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An Iterative Algorithm for Finding the Solution of a General Equilibrium Problem System

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Abstract. In this paper, we introduce a new iterative method for finding a common element of the set of solution of a general equilibrium problem system (GEPS) and the set of fixed points of a nonexpansive semigroup. Furthermore, we present some numerical examples (by using MATLAB software) to guarantee the main result of this paper.

1. Preliminaries

Throughout this paper, we assume that H is a real Hilbert space with inner product and norm are denoted by $\langle .,. \rangle$ and $\|.\|$, respectively and let C be a nonempty closed convex subset of H. A mapping $T : C \to C$ is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in C$. We use $F(T)$ to denote the set of fixed points of T, that is, $F(T) = \{Tx = x\}$. Recall that a self-mapping $f : H \to H$ is a contraction on H if there exists a constant $\rho \in (0, 1)$ and $x, y \in H$ such that $\|f(x) - f(y)\| \leq \rho\|x - y\|$. We denote weak convergence and strong convergence by notations $\rightharpoonup$ and $\to$, respectively. Moreover, H satisfies the Opial's condition [27], if for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|,$$

holds for every $y \in H$ with $x \neq y$.

A nonexpansive semigroup is a family $\{T(s) : s \in [0, \infty)\}$ of self-mappings on C such that:

(i) $T(0)x = x$, for all $x \in C$;

(ii) $T(s + t) = T(s)T(t)$, for all $s, t \geq 0$;

(iii) $\|T(s)x - T(s)y\| \leq \|x - y\|$, for all $x, y \in C$ and $s \geq 0$;

(iv) $s \mapsto T(s)x$ is continuous for all $x \in C$.

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We denoted by $F(S)$ the common fixed points set of nonexpansive semigroup S, that is, $F(S) = \bigcap_{s\geq 0} F(T(s))$. It is well known that $F(S)$ is closed and convex [1]. Recall that given a closed convex subset C of a real Hilbert space H, the nearest point projection P_C from H onto C assigns to each $x \in H$ its nearest point denoted by P_Cx in C from x to C; that is, P_Cx is the unique point in C with the property $\|x - P_Cx\| \leq \|x - y\|$ for all $y \in C$. P_C is call the metric projection of H into C.

Lemma 1.1. *([2]) Let H be a real Hilbert space. Then the following hold:*

(a) $\|x + y\|^2 \leq \|y\|^2 + 2\langle x, x + y\rangle$ *for all* $x, y \in H$;

(b) $\|\alpha x + (1 - \alpha)y\|^2 = \alpha\|x\|^2 + (1 - \alpha)\|y\|^2 - \alpha(1 - \alpha)\|x - y\|^2$;

(c) $\|x - y\|^2 = \|x\|^2 + \|y\|^2 - 2\langle x, y\rangle$.

Let A be a strongly positive linear bounded operator on H: that is, there exists a constant $\bar{\gamma} > 0$ such that

$$\langle Ax, x\rangle \geq \bar{\gamma}\|x\|^2, \text{ for all } x \in H.$$

Lemma 1.2. *([25]) Assume A is a strongly positive linear bounded operator on a Hilbert space H with coefficient* $\overline{\gamma} > 0$ *and* $f : H \to H$ *is a contractive mapping with coefficient* ρ , $0 < \rho < \|A\|^{-1}$. *Then* $\|I - \rho A\| \leq I - \rho\overline{\gamma}$.

A mapping $B : C \to H$ is called $\alpha-$inverse strongly monotone [26, 36] if there exists a positive real number $\alpha > 0$ such that for all $x, y \in C$

$$\langle Bx - By, x - y\rangle \geq \alpha\|Bx - By\|^2.$$

Shimizu et al. [30] studied the strongly convergent of the sequence $\{x_n\}$ which is defined by

$$x_{n+1} = \alpha_n x + (1 - \alpha_n)\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds, \quad x \in C,$$

in a real Hilbert space, where $\{T(s) : s \in [0, \infty)\}$ is a strongly continuous semigroup of nonexpansive mappings on a closed convex subset C of a Hilbert space and $\lim_{n\to\infty} t_n = \infty$.

Later, Plubtieng et al. [29] introduced the following iterative method for nonexpansive semigroup $\{T(s) : s \in [0, \infty)\}$.
Let $f : C \to C$ be a contraction and the sequence $\{x_n\}$ be defined by $x_1 \in C$,

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + (1 - \alpha_n - \beta_n)\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds,$$

where $\{\alpha_n\}, \{\beta_n\}$ are the sequences in $(0, 1)$ and $\{s_n\}$ is a positive real divergent sequence. Under the conditions $\sum_{n=1}^{\infty}\alpha_n = \infty$, $\alpha_n + \beta_n < 1$, $\lim_{n\to\infty}\alpha_n = 0$ and $\lim_{n\to\infty}\beta_n = 0$, they proved the strong convergence of the sequence.

Many authors used iterative method to find the solution of equilibrium problem, mixed equilibrium problem for nonexpansive semigroups ([6–9, 14, 15, 17–22, 38]).
In 2007, Plubtieng et al. [28] introduced the following iterative scheme for finding a common element of the set equilibrium problem (EP) and the set of fixed points of a nonexpansive mapping in a Hilbert space. Let $S : C \to H$ be a nonexpansive mapping, defined sequences $\{x_n\}$ and $\{u_n\}$ by

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0; \\ x_{n+1} = \alpha_n\gamma f(x_n) + (I - \alpha_n A)Su_n, \forall y \in H. \end{cases}$$

They proved, under the certain appropriate conditions, the sequence $\{x_n\}$ converges strongly to the unique solution of the variational inequality

$$\langle (A - \gamma f)z, x - z\rangle \geq 0, \forall x \in F(S) \cap EP(F),$$

which is the optimality condition for the minimization problem

$$\min_{x\in F(S)\cap EP(F)} \frac{1}{2}\langle Ax, x\rangle - h(x),$$

where h is a potential function for γf.

In 2009, Kumam [18] proved that the sequence x_n generated by

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \forall y \in C, \\ y_n = P_C(u_n - \lambda_n A u_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n SP_C(x_n - \lambda_n A y_n), \end{cases}$$

is strongly convergent to a common fixed point of a nonexpansive mapping, the set of solutions of an equilibrium problem, and the solution set of the variational inequality problem for a monotone k-Lipschitz-continues mapping in a real Hilbert space.

In 2010, Kumam et al. [24] introduced the following iterative scheme an iterative scheme by the shrinking projection method for finding a common element of the set of solutions of generalized mixed equilibrium problems, the set of fixed points of a finite family of quasi-nonexpansive mappings and the set of solutions of variational inclusion problems in a real Hilbert space. Starting with an arbitrary $x_0 \in H$, $x_1 = P_{C_1}x_0$, $u_n \in C$ define sequences $\{x_n\}, \{y_n\}, \{v_n\}$ and $\{z_n\}$ by

$$\begin{cases} F(u_n, y) + \varphi(y) - \varphi(u_n) + \langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_{n-x_n}\rangle \geq 0, \forall y \in C, \\ y_n = J_{M,\delta_n}(u_n - \delta_n B u_n), \\ v_n = J_{M,\lambda_n}(y_n - \lambda_n B y_n), \\ z_n = \alpha_n x_n + (1-\alpha_n)K_n v_n, \\ C_{n+1} = \{z \in C_n : \|z_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}x_0, n \in \mathbb{N}. \end{cases}$$

They proved that under certain appropriate conditions imposed on $\{\alpha_n\}$ and $\{\beta_n\}, \{\lambda_n\}, \{\delta_n\}$ and $\{r_n\}$, the sequence $\{x_n\}$ is strongly convergent to $z = P_{\bigcap_{i=1}^N F(T_i)\cap GMEP(F,\varphi,A)\cap I(B,M)}x_0$.

In the same year, Kumam and Jaiboom [23] used a new approximation iterative method to prove a strong convergence theorem for finding a common element of the set of fixed points of strict pseudocontractions, the set of common solutions of the system of generalized mixed equilibrium problems, and the set of a common solutions of the variational inequalities for inverse-strongly monotone mappings in a real Hilbert space. Furthermore, they obtained a strong convergence theorem for the sequences generated by these processes under some parameter controlling conditions.

Katchang et al.[13] introduced modified Mann iterative algorithms by the new hybrid projection method for finding a common element of the set of fixed points of a countable family of nonexpansive mappings, the set of solutions of the generalized mixed equilibrium problems and the set of solutions of the general system of the variational inequality for two inverse-strongly monotone mappings in a real Hilbert space. They proved some convergence theorems for the sequences generated by these process which connected with minimize problems.

In 2011, Sunthrayuth and Kumam [32] introduced a new iterative scheme for finding common element of the set of solutions of the variational inclusion with set-valued maximal monotone mapping and Lipschitzian relaxed cocoercive mapping and the set of fixed point of nonexpansive semigroups in a uniformly convex and 2-uniformly smooth Banach space. They proved the strong convergence of the proposed iterative method under some certain control conditions.

