*(\alpha_n \nabla f(u) + (1 - \alpha_n)\nabla f(Tx_n))) \\
&= V_f(p, \alpha_n \nabla f(u) + (1 - \alpha_n)\nabla f(Tx_n)) \\
&\leq V_f(p, \alpha_n \nabla f(u) + (1 - \alpha_n)\nabla f(Tx_n) - \alpha_n(\nabla f(u) - \nabla f(p))) \\
&\quad + \langle \alpha_n(\nabla f(u) - \nabla f(p)), y_n - p\rangle \\
&= V_f(p, \alpha_n \nabla f(p) + (1 - \alpha_n)\nabla f(Tx_n)) \\
&\quad + \alpha_n \langle \nabla f(u) - \nabla f(p), y_n - p\rangle \\
&\leq \alpha_n V_f(p, \nabla f(p) + (1 - \alpha_n) V_f(p, \nabla f(Tx_n)) \\
&\quad + \alpha_n \langle \nabla f(u) - \nabla f(p), y_n - p\rangle \\
&= (1 - \alpha_n) D_f(p, Tx_n) + \alpha_n \langle \nabla f(u) - \nabla f(p), y_n - p\rangle \\
&\leq (1 - \alpha_n) D_f(p, x_n) + \alpha_n \langle \nabla f(u) - \nabla f(p), y_n - p\rangle. \qquad (18)
\end{aligned}$$

Now, we consider two cases:

Case 1. Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{D_f(p, x_n)\}$ is decreasing for all $n \geq n_0$. Then, we get that, $\{D_f(p, x_n)\}$ is convergent and hence

$$D_f(p, x_n) - D_f(p, x_{n+1}) \to 0, \text{ as } n \to \infty. \tag{19}$$

It follows from (15), (19) and the fact $\alpha_n \to 0$, as $n \to \infty$, that

$$\begin{aligned} D_f(p, x_n) - D_f(p, Tx_n) &= D_f(p, x_n) - D_f(p, x_{n+1}) + D_f(p, x_{n+1}) - D_f(p, Tx_n) \\ &\leq D_f(p, x_n) - D_f(p, x_{n+1}) + \alpha_n(D_f(p, u) - D_f(p, Tx_n)) \\ &\to 0, \text{ as } n \to \infty. \end{aligned} \tag{20}$$

Now, since T_i, for each $i \in \{1, 2, ..., N\}$, and hence T (by Lemma 2.1) are Bregman strongly nonexpansive we get that

$$\lim_{n\to\infty} D_f(x_n, Tx_n) = 0. \tag{21}$$

This implies by Lemma 2.3 that

$$\lim_{n\to\infty} \|Tx_n - x_n\| = 0. \tag{22}$$

Since E is reflexive and $\{y_n\}$ is bounded, there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $y_{n_k} \rightharpoonup v \in C$ and

$$\begin{aligned} \limsup_{n\to\infty} \langle \nabla f(u) - \nabla f(p), y_n - p\rangle &= \limsup_{k\to\infty} \langle \nabla f(u) - \nabla f(p), y_{n_k} - p\rangle \\ &= \langle \nabla f(u) - \nabla f(p), v - p\rangle. \end{aligned} \tag{23}$$

Thus, from (17) and (22) we obtain that $x_{n_k} \rightharpoonup v$ and hence using the fact that T is Bregman strongly nonexpansive mapping and Lemma 2.4 we get that $v \in \widehat{F}(T) = F(T) = \cap_{i=1}^{N} F(T_i)$ and

$$\limsup_{n\to\infty} \langle \nabla f(u) - \nabla f(p), y_n - p\rangle = \langle \nabla f(u) - \nabla f(p), v - p\rangle \leq 0. \tag{24}$$

Therefore, it follows from (18), (24) and Lemma 2.7 that $D_f(p, x_n) \to 0$, as $n \to \infty$. Consequently, by Lemma 2.3 we obtain that $x_n \to p = P^f_{\mathcal{F}}(u)$.