Furthermore, Sunthrayuth and Kumam [31] proved that the sequence x_n generated by

$$\begin{cases} x_0 \in C \ \ \textit{chosen arbitriraly}, \\ z_n = \gamma_n x_n + (1-\gamma_n)\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds, \\ y_n = \alpha_n \gamma f(z_n) + (I - \alpha_n A)z_n, \\ x_{n+1} = \beta_n x_n + (1-\beta_n)y_n, \forall n \geq 0, \end{cases}$$

is strongly convergent to a common fixed point x^*, in which $x^* \in F(S)$, is the unique solution of the variational inequality $\langle \gamma f(x^*) - Ax^*, J_\varphi(x - x^*)\rangle \leq 0,\ \forall x \in F(S)$.
Jitpeera and Kumam [11] considered a shrinking projection method for finding the common element of the set of common fixed points for nonexpansive semigroups, the set of common fixed points for an infinite family of a ξ-strict pseudocontraction, the set of solutions of a system of mixed equilibrium problems, and the set of solutions of the variational inclusion problem. They proved strong convergence theorems of the iterative sequence generated by the shrinking projection method under some suitable conditions in a real Hilbert space.
Later, Kamraska et al. [12] introduced a new iterative by viscosity approximation methods in a Hilbert space. To be more precisely, they proved the following result:

Theorem 1.3. *Let $S = (T(s))_{s\geq 0}$ be a nonexpansive semigroup on a real Hilbert space H. Let $f : H \to H$ be an α-contraction, $A : H \to H$ a strongly positive linear bounded self-adjoint operator with coefficient $\overline{\gamma}$. Let γ be a real number such that $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$. Let $G : H \times H \to \mathbb{R}$ be a mapping satisfying hypotheses $(A1) - (A4)$ and $\Psi : H \to H$ an inverse-strongly monotone mapping with coefficients $\delta > 0$ such that $F(S) \cap GEP(G, \Psi) \neq \emptyset$. Let the sequences $\{x_n\}, \{u_n\}$ and $\{y_n\}$ be generated by*

$$\begin{cases} x_1 \in H \text{ chosen arbitrary}, \\ G(u_n, y) + \langle \Psi x_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle; \\ y_n = \beta_n x_n + (1 - \beta_n)\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds, x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)y_n, \forall n \geq 1. \end{cases}$$

Under the certain appropriate conditions, they proved that the sequences $\{x_n\}, \{u_n\}$ and $\{y_n\}$ is strongly convergent to z, which is a unique solution in $F(S) \cap GEP(G, \Psi)$ of the variational inequality

$$\langle(\gamma f - A)z, p - z\rangle \leq 0, \forall p \in F(S) \cap GEP(G, \Psi).$$

Recently, Wattanawitoon and Kumam [39] introduced the following new hybrid proximal-point algorithms defined by $x_1 = x \in C$:

$$\begin{cases} w_n = \prod_C J^{-1}(Jx_n - \lambda_n Ax_n), \\ z_n = J^{-1}(\beta_n J(x_n) + (1 - \beta_n)J(J_{r_n} w_n)), \\ y_n = J^{-1}(\alpha_n J(x_1) + (1 - \alpha_n)J(z_n)), \\ u_n \in C \text{ such that}, \\ \Theta(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \forall y \in C, \\ C_{n+1} = \{z \in C : \phi(z, u_n) \leq \alpha_n \phi(z, x_1) + (1 - \alpha_n)\phi(z, x_n)\}, \\ x_{n+1} = \prod_{C_{n+1}} x \end{cases}$$

and

$$\begin{cases} u_n \in C \text{ such that}, \\ \Theta(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \forall y \in C, \\ z_n = \prod_C J^{-1}(Ju_n - \lambda_n Au_n), \\ y_n = J^{-1}(\beta_n J(x_n) + (1 - \beta_n)J(J_{r_n} z_n)), \\ x_{n+1} = \prod_C J^{-1}(\alpha_n J(x_1) + (1 - \alpha_n)J(y_n)). \end{cases}$$

Under appropriate conditions, they proved that the sequence $\{x_n\}$ generated by above algorithms is strongly convergent to the point $\prod_{VI(C,A)\cap T^{-1}(0)\cap MEP(\Theta,\varphi)} x$ and converges weakly to the point $\lim_{n\to\infty} \prod_{VI(C,A)\cap T^{-1}(0)\cap MEP(\Theta,\varphi)} x_n$, respectively. Very recently, Sunthrayuth and Kumam [33] introduced a general implicit iterative scheme base on viscosity approximation method with a ϕ-strongly pseudocontractive mapping for finding a common element of the set of solutions for a system of mixed equilibrium problems, the set of common fixed point for a nonexpansive semigroup, and the set of solutions of system of variational inclusions with set-valued maximal monotone mapping and Lipschitzian relaxed cocoercive mappings in Hilbert spaces. They proved that the proposed iterative algorithm is strongly convergent to a common element of the above three sets, which is a solution of the optimization problem related to a strongly positive bounded linear

operator.
In this paper, by intuition from [5, 10, 12], a new iterative scheme is introduced. Indeed, we consider C_1 and C_2 be closed convex subsets in H. Let $F(x, y)$ be a bifunction satisfy conditions $(A1) - (A4)$ with C replaced by C_1 and let $\{T(s) : s \in [0, \infty)\}$ be a nonexpansive semigroup on C_2. By this scheme find a common element of the set of solution of the generalized equilibrium problem system (GEPS) and the set of all common fixed points of a nonexpansive semigroup in the framework of a real Hilbert space. Furthermore, by using these results, we obtain mean eragodic theorems for a nonexpansive mapping in a real Hilbert space. Finally, some numerical examples are also given.

2. Generalized Equilibrium Problems System

A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mappings on a real Hilbert space H:

$$\min \frac{1}{2}\langle Ax, x\rangle - h(x),$$

where A is strongly positive linear bounded operator and h is a potential function for γf, i. e., $h'(x) = \gamma f$, for all $x \in H$.

Let $A : H \to H$ be an inverse strongly monotone mapping and $F : C \times C \to \mathbb{R}$ be a bifunction. Then we consider the following GEPS

$$\text{Find } \tilde{x} \in C \text{ such that } F(\tilde{x}, y) + \langle Ax, y - x\rangle \geq 0, \text{ for all } y \in C. \tag{1}$$

The set of such $x \in C$ is denoted by $GEPS(F, A)$, i.e.,

$$GEPS(F, A) = \{x \in C : F(x, y) + \langle Ax, y - x\rangle \geq 0, \forall y \in C\}.$$

To study the generalized equilibrium problem (1), let F satisfies the following conditions:

(A1) $F(x, x) = 0$, for all $x \in C$;

(A2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$, $\limsup_{t\to 0^-} F(tz + (1 - t)x, y) \leq F(x, y)$;

(A4) for each $x \in C$ $y \mapsto F(x, y)$ is convex and weakly lower semi-continuous.

In case that $C_1 = H$, $F(x, y) = 0$, $C_2 = C$, and $T(s) = T$ is a nonexpansive mapping on C, (1) is the fixed point problem of a nonexpansive mapping.

Lemma 2.1. *([3]) Let C be a nonempty closed convex subset of H and $F : C \times C \to \mathbb{R}$ be a bifunction satisfying $(A1) - (A4)$. Then, for any $r > 0$ and $x \in H$ there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C.$$

Further, define

$$T_r x = \{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\}$$

for all $r > 0$ and $x \in H$. Then

(a) T_r is single-valued;

(b) T_r is firmly nonexpansive, i.e., for any $x, y \in H$

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle;$$

(c) $F(T_r) = GEP(F)$;

(d) $\|T_s x - T_r x\| \leq \frac{s-r}{s}\|T_s x - x\|$;

(e) $GEP(F)$ *is closed and convex.*

Remark 2.2. *It is clear that for any* $x \in H$ *and* $r > 0$, *by Lemma 2.1(a), there exists* $z \in H$ *such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \text{ for all } y \in H. \tag{2}$$

Replacing x *with* $x - r\Psi x$ *in (2), we obtain*

$$F(z, y) + \langle \Psi x, y - z\rangle + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \text{ for all } y \in H.$$

Lemma 2.3. *([35]) Let* $\{x_n\}$ *and* $\{y_n\}$ *be bounded sequences in a Banach space* E *and* $\{\beta_n\}$ *be a sequence in* $[0, 1]$ *with* $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$. *Suppose* $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$ *for all integers* $n \geq 1$ *and* $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. *Then* $\lim_{n\to\infty}\|y_n - x_n\| = 0$.

Lemma 2.4. *([30]) Let* C *be a nonempty bounded closed convex subset of* H *and let* $S = \{T(s) : s \in [0, \infty)\}$ *be a nonexpansive semigroup on* C. *Then for any* $h \in [0, \infty)$,

$$\limsup_{t\to\infty}\sup_{x\in C}\left\|\frac{1}{t}\int_0^t T(s)x ds - T(h)(\frac{1}{t}\int_0^t T(s)x ds)\right\| = 0,$$

for $x \in C$ *and* $t > 0$.

Lemma 2.5. *([40]) Assume* $\{a_n\}$ *is a sequence of nonnegative numbers such that*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \delta_n,$$

where $\{\alpha_n\}$ *is a sequence in* $(0, 1)$ *and* $\{\delta_n\}$ *is a sequence in real number such that*

(i) $\lim_{n\to\infty}\alpha_n = 0, \sum_{n=1}^{\infty}\alpha_n = \infty$;

(ii) $\limsup_{n\to\infty}\frac{\delta_n}{\alpha_n} \leq 0$ *or* $\sum_{n=1}^{\infty}|\delta_n| < \infty$;

Then $\lim_{n\to\infty}a_n = 0$.

Lemma 2.6. *([4]) If* C *is a closed convex subset of* H, T *is a nonexpansive mapping on* C, $\{x_n\}$ *is a sequence in* C *such that* $x_n \rightharpoonup x \in C$ *and* $x_n - Tx_n \to 0$, *then* $x - Tx = 0$

3. Strong Convergence for an Iterative Algorithm

In this section, we introduce a new iterative for finding a common element of the set of solution for an equilibrium problem involving a bifunction defined on a closed convex subset and the set of fixed points for a nonexpansive semigroup.

Theorem 3.1. *Let* H *be a real Hilbert space. Asuume that*

- C_1, C_2 *are two nonempty convex closed subsets* H,

- *$F_1, F_2, \dots, F_k$ be bifunctions from $C_1 \times C_1$ to $\mathbb{R}$ satisfying $(A1) - (A4)$,*
- *$\Psi_1, \Psi_2, \dots, \Psi_k$ is μ_i−inverse strongly monotone mapping on H,*
- *$f : H \to H$ is a ρ−contraction,*
- *A is a strongly positive linear bounded operator on H with coefficient λ and $0 < \gamma < \frac{\lambda}{\rho}$,*
- *$F(S) = \{T(s) : s \in [0, \infty)\}$ is a nonexpansive semigroup on C_2 such that $\bigcap_{i=1}^{k} F(S) \cap GEPS(F_i, \Psi_i) \neq \emptyset$,*
- *$\{x_n\}$ is a sequence generated in the following manner:*

$$\begin{cases} x_1 \in H, \text{ and } u_n^{(i)} \in C_1, \\ F_1(u_n^{(1)}, y) + \langle \Psi_1 x_n, y - u_n^{(1)} \rangle + \frac{1}{r_n} \langle y - u_n^{(1)}, u_n^{(1)} - x_n \rangle \geq 0, \text{ for all } y \in C_1, \\ F_2(u_n^{(2)}, y) + \langle \Psi_2 x_n, y - u_n^{(2)} \rangle + \frac{1}{r_n} \langle y - u_n^{(2)}, u_n^{(2)} - x_n \rangle \geq 0, \text{ for all } y \in C_1, \\ \vdots \\ F_k(u_n^{(k)}, y) + \langle \Psi_k x_n, y - u_n^{(k)} \rangle + \frac{1}{r_n} \langle y - u_n^{(k)}, u_n^{(k)} - x_n \rangle \geq 0, \text{ for all } y \in C_1, \\ \omega_n = \frac{1}{k} \sum_{i=1}^{k} u_n^{(i)}, \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n) I - \alpha_n A) \frac{1}{t_n} \int_0^{t_n} T(s) P_{C_2} \omega_n ds, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}$ are the sequences in $[0, 1]$ and $\{r_n\} \subset (0, \infty)$ is a real sequence which satisfy the following conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty;$

(C2) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1;$

(C3) $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$ *and* $0 < b < r_n < a < 2\mu_i$ *for* $i \in \{1, 2, \dots, k\};$

(C4) $\lim_{n\to\infty} t_n = \infty$, *and* $\sup |t_{n+1} - t_n|$ *is bounded.*

Then

(i) the sequence $\{x_n\}$ is bounded;

(ii) $\lim_{n\to\infty} \|\Psi_i x_n - \Psi_i q\| = 0$, *for* $q \in \bigcap_{i=1}^{k} F(S) \cap GEPS(F_i, \Psi_i)$, $i \in \{1, 2, \dots, k\};$

(iii) $\lim_{n\to\infty} \left\| x_n - \frac{1}{t_n} \int_0^{t_n} T(s) P_{C_2} \omega_n ds \right\| = 0$ *and* $\lim_{n\to\infty} \left\| \omega_n - \frac{1}{t_n} \int_0^{t_n} T(s) P_{C_2} \omega_n ds \right\| = 0.$

Proof. **(i)** By the same argument in [10, 16],

$$\|(1 - \beta_n) I - \alpha_n A\| \leq \quad 1 - \beta_n - \alpha_n \lambda.$$

Let $q \in \bigcap_{i=1}^{k} F(S) \cap GEPS(F_i, \Psi_i)$. Observe that $I - r_n \Psi_i$ for any $i = 1, 2, \dots, k$ is a nonexpansive mapping. Indeed, for any $x, y \in H$,

$$\begin{aligned} \|(I - r_n \Psi_i) x - (I - r_n \Psi_i) y\|^2 &= \|(x - y) - r_n (\Psi_i x - \Psi_i y)\|^2 \\ &= \|x - y\|^2 - 2 r_n \langle x - y, \Psi_i x - \Psi_i y \rangle + r_n^2 \|\Psi_i x - \Psi_i y\|^2 \\ &\leq \|x - y\|^2 - r_n (2\mu_i - r_n) \|\Psi_i x - \Psi_i y\|^2 \\ &\leq \|x - y\|^2. \end{aligned}$$

So

$$\|u_n^{(i)} - q\| \le \|x_n - q\|, \tag{3}$$

and hence

$$\|\omega_n - q\| \le \|x_n - q\|. \tag{4}$$

Thus

$$\begin{aligned}
\|x_{n+1} - q\| &= \|\alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)\frac{1}{t_n}\int_0^{t_n} T(s)P_{C_2}\omega_n ds - q\| \\
&\le \alpha_n \|\gamma f(x_n) - Aq\| + \beta_n\|x_n - q\| \\
&\quad + \|(1-\beta_n)I - \alpha_n A)\|\|\frac{1}{t_n}\int_0^{t_n} T(s)P_{C_2}\omega_n ds - q\| \\
&\le \alpha_n\{\|\gamma f(x_n) - \gamma f(q)\| + \|\gamma f(q) - Aq\|\} + \beta_n\|x_n - q\| \\
&\quad + (1-\beta_n - \alpha_n\lambda)\frac{1}{t_n}\int_0^{t_n} \|T(s)P_{C_2}\omega_n - P_{C_2}q\| ds \\
&\le \alpha_n\rho\gamma\|x_n - q\| + \alpha_n\|\gamma f(q) - Aq\| + \beta_n\|x_n - q\| \\
&\quad + (1-\beta_n - \alpha_n\lambda)\|\omega_n - q\| \\
&\le \alpha_n\rho\gamma\|x_n - q\| + \alpha_n\|\gamma f(q) - Aq\| + \beta_n\|x_n - q\| \\
&\quad + (1-\beta_n - \alpha_n\lambda)\|x_n - q\| \\
&= (1 - \alpha_n(\lambda - \gamma\rho))\|x_n - q\| + \alpha_n\|\gamma f(q) - Aq\| \\
&\le \max\{\|x_n - q\|, \frac{\|\gamma f(q) - Aq\|}{\lambda - \gamma\rho}\}.
\end{aligned}$$

By induction

$$\|x_n - q\| \le \max\{\|x_1 - q\|, \frac{\|\gamma f(q) - Aq\|}{\lambda - \gamma\rho}\}.$$

Therefore, the sequence $\{x_n\}$ is bounded and also $\{f(x_n)\}, \{\omega_n\}$ and $\{\frac{1}{t_n}\int_0^{t_n} T(s)P_{C_2}\omega_n ds\}$ are bounded.
(ii) Note that $u_n^{(i)}$ can be written as $u_n^{(i)} = T_{r_n^{(i)}}(x_n - r_n\Psi_i x_n)$. By Lemma 2.1, for any $i = 1, 2, \ldots, k$,

$$\begin{aligned}
\|u_{n+1}^{(i)} - u_n^{(i)}\| &\le \|T_{r_{n+1}^{(i)}}(I - r_{n+1}\Psi_i)x_{n+1} - T_{r_{n+1}^{(i)}}(I - r_n\Psi_i)x_n\| \\
&\quad + \|T_{r_{n+1}^{(i)}}(I - r_n\Psi_i)x_n - T_{r_n^{(i)}}(I - r_n\Psi_i)x_n\| \\
&\le \|(I - r_{n+1}\Psi_i)x_{n+1} - (I - r_n\Psi_i)x_n\| \\
&\quad + \|T_{r_{n+1}^{(i)}}(I - r_n\Psi_i)x_n - T_{r_n^{(i)}}(I - r_n\Psi_i)x_n\| \\
&\le \|x_{n+1} - x_n\| + |r_{n+1} - r_n|\|\Psi_i x_n\| \\
&\quad + \frac{r_{n+1} - r_n}{r_{n+1}}\|T_{r_{n+1}^{(i)}}(I - r_n\Psi_i)x_n - T_{r_n^{(i)}}(I - r_n\Psi_i)x_n\|.
\end{aligned}$$

Then

$$\|u_{n+1}^{(i)} - u_n^{(i)}\| \le \|x_{n+1} - x_n\| + 2M_i|r_{n+1} - r_n|, \tag{5}$$

where

$$M_i = \max\{\sup\{\frac{\|T_{r_{n+1}^{(i)}}(I - r_n\Psi_i)x_n - T_{r_n^{(i)}}(I - r_n\Psi_i)x_n\|}{r_{n+1}}, \sup\{\|\Psi_i x_n\|\}\}.$$

Also

$$\begin{aligned}
&\left\|\frac{1}{t_{n+1}}\int_0^{t_{n+1}} T(s)P_{C_2}\omega_{n+1}ds - \frac{1}{t_n}\int_0^{t_n} T(s)P_{C_2}\omega_n ds\right\| \\
= &\left\|\frac{1}{t_{n+1}}\int_0^{t_{n+1}} [T(s)\omega_{n+1} - T(s)\omega_n]ds + (\frac{1}{t_{n+1}} - \frac{1}{t_n})\int_0^{t_n} [T(s)\omega_n - T(s)q]ds \right.\\
&\left. + \frac{1}{t_{n+1}}\int_{t_n}^{t_{n+1}} [T(s)\omega_n - T(s)q]ds\right\| \\
\leq & \|\omega_{n+1} - \omega_n\| + \frac{2|t_{n+1} - t_n|}{t_{n+1}}\|\omega_n - q\|.
\end{aligned}$$

Let $M = \frac{1}{k}\sum_{i=1}^{k} 2M_i < \infty$, since

$$\|\omega_{n+1} - \omega_n\| \leq \frac{1}{k}\sum_{i=1}^{k}\|u_{n+1}^{(i)} - u_n^{(i)}\| \leq \|x_{n+1} - x_n\| + M|r_{n+1} - r_n|,$$

hence

$$\begin{aligned}
&\left\|\tfrac{1}{t_{n+1}}\int_0^{t_{n+1}} T(s)P_{C_2}\omega_{n+1}ds - \tfrac{1}{t_n}\int_0^{t_n} T(s)P_{C_2}\omega_n ds\right\| \\
&\leq \|x_{n+1} - x_n\| + M|r_{n+1} - r_n| + \tfrac{2|t_{n+1} - t_n|}{t_{n+1}}\|\omega_n - q\|.
\end{aligned} \tag{6}$$