Case 2. Suppose that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that

$$D_f(p, x_{n_i}) < D_f(p, x_{n_i+1}),$$

for all $i \in \mathbb{N}$. Then, by Lemma 2.8, there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$, and

$$D_f(p, x_{m_k}) \leq D_f(p, x_{m_k+1}) \text{ and } D_f(p, x_k) \leq D_f(p, x_{m_k+1}), \tag{25}$$

for all $k \in \mathbb{N}$. Thus, we get that

$$\begin{aligned} D_f(p, x_{m_k}) - D_f(p, Tx_{m_k}) &\leq D_f(p, x_{m_k}) - D_f(p, x_{m_k+1}) \\ &\quad + D_f(p, x_{m_k+1}) - D_f(p, Tx_{m_k}) \\ &\leq \alpha_{m_k}\Big(D_f(p, u) - D_f(p, Tx_{m_k})\Big) \to 0. \end{aligned} \tag{26}$$

This implies that $D_f(Tx_{n_k}, x_{n_k}) \to 0$, as $n \to \infty$. In addition, following the method in case 1, we obtain that

$$\limsup_{k\to\infty} \langle \nabla f(u) - \nabla f(p), y_{m_k} - p\rangle \leq 0. \tag{27}$$

Now, from (18) we have that

$$D_f(p, x_{m_k+1}) \leq (1 - \alpha_{m_k})D_f(p, x_{m_k}) + \alpha_{m_k}\langle \nabla f(u) - \nabla f(p), y_{m_k} - p\rangle. \tag{28}$$

But this implies that

$$\begin{aligned}\alpha_{m_k} D_f(p, x_{m_k}) &\leq D_f(p, x_{m_k}) - D_f(p, x_{m_k+1}) + \alpha_{m_k}\langle \nabla f(u) - \nabla f(p), y_{m_k} - p\rangle \\ &\leq \alpha_{m_k}\langle \nabla f(u) - \nabla f(p), y_{m_k} - p\rangle,\end{aligned}$$

and noting that $\alpha_{m_k} > 0$, we get that

$$D_f(p, x_{m_k}) \leq \langle \nabla f(u) - \nabla f(p), y_{m_k} - p\rangle.$$

Thus, using (27) we get that $D_f(p, x_{m_k}) \to 0$. This together with (28) imply that $D_f(p, x_{m_k+1}) \to 0$, as $k \to \infty$. But $D_f(p, x_k) \leq D_f(p, x_{m_k+1})$, for all $k \in N$, thus we obtain that $D_f(p, x_k) \to 0$ and hence by Lemma 2.3 we have that $x_k \to p$. Therefore, from the above two cases, we can conclude that $\{x_n\}$ converges strongly to $p = P^f_{\mathcal{F}}(u)$ and the proof is complete. □

If, in Theorem 3.1, we assume $u = 0$, then the scheme converges strongly to the common minimum norm fixed point of a finite family of Bregman strongly nonexpansive mappings. In fact, have the following corollary.

Corollary 3.2. *Let $f : E \to \mathbb{R}$ be a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E. Let C be a nonempty, closed and convex subset of int(dom f) and $T_i : C \to C$, for $i = 1, 2, ..., N$, be a finite family of Bregman strongly nonexpansive mappings such that $F(T_i) = \widehat{F}(T_i)$, for each $i \in \{1, 2, ..., N\}$. Assume that $\mathcal{F} := \cap_{i=1}^N F(T_i)$ is nonempty. For $x_1 \in C$ let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = P^f_C \nabla f^*(\alpha_n \nabla f(0) + (1 - \alpha_n)\nabla f(Tx_n)), n = 1, 2, ...,$$

where $T = T_N \circ T_{N-1} \circ ... \circ T_1$, $\{\alpha_n\} \subset (0, 1)$ satisfying $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then, $\{x_n\}$ converges strongly to $p = P^f_{\mathcal{F}}(0)$, which is the common minimum-norm (with respect to the Bregman distance) fixed point of T_i, $i = 1, 2, ..., N$.

If, in Theorem 3.1, we consider a single Bregman strongly nonexpansive mapping, we get the following corollary.

Corollary 3.3. *Let $f : E \to \mathbb{R}$ be a coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E. Let C be a nonempty, closed and convex subset of int (dom f) and $T : C \to C$ be a Bregman strongly nonexpansive mapping such that $\widehat{F}(T) = F(T) \neq \emptyset$. For $u, x_1 \in C$ let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = P^f_C \nabla f^*(\alpha_n \nabla f(u) + (1 - \alpha_n)\nabla f(Tx_n)), n = 1, 2, ...,$$

where $\{\alpha_n\} \subset (0, 1)$ satisfying $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then, $\{x_n\}$ converges strongly to $p = P^f_{F(T)}(u)$.

If, in Theorem 3.1, we assume that each T_i, $(i = 1, 2, ..., N)$ is Bregman firmly nonexpansive, then we have that $T = T_N \circ T_{N-1} \circ ... \circ T_1$ is Bregman firmly nonexpansive with $\widehat{F}(T) = F(T) = \cap_{i=1}^N F(T_i)$. Thus, we have the following.