Suppose $z_n = \frac{\alpha_n\gamma f(x_n) + ((1-\beta_n)I - \alpha_n A)\Lambda_n}{1-\beta_n}$, where $\Lambda_n := \frac{1}{t_n}\int_0^{t_n} T(s)P_{C_2}\omega_n ds$. It follows from (5), (6)

$$\begin{aligned}
\|z_{n+1} - z_n\| &= \|\frac{\alpha_{n+1}\gamma f(x_{n+1}) + ((1-\beta_{n+1})I - \alpha_{n+1}A)\Lambda_{n+1}}{1-\beta_{n+1}} \\
&\quad - \frac{\alpha_n\gamma f(x_n) + ((1-\beta_n)I - \alpha_n A)\Lambda_n}{1-\beta_n}\| \\
&= \|\frac{\alpha_{n+1}\gamma f(x_{n+1})}{1-\beta_{n+1}} + \frac{(1-\beta_{n+1})\Lambda_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_{n+1}A\Lambda_{n+1}}{1-\beta_{n+1}} \\
&\quad - \frac{\alpha_n\gamma f(x_n)}{1-\beta_n} - \frac{(1-\beta_n)\Lambda_n}{1-\beta_n} + \frac{\alpha_n A\Lambda_n}{1-\beta_n}\| \\
&= \|\frac{\alpha_{n+1}}{1-\beta_{n+1}}(\gamma f(x_{n+1}) - A\Lambda_{n+1}) + \frac{\alpha_n}{1-\beta_n}(A\Lambda_n - \gamma f(x_n)) + (\Lambda_{n+1} - \Lambda_n)\| \\
&\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|\gamma f(x_{n+1}) - A\Lambda_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|A\Lambda_n - \gamma f(x_n)\| + \|\Lambda_{n+1} - \Lambda_n\| \\
&\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|\gamma f(x_{n+1}) - A\Lambda_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|A\Lambda_n - \gamma f(x_n)\| + \|x_{n+1} - x_n\| \\
&\quad + M|r_{n+1} - r_n| + \frac{2|t_{n+1} - t_n|}{t_{n+1}}\|\omega_n - q\|.
\end{aligned}$$

(C1), (C3) and (C4) implies that

$$\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

By Lemma 2.3, $\lim_{n\to\infty}\|z_n - x_n\| = 0$.
Consequently

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = \lim_{n\to\infty}(1-\beta_n)\|z_n - x_n\| = 0. \tag{7}$$

Moreover, for any $i \in \{1, 2, \dots, k\}$,

$$\begin{aligned} \|u_n^{(i)} - q\|^2 &\leq \|(x_n - q) - r_n(\Psi_i x_n - \Psi_i q)\|^2 \\ &= \|x_n - q\|^2 - 2r_n\langle x_n - q, \Psi_i x_n - \Psi_i q\rangle + r_n^2\|\Psi_i x_n - \Psi_i q\|^2 \\ &\leq \|x_n - q\|^2 - r_n(2\mu_i - r_n)\|\Psi_i x_n - \Psi_i q\|^2, \end{aligned}$$

and then

$$\begin{aligned} \|\omega_n - q\|^2 = &\ \|\sum_{i=1}^{k}\frac{1}{k}(u_n^{(i)} - q)\|^2 \leq \frac{1}{k}\sum_{i=1}^{k}\|u_n^{(i)} - q\|^2 \\ \leq &\ \|x_n - q\|^2 - \tfrac{1}{k}\sum_{i=1}^{k} r_n(2\mu_i - r_n)\|\Psi_i x_n - \Psi_i q)\|^2. \end{aligned} \tag{8}$$

By (8), we have

$$\begin{aligned} \|x_{n+1} - q\|^2 &= \|\alpha_n(\gamma f(x_n) - Aq) + \beta_n(x_n - q) + ((1 - \beta_n)I - \alpha_n A)(\Lambda_n - q)\|^2 \\ &\leq \alpha_n\|\gamma f(x_n) - Aq\|^2 + \beta_n\|x_n - q\|^2 + (1 - \beta_n - \alpha_n\lambda)\|\Lambda_n - q\|^2 \\ &\leq \alpha_n\|\gamma f(x_n) - Aq\|^2 + \beta_n\|x_n - q\|^2 + (1 - \beta_n - \alpha_n\lambda)\|\omega_n - q\|^2 \\ &\leq \alpha_n\|\gamma f(x_n) - Aq\|^2 + \beta_n\|x_n - q\|^2 + (1 - \beta_n - \alpha_n\lambda)\{\|x_n - q\|^2 \\ &\quad -\frac{1}{k}\sum_{i=1}^{k} r_n(2\mu_i - r_n)\|\Psi_i x_n - \Psi_i q)\|^2\} \\ &\leq \alpha_n\|\gamma f(x_n) - Aq\|^2 + \|x_n - q\|^2 \\ &\quad -(1 - \beta_n - \alpha_n\lambda)\frac{1}{k}\sum_{i=1}^{k} r_n(2\mu_i - r_n)\|\Psi_i x_n - \Psi_i q\|^2, \end{aligned}$$

and hence

$$\begin{aligned} (1 - \beta_n - \alpha_n\lambda)\tfrac{1}{k} &\sum_{i=1}^{k} b(2\mu_i - a)\|\Psi_i x_n - \Psi_i q\|^2 \\ &\leq \alpha_n\|\gamma f(x_n) - Aq\|^2 + \|x_n - q\|^2 - \|x_{n+1} - q\|^2 \\ &\leq \alpha_n\|\gamma f(x_n) - Aq\|^2 + \|x_{n+1} - x_n\|(\|x_{n+1} - q\| - \|x_n - q\|). \end{aligned}$$

Since $\alpha_n \to 0$ and $\|x_{n+1} - x_n\| \to 0$, it follows that

$$\lim_{n\to\infty}\|\Psi_i x_n - \Psi_i q\| = 0, \forall i = 1, 2, \dots, k. \tag{9}$$

(iii) By Lemma 2.1, for any $i = 1, 2, \dots, k$,

$$\begin{aligned} \|u_n^{(i)} - q\|^2 &\leq \langle (I - r_n\Psi_i)x_n - (I - r_n\Psi_i)q, u_n^{(i)} - q\rangle \\ &= \frac{1}{2}\{\|(I - r_n\Psi_i)x_n - (I - r_n\Psi_i)q\|^2 + \|u_{n,i} - q\|^2 \\ &\quad -\|(I - r_n\Psi_i)x_n - (I - r_n\Psi_i)q - (u_n^{(i)} - q)\|^2\} \\ &\leq \frac{1}{2}\{\|x_n - q\|^2 + \|u_n^{(i)} - q\|^2 - \|x_n - u_n^{(i)} - r_n(\Psi_i x_n - \Psi_i q)\|^2\} \\ &= \frac{1}{2}\{\|x_n - q\|^2 + \|u_n^{(i)} - q\|^2 - (\|x_n - u_n^{(i)}\|^2 \\ &\quad -2r_n\langle x_n - u_n^{(i)}, \Psi_i x_n - \Psi_i q\rangle + r_n^2\|\Psi_i x_n - \Psi_i q\|^2). \end{aligned}$$

This implies

$$\|u_n^{(i)} - q\|^2 \leq \|x_n - q\|^2 - \|x_n - u_n^{(i)}\|^2 + 2r_n\|x_n - u_n^{(i)}\|\|\Psi_i x_n - \Psi_i q\|, \tag{10}$$

and hence

$$\begin{aligned} \|\omega_n - q\|^2 = & \quad \|\sum_{i=1}^{k}\frac{1}{k}(u_n^{(i)} - q)\|^2 \\ \leq & \quad \tfrac{1}{k}\sum_{i=1}^{k}\|u_n^{(i)} - q\|^2 \\ \leq & \quad \|x_n - q\|^2 - \tfrac{1}{k}\sum_{i=1}^{k}\|u_n^{(i)} - x_n\|^2 + \frac{1}{k}\sum_{i=1}^{k}2r_n\|x_n - u_n^{(i)}\|\|\Psi_i x_n - \Psi_i q\|. \end{aligned} \tag{11}$$

Observe that

$$\begin{aligned} \|x_{n+1} - q\|^2 \quad \leq & \quad \alpha_n\|\gamma f(x_n) - Aq\|^2 + \beta_n\|x_n - q\|^2 + (1 - \beta_n - \alpha_n\lambda)\|\omega_n - q\|^2 \\ \leq & \quad \alpha_n\|\gamma f(x_n) - Aq\|^2 + \beta_n\|x_n - q\|^2 + (1 - \beta_n - \alpha_n\lambda)\{\|x_n - q\|^2 \\ & -\frac{1}{k}\sum_{i=1}^{k}\|u_n^{(i)} - x_n\|^2 + \frac{1}{k}\sum_{i=1}^{k}2r_n\|x_n - u_n^{(i)}\|\|\Psi_i x_n - \Psi_i q\|\}. \end{aligned}$$

It follows that

$$\begin{aligned} (1 - \beta_n - \alpha_n\lambda)\tfrac{1}{k} \quad & \sum_{i=1}^{k}\|u_n^{(i)} - x_n\|^2 \leq \alpha_n\|\gamma f(x_n) - Aq\|^2 + \|x_n - q\|^2 - \|x_{n+1} - q\|^2 \\ & +(1 - \beta_n - \alpha_n\lambda)\tfrac{1}{k}\sum_{i=1}^{k}2r_n\|x_n - u_n^{(i)}\|\|\Psi_i x_n - \Psi_i q\| \\ & \leq \alpha_n\|\gamma f(x_n) - Aq\|^2 + \|x_{n+1} - x_n\|(\|x_n - q\| - \|x_{n+1} - q\|) \\ & +(1 - \beta_n - \alpha_n\lambda)\tfrac{1}{k}\sum_{i=1}^{k}2r_n\|x_n - u_n^{(i)}\|\|\Psi_i x_n - \Psi_i q\|. \end{aligned}$$

Since $\alpha_n \to 0$ and $\|x_{n+1} - x_n\| \to 0$, we have

$$\lim_{n\to\infty}\|u_n^{(i)} - x_n\| = 0. \tag{12}$$

It is easy to prove

$$\lim_{n\to\infty}\|\omega_n - x_n\| = 0. \tag{13}$$

The definition of $\{x_n\}$ shows

$$\begin{aligned} \|\Lambda_n - x_n\| \quad \leq & \quad \|x_{n+1} - x_n\| + \|x_{n+1} - \Lambda_n\| \\ \leq & \quad \|x_{n+1} - x_n\| + \|\alpha_n\gamma f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n A)\Lambda_n - \Lambda_n\| \\ \leq & \quad \|x_{n+1} - x_n\| + \alpha_n\|\gamma f(x_n) - A\Lambda_n\| + \beta_n\|x_n - \Lambda_n\|. \end{aligned}$$

That is

$$\|\Lambda_n - x_n\| \leq \frac{1}{1 - \beta_n}\|x_{n+1} - x_n\| + \frac{\alpha_n}{1 - \beta_n}\|\gamma f(x_n) - A\Lambda_n\|.$$

The condition (C1) together (7) implies that

$$\lim_{n\to\infty}\|\Lambda_n - x_n\| = 0. \tag{14}$$

Moreover, $\|\omega_n - \Lambda_n\| \leq \|\omega_n - x_n\| + \|x_n - \Lambda_n\|$, we get

$$\lim_{n\to\infty}\|\Lambda_n - \omega_n\| = 0. \tag{15}$$

Then,

$$\lim_{n\to\infty}\left\|x_n - \frac{1}{t_n}\int_0^{t_n} T(s)P_{C_2}\omega_n ds\right\| = 0,$$

$$\lim_{n\to\infty}\left\|\omega_n - \frac{1}{t_n}\int_0^{t_n} T(s)P_{C_2}\omega_n ds\right\| = 0.$$

□

Theorem 3.2. *Suppose all assumptions of Theorem 3.1 are hold. Then the sequence $\{x_n\}$ is strongly convergent to a point $\bar{x}$, where $\bar{x} \in \bigcap_{i=1}^k F(S) \cap GEPS(F_i, \Psi_i)$, which solves the variational inequality*

$$\langle (A - \gamma f)\bar{x}, \bar{x} - x\rangle \le 0.$$

Equivalently, $\bar{x} = P_{\bigcap_{i=1}^k F(S)\cap GEPS(F_i,\Psi_i)}(I - A + \gamma f)(\bar{x})$.

Proof. For all $x, y \in H$, we have

$$\begin{aligned} & \|P_{\bigcap_{i=1}^k F(S)\cap GEPS(F_i,\Psi_i)}(I - A + \gamma f)(x) - P_{\bigcap_{i=1}^k F(S)\cap GEPS(F_i,\Psi_i)}(I - A + \gamma f)(y)\| \\ \le\ & \|(I - A + \gamma f)(x) - (I - A + \gamma f)(y)\| \\ \le\ & \|I - A\|\|x - y\| + \gamma\|f(x) - f(y)\| \\ \le\ & (1 - \lambda)\|x - y\| + \gamma\rho\|x - y\| \\ =\ & (1 - (\lambda - \gamma\rho))\|x - y\|. \end{aligned}$$

This implies that $P_{\bigcap_{i=1}^k F(S)\cap GEPS(F_i,\Psi_i)}(I - A + \gamma f)$ is a contraction of H into itself. Since H is complete, then there exists a unique element $\bar{x} \in H$ such that

$$\bar{x} = P_{\bigcap_{i=1}^k F(S)\cap GEPS(F_i,\Psi_i)}(I - A + \gamma f)(\bar{x}).$$

Next, we prove

$$\limsup_{n\to\infty}\langle (A - \gamma f)\bar{x}, \bar{x} - \frac{1}{t_n}\int_0^{t_n} T(s)P_{C_2}\omega_n ds\rangle \le 0.$$

Let $\tilde{x} = P_{\bigcap_{i=1}^k F(S)\cap GEPS(F_i,\Psi_i)}x_1$, set

$$\Sigma = \{\bar{y} \in C_2 : \|\bar{y} - \tilde{x}\| \le \|x_1 - \tilde{x}\| + \frac{\|\gamma f(\tilde{x}) - A\tilde{x}\|}{\lambda - \gamma\rho}\}.$$

It is clear, Σ is nonempty closed bounded convex subset of C_2 and $S = \{T(s) : s \in [0, \infty)\}$ is a nonexpansive semigroup on Σ.

Let $\Lambda_n = \frac{1}{t_n}\int_0^{t_n} T(s)P_{C_2}\omega_n ds$, since $\{\Lambda_n\} \subset \Sigma$ is bounded, there is a subsequence $\{\Lambda_{n_j}\}$ of $\{\Lambda_n\}$ such that

$$\limsup_{n\to\infty}\langle (A - \gamma f)\bar{x}, \bar{x} - \Lambda_n\rangle = \lim_{j\to\infty}\langle (A - \gamma f)\bar{x}, \bar{x} - \Lambda_{n_j}\rangle. \tag{16}$$

As $\{\omega_{n_j}\}$ is also bounded, there exists a subsequence $\{\omega_{n_{j_l}}\}$ of $\{\omega_{n_j}\}$ such that $\omega_{n_{j_l}} \rightharpoonup \xi$. Without loss of generality, let $\omega_{n_j} \rightharpoonup \xi$. From *(iii)* in Theorem 3.1, we have $\Lambda_{n_j} \to \xi$.
Since $\{\omega_n\} \subset C_1$ and $\{\Lambda_n\} \subset C_2$ and C_1, C_2 are two closed convex subsets in H, we obtain that $\xi \in C_1 \cap C_2$.
Now, we prove the following items:

(i) $\xi \in F(S) = \bigcap_{s\geq 1} F(T(s))$.
Since $\{\Lambda_n\} \subset C_2$, we have

$$\begin{aligned} \|\Lambda_n - P_{C_2}\omega_n\| &= \|P_{C_2}\Lambda_n - P_{C_2}\omega_n\| \\ &\leq \|\Lambda_n - \omega_n\|. \end{aligned}$$

By (*iii*) in Theorem 3.1, we have

$$\lim_{n\to\infty} \|\Lambda_n - P_{C_2}\omega_n\| = 0. \tag{17}$$

By using (*iii*) in Theorem 3.1 and (17), we obtain

$$\lim_{n\to\infty} \|\omega_n - P_{C_2}\omega_n\| = 0. \tag{18}$$

This shows that the sequence $P_{C_2}\omega_{n_j} \rightharpoonup \xi$ as $j \to \infty$.
For each $h > 0$, we have

$$\begin{aligned} \|T(h)P_{C_2}\omega_n - P_{C_2}\omega_n\| &\leq \|T(h)P_{C_2}\omega_n - T(h)\Lambda_n\| + \|T(h)\Lambda_n - \Lambda_n\| + \|\Lambda_n - P_{C_2}\omega_n\| \\ &\leq 2\|\Lambda_n - P_{C_2}\omega_n\| + \|T(h)\Lambda_n - \Lambda_n\|. \end{aligned}$$

The lemma 2.4 implies that

$$\lim_{n\to\infty} \|T(h)\Lambda_n - \Lambda_n\| = 0, \tag{19}$$

the equalities (17, 18) and (19)implies that

$$\lim_{n\to\infty} \|T(h)P_{C_2}\omega_n - \omega_n\| = 0.$$

Note that $F(TP_C) = F(T)$ for any mapping $T : C \to C$. The Lemma 2.6 implies that $\xi \in F(T(h)P_{C_2}) = F(T(h))$ for all $h > 0$. This shows that $\xi \in F(S)$.

(ii) $\xi \in \bigcap_{i=1}^{k} GEPS(F_i, \Psi_i)$.

Since $\{\omega_n\}$ is bounded and as respects (15), there exists a subsequence $\{\omega_{n_j}\}$ of $\{\omega_n\}$ such that $\omega_{n_j} \rightharpoonup \xi$.
By intuition from [12],

$$F_i(u_n^{(i)}, y) + \langle \Psi_i x_n, y - u_n^{(i)} \rangle + \frac{1}{r_n} \langle y - u_n^{(i)}, u_n^{(i)} - x_n \rangle \geq 0, \text{ for all } y \in C_1.$$

By $(A2)$, we have

$$\langle \Psi_i x_n, y - u_n^{(i)} \rangle + \frac{1}{r_n} \langle y - u_n^{(i)}, u_n^{(i)} - x_n \rangle \geq F_i(y, u_n^{(i)}).$$

Substitute n by n_j, we get

$$\langle \Psi_i x_{n_j}, y - u_{n_j}^{(i)} \rangle + \langle y - u_{n_j}^{(i)}, \frac{u_{n_j}^{(i)} - x_{n_j}}{r_{n_j}} \rangle \geq F_i(y, u_{n_j}^{(i)}). \tag{20}$$

For $0 < l \le 1$ and $y \in C_1$, set $y_l = ly + (1-l)\xi$. We have $y_l \in C_1$ and

$$\begin{aligned}\langle y_l - u_{n_j}^{(i)}, \Psi_i y_l\rangle &\ge \langle y_l - u_{n_j}^{(i)}, \Psi_i y_l\rangle - \langle \Psi_i x_{n_j}, y_l - u_{n_j}^{(i)}\rangle \\ &\quad -\langle y_l - u_{n_j}^{(i)}, \frac{u_{n_j}^{(i)} - x_{n_j}}{r_{n_j}}\rangle + F_i(y_l, u_{n_j}^{(i)}) \\ &= \langle y_l - u_{n_j}^{(i)}, \Psi_i y_l - \Psi_i u_{n_j}^{(i)}\rangle + \langle y_l - u_{n_j}^{(i)}, \Psi_i u_{n_j}^{(i)} - \Psi_i x_{n_j}\rangle \\ &\quad -\langle y_l - u_{n_j}^{(i)}, \frac{u_{n_j}^{(i)} - x_{n_j}}{r_{n_j}}\rangle + F_i(y_l, u_{n_j}^{(i)}).\end{aligned}$$

The condition $(A4)$, monotonicity of Ψ_i and (12) implies that
$\langle y_l - u_{n_j}^{(i)}, \Psi_i y_l - \Psi_i u_{n_j}^{(i)}\rangle \ge 0$ and $\|\Psi_i u_{n_j}^{(i)} - \Psi_i x_{n_j}\| \to 0$ as $j \to \infty$. Hence

$$\langle y_l - \xi, \Psi_i y_l\rangle \ge F_i(y_l, \xi). \tag{21}$$

Now, $(A1)$ and $(A4)$ together (21) show

$$\begin{aligned}0 = F_i(y_l, y_l) &\le lF_i(y_l, y) + (1-l)F_i(y_l, \xi) \\ &\le lF_i(y_l, y) + (1-l)\langle y_l - \xi, \Psi_i y_l\rangle \\ &= lF_i(y_l, y) + (1-l)l\langle y - \xi, \Psi_i y_l\rangle,\end{aligned}$$

which yields $F_i(y_l, y) + (1-l)\langle y - \xi, \Psi_i y_l\rangle \ge 0$.
By taking $l \to 0$, we have

$$F_i(\xi, y) + \langle y - \xi, \Psi_i \xi\rangle \ge 0.$$

This shows $\xi \in GEPS(F_i, \Psi_i)$, for all $i = 1, 2, \ldots, k$. Then, $\xi \in \bigcap_{i=1}^{k} GEPS(F_i, \Psi_i)$.