Corollary 3.4. *Let $f : E \to \mathbb{R}$ be a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E. Let C be a nonempty, closed and convex subset of int(dom f) and $T_i : C \to C$, for $i = 1, 2, ..., N$, be a finite family of Bregman firmly nonexpansive mappings with $\cap_{i=1}^{N} F(T_i) \neq \emptyset$. For $u, x_1 \in C$ let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = P_C^f \nabla f^*(\alpha_n \nabla f(u) + (1 - \alpha_n)\nabla f(Tx_n)), n = 1, 2, ...,$$

where $T = T_N \circ T_{N-1} \circ ... \circ T_1$, $\{\alpha_n\} \subset (0, 1)$ satisfying $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then, $\{x_n\}$ converges strongly to $p = P_{\mathcal{F}}^f(u)$.

If, in Theorem 3.1, we assume that E is a uniformly smooth and uniformly convex Banach space and $f(x) := \frac{1}{p}\|x\|^p$ $(1 < p < \infty)$, we have that $\nabla f = J_p$, where J_p is the generalized duality mapping from E onto E^*. Thus, we get the following corollary.

Corollary 3.5. *Let E be a uniformly smooth and uniformly convex Banach space and $f : E \to \mathbb{R}$ be defined by $f(x) := \frac{1}{p}\|x\|^p$ $(1 < p < \infty)$. Let C be a nonempty, closed and convex subset of int(dom f) and $T_i : C \to C$, for $i = 1, 2, ..., N$, be a finite family of Bregman strongly nonexpansive mappings on C such that $F(T_i) = \widehat{F}(T_i)$, for each $i = 1, 2, ..., N$. Assume that $\mathcal{F} := \cap_{i=1}^{N} F(T_i)$ is nonempty. For $x_1, u \in C$ let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = P_C^f J_p^{-1}(\alpha_n J_p(u) + (1 - \alpha_n)J_p(Tx_n)), n = 1, 2, ...,$$

where $T = T_N \circ T_{N-1} \circ ... \circ T_1$, $\{\alpha_n\} \subset (0, 1)$ satisfying $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then, $\{x_n\}$ converges strongly to $p = P_{\mathcal{F}}^f(u)$.

If, in Corollary 3.5, we assume $u = 0$, then the scheme converges strongly to the common minimun-norm fixed point of T_i, $i = 1, 2, ..., N$. In fact, we have the following corollary.

Corollary 3.6. *Let E be a uniformly smooth and uniformly convex Banach space and $f : E \to \mathbb{R}$ be defined by $f(x) := \frac{1}{p}\|x\|^p$ $(1 < p < \infty)$. Let C be a nonempty, closed and convex subset of int dom f and $T_i : C \to C$, for $i = 1, 2, ..., N$, be a finite family of Bregman strongly nonexpansive mappings on C such that $F(T_i) = \widehat{F}(T_i)$, for each $i = 1, 2, ..., N$. Assume that $\mathcal{F} := \cap_{i=1}^{N} F(T_i)$ is nonempty. For $x_1 \in C$ let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = P_C^f J_p^{-1}((1 - \alpha_n)J_p(Tx_n)), n = 1, 2, ...,$$

where $T = T_N \circ T_{N-1} \circ ... \circ T_1$, $\{\alpha_n\} \subset (0, 1)$ satisfying $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then $\{x_n\}$ converges strongly to $p = \Pi_{\mathcal{F}}(0)$, which is the common minimum-norm (with respect to the Bregman distance) fixed point of T_i, $i = 1, 2, ..., N$.

4. Applications

4.1. Zeros of Bregman inverse strongly monotone mappings

Using our method, we can find common zero for a class of mappings introduced by Butnariu and Kassay (see [10]). Let $A : E \to 2^{E^*}$ be any mapping and f be a Legendre function. We assume that the Legendre function f satisfies the following range condition:

$$ran(\nabla f - A) \subseteq ran(\nabla f). \tag{29}$$

The mapping A is called *Bregman inverse strongly monotone* if and only if $(domA) \cap int(domf) \neq \emptyset$ for any $x, y \in int(domf)$, and for each $x^* \in Ax$, $y^* \in Ay$, we have

$$\langle x^* - y^*, \nabla f^*(\nabla f(x) - x^*) - \nabla f^*(\nabla f(y) - y^*)\rangle \geq 0.$$

The *anti-resolvent* $A^f : E \to 2^E$ of A is defined by

$$A^f := \nabla f^* \circ (\nabla f - A).$$

Observe that $domA^f \subset (domA) \cap int(domf)$ and $ranA^f \subset int(domf)$.