Now, in view of (16), we see

$$\limsup_{n\to\infty}\langle (A - \gamma f)\bar{x}, \bar{x} - \Lambda_n\rangle = \langle (A - \gamma f)\bar{x}, \bar{x} - \xi\rangle \le 0. \tag{22}$$

Finally, we prove $\{x_n\}$ is strongly convergent to $\bar{x}$.

$$\begin{aligned}\|x_{n+1} - \bar{x}\|^2 &= \|\alpha_n\gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)\Lambda_n - \bar{x}\|^2 \\ &= \|\alpha_n(\gamma f(x_n) - A\bar{x}) + \beta_n(x_n - \bar{x}) + ((1-\beta_n)I - \alpha_n A)(\Lambda_n - \bar{x})\|^2 \\ &= \|\beta_n(x_n - \bar{x}) + ((1-\beta_n)I - \alpha_n A)(\Lambda_n - \bar{x})\|^2 + \alpha_n^2\|\gamma f(x_n) - A\bar{x}\|^2 \\ &\quad +2\alpha_n\beta_n\langle x_n - \bar{x}, \gamma f(x_n) - A\bar{x}\rangle \\ &\quad +2\alpha_n\langle ((1-\beta_n)I - \alpha_n A)(\Lambda_n - \bar{x}), \gamma f(x_n) - A\bar{x}\rangle \\ &\le \{(1-\beta_n-\alpha_n\lambda)\|\Lambda_n - \bar{x}\| + \beta_n\|x_n - \bar{x}\|\}^2 + \alpha_n^2\|\gamma f(x_n) - A\bar{x}\|^2 \\ &\quad +2\alpha_n\beta_n\gamma\langle x_n - \bar{x}, f(x_n) - f(\bar{x})\rangle + 2\alpha_n\beta_n\langle x_n - \bar{x}, \gamma f(\bar{x}) - A\bar{x}\rangle \\ &\quad +2(1-\beta_n)\gamma\alpha_n\langle \Lambda_n - \bar{x}, f(x_n) - f(\bar{x})\rangle \\ &\quad +2(1-\beta_n)\alpha_n\langle \Lambda_n - \bar{x}, \gamma f(\bar{x}) - A\bar{x}\rangle - 2\alpha_n^2\langle A(\Lambda_n - \bar{x}), \gamma f(x_n) - A\bar{x}\rangle.\end{aligned}$$

Consequently

$$\begin{aligned}
\|x_{n+1}-\bar{x}\|^2 &\le \{(1-\alpha_n\lambda)^2+2\rho\alpha_n\beta_n\gamma+2\rho(1-\beta_n)\gamma\alpha_n\}\|x_n-\bar{x}\|^2\\
&\quad+\alpha_n^2\|\gamma f(x_n)-A\bar{x}\|^2+2\alpha_n\beta_n\langle x_n-\bar{x},\gamma f(\bar{x})-A\bar{x}\rangle\\
&\quad+2\alpha_n(1-\beta_n)\langle\Lambda_n-\bar{x},\gamma f(\bar{x})-A\bar{x}\rangle-2\alpha_n^2\langle A(\Lambda_n-\bar{x}),\gamma f(x_n)-A\bar{x}\rangle\\
&\le (1-2\alpha_n(\lambda-\rho\gamma))\|x_n-\bar{x}\|^2+\lambda^2\alpha_n^2\|x_n-\bar{x}\|^2+\alpha_n^2\|\gamma f(x_n)-A\bar{x}\|^2\\
&\quad+2\alpha_n\beta_n\langle x_n-\bar{x},\gamma f(\bar{x})-A\bar{x}\rangle+2\alpha_n(1-\beta_n)\langle\Lambda_n-\bar{x},\gamma f(\bar{x})-A\bar{x}\rangle\\
&\quad+2\alpha_n^2\|A(\Lambda_n-\bar{x})\|\|\gamma f(x_n)-A\bar{x}\|\\
&= (1-2\alpha_n(\lambda-\rho\gamma))\|x_n-\bar{x}\|^2+\alpha_n\{\alpha_n(\lambda^2\|x_n-\bar{x}\|^2\\
&\quad+\|\gamma f(x_n)-A\bar{x}\|^2+2\|A(\Lambda_n-\bar{x})\|\|\gamma f(x_n)-A\bar{x}\|\\
&\quad+2\beta_n\langle x_n-\bar{x},\gamma f(\bar{x})-A\bar{x}\rangle+2(1-\beta_n)\langle\Lambda_n-\bar{x},\gamma f(\bar{x})-A\bar{x}\rangle\}.
\end{aligned}$$

Since $\{x_n\}$, $\{f(x_n)\}$ and $\{\Lambda_n\}$ are bounded, one can take a constant $\Gamma > 0$ such that

$$\Gamma \ge \lambda^2\|x_n-\bar{x}\|^2+\|\gamma f(x_n)-A\bar{x}\|^2+2\|A(\Lambda_n-\bar{x})\|\|\gamma f(x_n)-A\bar{x}\|.$$

Let

$$\Xi_n = 2\beta_n\langle x_n-\bar{x},\gamma f(\bar{x})-A\bar{x}\rangle+2(1-\beta_n)\langle\Lambda_n-\bar{x},\gamma f(\bar{x})-A\bar{x}\rangle+\Gamma\alpha_n.$$

Hence

$$\|x_{n+1}-\bar{x}\|^2 \le (1-2\alpha_n(\lambda-\rho\gamma))\|x_n-\bar{x}\|^2+\alpha_n\Xi_n. \tag{23}$$

With respect to (22), $\limsup_{n\to\infty}\Xi_n \le 0$ and so all conditions of Lemma 2.5 are satisfied for (23). Consequently, the sequence $\{x_n\}$ is strongly convergent to $\bar{x}$. □

As a result, by intuition from [12], the following mean ergodic theorem for a nonexpansive mapping in Hilbert space is proved.

Corollary 3.3. *Suppose all assumptions of Theorem 3.1 are holds. Let $\{T^i\}$ be a family of nonexpansive mappings on C_1 for all $i = 1, 2, \dots, k$ such that $\bigcap_{i=1}^k F(T^i) \cap GEPS(F_i, \Psi_i) \neq \emptyset$. Let $\{x_n\}$ and $\{u_n^{(i)}\} \subset C_1$ be sequences generated in the following manner:*

$$\begin{cases}
x_1 \in H \ \text{choosen arbitrary},\\
F_1(u_n^{(1)}, y)+\langle\Psi_1 x_n, y-u_n^{(1)}\rangle+\frac{1}{r_n}\langle y-u_n^{(1)}, u_n^{(1)}-x_n\rangle \ge 0, \text{ for all } y\in C_1,\\
F_2(u_n^{(2)}, y)+\langle\Psi_2 x_n, y-u_n^{(2)}\rangle+\frac{1}{r_n}\langle y-u_n^{(2)}, u_n^{(2)}-x_n\rangle \ge 0, \text{ for all } y\in C_1,\\
\vdots\\
F_k(u_n^{(k)}, y)+\langle\Psi_k x_n, y-u_n^{(k)}\rangle+\frac{1}{r_n}\langle y-u_n^{(k)}, u_n^{(k)}-x_n\rangle \ge 0, \text{ for all } y\in C_1,\\
\omega_n = \frac{1}{k}\sum_{i=1}^k u_n^{(i)},\\
x_{n+1} = \alpha_n\gamma f(x_n)+\beta_n x_n+((1-\beta_n)I-\alpha_n A)\frac{1}{n+1}\sum_{i=0}^n P_{C_2}T^i\omega_n,
\end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}$ are the sequences in $[0, 1]$ and $\{r_n\} \subset (0,\infty)$ is a real sequence. Suppose the following conditions are satisfied:

(C1) $\lim_{n\to\infty}\alpha_n = 0, \sum_{n=1}^{\infty}\alpha_n = \infty$;

(C2) $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$;

(C3) $\lim_{n\to\infty}|r_{n+1}-r_n| = 0$ *and* $0 < b < r_n < a < 2\mu_i$ *for* $i \in \{1, 2, \dots, k\}$.

Then the sequence $\{x_n\}$ is strongly convergent to a point $\bar{x}$, where

$$\bar{x} = P_{\bigcap_{i=1}^{k} F(T^i)\cap GEPS(F_i,\Psi_i)}(I - A + \gamma f)(\bar{x})$$

, is the unique solution of the variational inequality

$$\langle (A - \gamma f)\bar{x}, \bar{x} - x\rangle \leq 0, \forall x \in \bigcap_{i=1}^{k} F(T^i) \cap GEPS(F_i, \Psi_i).$$

4. Application

If $T(s) = T$ for all $s > 0$ and $C_1 = C_2 = C$, then we have the following corollary.

Corollary 4.1. *Let H be a real Hilbert space, $F_1, F_2, \ldots, F_k$ be bifunctions from $C \times C$ to $\mathbb{R}$ satisfying $(A1) - (A4)$, $\Psi_1, \Psi_2, \ldots, \Psi_k$ be μ_i-inverse strongly monotone mapping on H, A be a strongly positive linear bounded operator on H with coefficient λ and $0 < \gamma < \frac{\lambda}{\rho}$, $f : H \to H$ be a $\rho-$contraction. Suppose that T be a nonexpansive mapping on C such that $\bigcap_{i=1}^{k} F(T) \cap GEPS(F_i, \Psi_i) \neq \emptyset$. Define the sequence $\{x_n\}$ as follows.*

$$\begin{cases} x_1 \in H, \text{ and } u_n^{(i)} \in C, \\ F_1(u_n^{(1)}, y) + \langle \Psi_1 x_n, y - u_n^{(1)}\rangle + \frac{1}{r_n}\langle y - u_n^{(1)}, u_n^{(1)} - x_n\rangle \geq 0, \text{ for all } y \in C, \\ F_2(u_n^{(2)}, y) + \langle \Psi_2 x_n, y - u_n^{(2)}\rangle + \frac{1}{r_n}\langle y - u_n^{(2)}, u_n^{(2)} - x_n\rangle \geq 0, \text{ for all } y \in C, \\ \vdots \\ F_k(u_n^{(k)}, y) + \langle \Psi_k x_n, y - u_n^{(k)}\rangle + \frac{1}{r_n}\langle y - u_n^{(k)}, u_n^{(k)} - x_n\rangle \geq 0, \text{ for all } y \in C, \\ \omega_n = \frac{1}{k}\sum_{i=1}^{k} u_n^{(i)}, \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n A)TP_C\omega_n ds, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$ and $\{r_n\} \subset (0, \infty)$ are the sequences satisfying the conditions $(C1) - (C3)$ in Theorem 3.2. Then the sequence $\{x_n\}$ converges strongly to a point $\bar{x}$, where $\bar{x} \in \bigcap_{i=1}^{k} F(T) \cap GEPS(F_i, \Psi_i)$ solves the variational inequality $\langle (A - \gamma f)\bar{x}, \bar{x} - x\rangle \leq 0$.

We apply Theorem 3.2 for finding a common fixed point of a nonexpansive semigroup mappings and strictly pseudo-contractive mapping and inverse strongly monotone mapping.
Recall that, a mapping $T : C \to C$ is called strictly pseudo-contractive if there exists k with $0 \leq k \leq 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \text{ for all } x, y \in C.$$

If $k = 0$, then T is nonexpansive. Put $J = I - T$, where $T : C \to C$ is a strictly pseudo-contractive mapping. J is $\frac{1-k}{2}$-inverse strongly monotone and $J^{-1}(0) = F(T)$. Indeed, for all $x, y \in C$ we have

$$\|(I - J)x - (I - J)y\|^2 \leq \|x - y\|^2 + k\|Jx - Jy\|^2.$$

Also

$$\|(I - J)x - (I - J)y\|^2 \leq \|x - y\|^2 + \|Jx - Jy\|^2 - 2\langle x - y, Jx - Jy\rangle.$$

So, we have

$$\langle x - y, Jx - Jy\rangle \geq \frac{1 - k}{2}\|Jx - Jy\|^2.$$

Corollary 4.2. *Let H be a real Hilbert space, $F_1, F_2, \ldots, F_k$ be bifunctions from $C \times C$ to $\mathbb{R}$ satisfying $(A1)-(A4)$, $\Psi_1, \Psi_2, \ldots, \Psi_k$ be μ_i−inverse strongly monotone mapping on H, A be a strongly positive linear bounded operator on H with coefficient λ and $0 < \gamma < \frac{\lambda}{\rho}$, $f : H \to H$ be a ρ−contraction. Suppose that $T : C \to H$ be a k-strictly pseudo-contractive mapping for some $0 \le k < 1$ such that $\bigcap_{i=1}^{k} F(T) \cap GEPS(F_i, \Psi_i) \neq \emptyset$. Define the sequence $\{x_n\}$ as follows.*

$$\begin{cases} x_1 \in H, \text{ and } u_n^{(i)} \in C, \\ F_1(u_n^{(1)}, y) + \langle \Psi_1 x_n, y - u_n^{(1)} \rangle + \frac{1}{r_n}\langle y - u_n^{(1)}, u_n^{(1)} - x_n \rangle \ge 0, \text{ for all } y \in C, \\ F_2(u_n^{(2)}, y) + \langle \Psi_2 x_n, y - u_n^{(2)} \rangle + \frac{1}{r_n}\langle y - u_n^{(2)}, u_n^{(2)} - x_n \rangle \ge 0, \text{ for all } y \in C, \\ \vdots \\ F_k(u_n^{(k)}, y) + \langle \Psi_k x_n, y - u_n^{(k)} \rangle + \frac{1}{r_n}\langle y - u_n^{(k)}, u_n^{(k)} - x_n \rangle \ge 0, \text{ for all } y \in C, \\ \omega_n = \frac{1}{k}\sum_{i=1}^{k} u_n^{(i)}, \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)P_C J\omega_n ds, \end{cases}$$

where $J : C \to H$ is a mapping defined by $Jx = kx + (1-k)Tx$ and $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$ and $\{r_n\} \subset (0,\infty)$ are the sequences satisfying the conditions $(C1)-(C3)$ in Theorem 3.1. Then the sequence $\{x_n\}$ is strongly convergent to a point $\bar{x}$, where $\bar{x} \in \bigcap_{i=1}^{k} F(T) \cap GEPS(F_i, \Psi_i)$ solves the variational inequality $\langle (A - \gamma f)\bar{x}, \bar{x} - x \rangle \le 0$.

Proof. Note that $S : C \to H$ is a nonexpansinve mapping and $F(T) = F(S)$. By Lemma 2.3 in [41] and Lemma 2.2 in [37], we have $P_C S : C_2 \to C$ is a nonexpansive mapping and $F(P_C S) = F(S) = F(S)$. Therefore, the result follows from Corollary 4.1. □

5. Numerical Examples

In this section, we show numerical examples which grantee the main theorem. The programming has been provided with Matlab according to the following algorithm.

Example 5.1. Suppose that $H = \mathbb{R}, C_1 = [-1,1], C_2 = [0,1]$ and

$$F_1(x,y) = -3x^2 + xy + 2y^2, F_2(x,y) = -4x^2 + xy + 3y^2, F_3(x,y) = -5x^2 + xy + 4y^2.$$

Also, we consider $\Psi_1(x) = x, \Psi_2 = 2x$ and $\Psi_3(x) = \frac{x}{10}$. Suppose that $A = \frac{x}{10}, f(x) = \frac{x}{10}$ with coefficient $\gamma = 1$ and $T(s) = e^{-s}$ is a nonexpansive semigroup on C_2. It is easy to check that $\Psi_1, \Psi_2, \Psi_3, A, f$ and $T(s)$ satisfy all conditions in Theorem 3.1. For each $y \in C_1$ there exists $z \in C_1$ such that

$$\begin{aligned} & F_1(z,y) + \langle \Psi_1 x, y - z \rangle + \frac{1}{r}\langle y - z, z - x \rangle \ge 0 \\ \Leftrightarrow \quad & -3z^2 + zy + 2y^2 + x(y-z) + \frac{1}{r}(y-z)(z-x) \ge 0 \\ \Leftrightarrow \quad & 2ry^2 + ((r+1)z - (r-1)x)y - 3rz^2 - xzr - z^2 + zx \ge 0. \end{aligned}$$

Set $G(y) = 2ry^2 + ((r+1)z - (r-1)x)y - 3rz^2 - xzr - z^2 + zx$. Then $G(y)$ is a quadratic function of y with coefficients $a = 2r, b = (r+1)z - (r-1)x$ and $c = -3rz^2 - xzr - z^2 + zx$. So

$$\begin{aligned} \Delta &= b^2 - 4ac \\ &= [(r+1)z - (r-1)x]^2 - 8r(-3rz^2 - xzr - z^2 + zx) \\ &= x^2(r-1)^2 + 2zx(r-1)(5r+1) + z^2(5r+1)^2 \\ &= [(x(r-1) + z(5r+1))]^2. \end{aligned}$$

Since $G(y) \ge 0$ for all $y \in C_1$, if and only if $\Delta = [(x(r-1) + z(5r+1))]^2 \le 0$. Therefore, $z = \dfrac{1-r}{5r+1}x$, which yields $T_{r_n^{(1)}} = u_n^{(1)} = \dfrac{1-r_n}{5r_n+1}x_n$.

By the same argument, for F_2 and F_3, one can conclude

$$\begin{aligned} T_{r_n^{(2)}} &= u_n^{(2)} = \frac{1-2r_n}{7r_n+1}x_n, \\ T_{r_n^{(3)}} &= u_n^{(3)} = \frac{10-r_n}{90r_n+10}x_n. \end{aligned}$$

Then

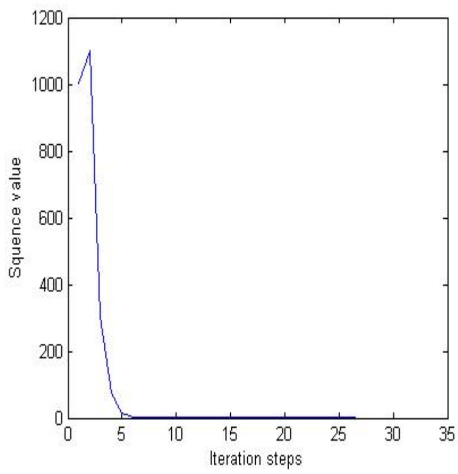
$$\begin{aligned} \omega_n &= \frac{u_n^{(1)}+u_n^{(2)}+u_n^{(3)}}{3} \\ &= \frac{1}{3}[\frac{1-r_n}{5r_n+1}+\frac{1-2r_n}{7r_n+1}+\frac{10-r_n}{90r_n+10}]x_n. \end{aligned}$$

By choosing $r_n = \frac{n+8}{n}$, $t_n = n$, and $\alpha_n = \frac{9}{10n}, \beta_n = \frac{2n-1}{10n-9}$, we have the following algorithm for the sequence $\{x_n\}$

$$x_{n+1} = \frac{200n^2-10n-81}{100n^2-90n}x_n + \frac{800n^2-890n+81}{1000n^2-900n}\left(\frac{1-e^{-n}}{n}\right)\omega_n.$$

Choose $x_1 = 1000$. By using MATLAB software, we obtain the following table and figure of the result.