It is known (see [23], Lemma 1.3.2) that if the Legendre function f is uniformly Fréchet differentiable and bounded on bounded subsets of E, then the anti-resolvent A^f is a single-valued Bregman strongly nonexpansive mapping which satisfies $F(A^f) = \widehat{F}(A^f)$ (see [23], Lemma 1.3.2). Now, by replacing $T_i = A_i^f$, for $i = 1, 2, ..., N$, in Theorem 3.1, we get the following result.

Theorem 4.1. *Let $f : E \to \mathbb{R}$ be a Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E. Let C be a nonempty, closed and convex subsets of $int(domf)$. Let $A_i : C \to 2^{E^*}$, $i = 1, 2, ..., N$, be Bregman inverse strongly monotone mappings such that $\mathcal{F} = \cap_{i=1}^N A^{-1}(0) \neq \emptyset$. Assume that the range condition (29) is satisfied for each A_i, $i = 1, 2, ..., N$. For $u, x_1 \in C$ let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = P_C^f \nabla f^*(\alpha_n \nabla f(u) + (1 - \alpha_n)\nabla f(Tx_n)), n = 1, 2, ...,$$

where $T = T_N \circ T_{N-1} \circ ... \circ T_1$, for $T_i = A_i^f$, $\{\alpha_n\} \subset (0, 1)$ satisfying $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then, $\{x_n\}$ converges strongly to $p = P_{\mathcal{F}}^f(u)$.

4.2. Solutions of variational inequalities

Let $A : E \to E^*$ be a Bregman inverse strongly monotone mapping and let C be a nonempty, closed and convex subset of $domA$. The variational inequality problem corresponding to A is to find $p \in C$ such that

$$\langle Ap, y - p\rangle \geq 0, \forall y \in C. \tag{30}$$

The set of solutions of (30) is denoted by $VI(A, C)$. The following lemma is proved in [22].

Lemma 4.2. *Let $f : E \to (-\infty, +\infty]$ be a Legendre and totally convex function which satisfies the range condition (29). Let $A : C \to E^*$ be a Bregman inverse strongly monotone mapping. If C is a nonempty, closed and convex subset of dom $(A) \cap int(domf)$, then $VI(C, A) = F(P_K^f \circ A^f)$.*

Thus, if the Legendre function f is uniformly Fréchet differentiable and bounded on bounded subsets of E, then the anti-resolvent A^f Bregman strongly nonexpansive operator ([10], Lemma 3.5(c), p. 2109) which satisfies $F(A^f) = \widehat{F}(A^f)$ ([23], Lemma 1.3.2). Since the Bregman projection P_K^f is a Bregman firmly nonexpansive mapping it is also a Bregman strongly nonexpansive mapping which satisfies $F(P_K^f) = \widehat{F}(P_K^f)$. In addition, from Lemma 2 of [19] we have that $P_K^f \circ A^f$ is a Bregman strongly nonexpansive mapping which satisfies $F(P_K^f \circ A^f) = \widehat{F}(P_K^f \circ A^f)$. We also know from Lemma 4.2 that $F(P_K^f \circ A^f) = VI(C, A)$. Now, by replacing $T_i = P_K^f \circ A_i^f$, for $i = 1, 2, ..., N$, in Theorem 3.1, we get the following result.

Theorem 4.3. *Let $f : E \to \mathbb{R}$ be a Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E. Let C be a nonempty, closed and convex subsets of $int(dom f)$. Let $A_i : C \to E^*$, $i = 1, 2, ..., N$, be Bregman inverse strongly monotone mappings such that $\mathcal{F} = \cap_{i=1}^{N} VI(C, A_i) \neq \emptyset$. Assume that the range condition (29) is satisfied for each A_i, $i = 1, 2, ..., N$. For $u, x_1 \in C$ let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = P_C^f \nabla f^*(\alpha_n \nabla f(u) + (1 - \alpha_n)\nabla f(Tx_n)), n = 1, 2, ...,$$

where $T = T_N \circ T_{N-1} \circ ... \circ T_1$, for $T_i = P_K^f \circ A_i^f$, $\{\alpha_n\} \subset (0, 1)$ satisfying $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then, $\{x_n\}$ converges strongly to $p = P_{\mathcal{F}}^f(u)$.

Remark 4.4. *Theorem 3.1 improves Theorem 1 and 2 of Reich and Sabach [22] in the sense that at each stage the computation of C_n or Q_n is not required.*

Remark 4.5. *Theorem 3.1 improves Theorem 3.2 of Suantai et al. [25] and Theorem 3.2 of Zhang and Cheng [32] in the sense that our scheme is applicable for Bregman strongly nonexpansive self-mappings on C, where C is nonempty, closed and convex subset of E.*

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