n	x_n	n	x_n	n	x_n
1	1000	11	0.001780587335	21	0.0000000003224455074
2	1098.409511	12	0.0003806958425	22	0.00000000006766467135
3	305.4917461	13	0.00008116570943	23	0.00000000001418205647
4	73.2742215	14	0.00001726216226	24	0.000000000002969041259
5	16.71274356	15	0.000003663196562	25	0.0000000000006208956961
6	3.722532274	16	0.0000007758152752	26	0.00000000000007698339387
7	0.8179356809	17	0.0000001640073634	27	0.0000000000001297089212
8	0.1781302867	18	0.00000003461280635	28	0.00000000000002707018951
9	0.03854703081	19	0.000000007293416021	29	0.000000000000005644205174
10	0.008300951942	20	0.000000001534585156	30	0.000000000000001175770771



Example 5.2. *Theorem 3.2 can be illustrated by the following numerical example where the parameters are given as follows:*

$$\begin{aligned} H = [-10, 10] \quad &, \quad C_1 = [-1, 1], C_2 = [0, 1], A = I, f(x) = \frac{x}{5} \\ \Psi_1(x) = x \quad &, \quad \Psi_2 = 2x, \Psi_3(x) = \frac{x}{10}, \Psi_4(x) = 3x, \Psi_5(x) = 4x \\ \alpha_n = \frac{1}{2n} \quad &, \quad \beta_n = \frac{n}{2n+1}, r_n = \frac{n+1}{n} \\ T(s) = e^{-s} \quad &, \quad \gamma = 1, t_n = n. \end{aligned}$$

Moreover,

$$\begin{aligned} F_1(x, y) = -3x^2 + xy + 2y^2 \quad &, \quad F_4(x, y) = -6x^2 + xy + 5y^2 \\ F_2(x, y) = -4x^2 + xy + 3y^2 \quad &, \quad F_5(x, y) = -8x^2 + xy + 7y^2 \\ F_3(x, y) = -5x^2 + xy + 4y^2. & \end{aligned}$$

By the same argument in Example 5.1, we compute $u_n^{(i)}$ for $i = 1, 2, 3, 4, 5$ as follows:

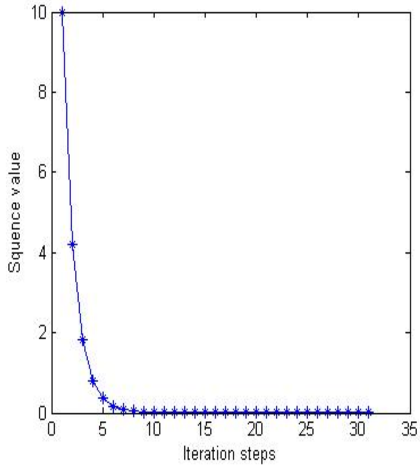
$$\begin{aligned} T_{r_n^{(1)}} = u_n^{(1)} &= \frac{1 - r_n}{5r_n + 1} x_n, \\ T_{r_n^{(2)}} = u_n^{(2)} &= \frac{1 - 2r_n}{7r_n + 1} x_n, \\ T_{r_n^{(3)}} = u_n^{(3)} &= \frac{10 - r_n}{90r_n + 10} x_n, \\ T_{r_n^{(4)}} = u_n^{(4)} &= \frac{1 - 3r_n}{11r_n + 1} x_n, \\ T_{r_n^{(5)}} = u_n^{(3)} &= \frac{1 - 4r_n}{15r_n + 1} x_n. \end{aligned}$$

Then

$$\begin{aligned} \omega_n &= \frac{u_n^{(1)} + u_n^{(2)} + \ldots + u_n^{(5)}}{5} \\ &= \frac{1}{5}[\frac{1 - r_n}{5r_n + 1} + \frac{1 - 2r_n}{7r_n + 1} + \frac{10 - r_n}{90r_n + 10} + \frac{1 - 3r_n}{11r_n + 1} + \frac{1 - 4r_n}{15r_n + 1}]x_n. \end{aligned}$$

Choose $x_1 = 10$. The detailed results of proposed iterative in Theorem 3.2 are presented in the following table and figure.

n	x_n	n	x_n	n	x_n
1	10	11	0.004288752976	21	0.000003269796382
2	4.187523857	12	0.002073897432	22	0.000001606140362
3	1.810011289	13	0.001005587742	23	0.0000007895721035
4	0.8111798915	14	0.0004887121222	24	0.0000003884328747
5	0.37213169	15	0.0002379837774	25	0.000000191218571
6	0.173392393	16	0.0001160888527	26	0.00000009419138076
7	0.08167872103	17	0.00005671409128	27	0.00000004642361668
8	0.03878661445	18	0.00002774424594	28	0.00000002289261628
9	0.0185321898	19	0.00001358854452	29	0.0000000112944317
10	0.008897703796	20	0.000006662503243	30	0.00000000557482625



Example 5.3. *Let*

$$\begin{aligned} H = [-10,10] \quad &, \quad C_1 = [0,1], C_2 = [-1,1], A = \frac{x}{10}, f(x) = \frac{x}{10} \\ \Psi_1(x) = \Psi_2(x) = 0 \quad &, \quad \Psi_3(x) = x, \Psi_4(x) = 2x, \Psi_5(x) = \frac{x}{10} \\ \Psi_6(x) = 3x \quad &, \quad \Psi_7(x) = 4x \\ \alpha_n = \frac{1}{n} \quad &, \quad \beta_n = \frac{n}{3n+1}, r_n = \frac{n+1}{n} \\ T(s) = e^{-s} \quad &, \quad \gamma = 1, t_n = n. \end{aligned}$$

Moreover,

$$\begin{aligned} F_1(x,y) = (1-x^2)(x-y) \quad &, \quad F_5(x,y) = -5x^2 + xy + 4y^2; \\ F_2(x,y) = -x^2(x-y)^2 \quad &, \quad F_6(x,y) = -6x^2 + xy + 5y^2; \\ F_3(x,y) = -3x^2 + xy + 2y^2 \quad &, \quad F_7(x,y) = -8x^2 + xy + 7y^2; \\ F_4(x,y) = -4x^2 + xy + 3y^2. & \end{aligned}$$

By the same argument in Example 5.1, we compute $u_n^{(i)}$ *for* $i = 1,2$ *as follows: For any* $y \in C_1$ *and* $r > 0$*, we have*

$$F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0 \Leftrightarrow (y-z)(rz^2 + z - r - x) \geq 0.$$

This implies that $rz^2 + z - r - x = 0$*. Therefore,* $z = \frac{-1+\sqrt{1+4r(r+x)}}{2r}$ *which yields* $T_{r_n^{(1)}} = \frac{-1+\sqrt{1+4r_n(r_n+x_n)}}{2r_n}$.
Also, we have

$$F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0 \Leftrightarrow -rz^2y^2 + (2rz^3 + z - x)y - rz^4 - z^2 + zx \geq 0.$$

Set $J(y) = -rz^2y^2 + (2rz^3 + z - x)y - rz^4 - z^2 + zx$*. Then* $J(y)$ *is a quadratic function of* y *with coefficients* $a = -rz^2, b = 2rz^3 + z - x$ *and* $c = -rz^4 - z^2 + zx$*. So*

$$\begin{aligned} \Delta &= [2rz^3 + z - x]^2 + 4rz^2(-rz^4 - z^2 + zx) \\ &= (z-x)^2. \end{aligned}$$

Since $J(y) \geq 0$ *for all* $y \in H$*, if and only if* $\Delta = (z-x)^2 = 0$*. Therefore,* $z = T_{r_n^{(2)}} = x$*.*
Then

$$\begin{aligned}
T_{r_n^{(1)}} = u_n^{(1)} &= \frac{-1+\sqrt{1+4r_n(r_n+x_n)}}{2r_n},\\
T_{r_n^{(2)}} = u_n^{(2)} &= x_n,\\
T_{r_n^{(3)}} = u_n^{(3)} &= \frac{1-r_n}{5r_n+1}x_n,\\
T_{r_n^{(4)}} = u_n^{(4)} &= \frac{1-2r_n}{7r_n+1}x_n,\\
T_{r_n^{(5)}} = u_n^{(5)} &= \frac{10-r_n}{90r_n+10}x_n,\\
T_{r_n^{(6)}} = u_n^{(6)} &= \frac{1-3r_n}{11r_n+1}x_n,\\
T_{r_n^{(7)}} = u_n^{(7)} &= \frac{1-4r_n}{15r_n+1}x_n.
\end{aligned}$$

Then

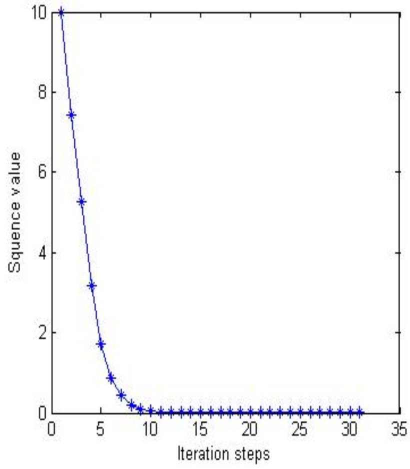
$$\omega_n = \frac{u_n^{(1)}+u_n^{(2)}+\ldots+u_n^{(7)}}{7}.$$

We have

$$x_{n+1} = \frac{10n^2+3n+1}{30n^2-10n}x_n + \frac{20n^2+7n-1}{30n^2+10n}\left(\frac{1-e^{-n}}{n}\right)\omega_n.$$

Choose $x_1 = 10$. The detailed results of proposed iterative in Theorem 3.2 are presented in the following table and figure.

n	x_n	n	x_n	n	x_n
1	10	11	0.0239156544	21	0.00001930391707
2	7.414282455	12	0.01166047726	22	0.000009519419651
3	5.25738677	13	0.005695329069	23	0.000004697209017
4	3.188403646	14	0.002786256225	24	0.000002319056949
5	1.720431319	15	0.001365047089	25	0.000001145527389
6	0.8702902283	16	0.000669619601	26	0.0000005661151488
7	0.4279164132	17	0.0003288520367	27	0.0000002798942327
8	0.2082805371	18	0.0001616632095	28	0.000000138439269
9	0.1011375703	19	0.00007954524563	29	0.00000006849968096
10	0.04914122712	20	0.00003917150569	30	0.00000003390554052



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