Boundary Value Problems of Nonlinear Fractional q-Difference (Integral) Equations with Two Fractional Orders and Four-point Nonlocal Integral Boundary Conditions

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Abstract. This paper investigates the existence of solutions for nonlinear fractional q-difference equations and q-difference integral equations involving two fractional orders with four-point nonlocal integral boundary conditions. The existence results are obtained by applying some traditional tools of fixed point theory, and are illustrated with examples.

1. Introduction

Boundary value problems of fractional-order have recently been studied by many researchers. Fractional derivatives appear naturally in the mathematical modelling of dynamical systems involving fractals and chaos. In fact, the concept of fractional calculus has played an important role in improving the work based on integer-order (classical) calculus in several diverse disciplines of science and engineering. It has been probably due to the reason that differential operators of fractional-order help to understand the hereditary phenomena in many materials and processes in a better way than the corresponding integer-order differential operators. Examples include physics, chemistry, biology, biophysics, blood flow phenomena, control theory, wave propagation, signal and image processing, viscoelasticity, percolation, identification, fitting of experimental data, economics, etc. [1]-[4]. For some recent work on fractional differential equations, we refer to [5]-[14] and the references therein.

Fractional q-difference equations, regarded as fractional analogue of q-difference equations, have recently been studied by several researchers. For some earlier work on the topic, we refer to [15]-[16], whereas some recent work on the existence theory of fractional q-difference equations can be found in [17]-[27].

In this paper, we discuss the existence and uniqueness of solutions for a nonlocal boundary value problem of nonlinear fractional q-difference equations and q-difference integral equations with four-point nonlocal integral boundary conditions. As a first problem, we consider

$$^cD_q^\beta(^cD_q^\gamma + \lambda)x(t) = f(t, x(t)), \ \ 0 \le t \le 1, \ 0 < q < 1, \ 0 < \beta \le 1, \ \ 0 < \gamma \le 1, \ \ \lambda \in \mathbb{R}, \tag{1}$$

$$\begin{cases} x(0) = aI_q^{\alpha-1}x(\eta) = a\int_0^\eta \dfrac{(\eta - qs)^{(\alpha-2)}}{\Gamma_q(\alpha - 1)}x(s)d_qs, \\ x(1) = bI_q^{\alpha-1}x(\sigma) = b\int_0^\sigma \dfrac{(\sigma - qs)^{(\alpha-2)}}{\Gamma_q(\alpha - 1)}x(s)d_qs, \ \alpha > 2, \ 0 < \eta, \sigma < 1, \end{cases} \tag{2}$$

where $f \in C([0, 1] \times \mathbb{R}, \mathbb{R})$, $^cD_q^\beta$ and $^cD_q^\gamma$ are the fractional q-derivative of the Caputo type, a and b are real numbers.

In the second problem, we consider the following nonlinear fractional q-difference integral equation supplemented with boundary conditions (2):

$$^cD_q^\beta(^cD_q^\gamma + \lambda)x(t) = pf(t, x(t)) + kI_q^\xi g(t, x(t)), \ \ 0 \le t \le 1, \ 0 < q < 1, \ \ \lambda \in \mathbb{R}, \tag{3}$$

where $I_q^\xi(.)$ denotes Riemann-Liouville integral with $0 < \xi < 1$, f, g are given continuous functions, and p, k are real constants.

The paper is organized as follows. Section 2 contains some necessary background material on the topic, while the main results for the problem (1)-(2) are presented in Section 3. We make use of Banach's contraction principle, Krasnoselskii's fixed point theorem and Leray-Schauder nonlinear alternative to establish the existence results for the problem at hand. Although these tools are standard, yet their exposition in the framework of the present problem is new. In Section 4, we present some existence results for the problem (3)-(2).

2. Preliminaries on Fractional q-Calculus

Here we recall some definitions and fundamental results on fractional q-calculus [28]-[30].

Let a q-real number denoted by $[a]_q$ be defined by

$$[a]_q = \frac{1 - q^a}{1 - q}, \ \ a \in \mathbb{R}, \ q \in \mathbb{R}^+ \setminus \{1\}.$$

The q-analogue of the Pochhammer symbol (q-shifted factorial) is defined as

$$(a; q)_0 = 1, \ \ (a; q)_k = \prod_{i=0}^{k-1}(1 - aq^i), \ k \in \mathbb{N} \cup \{\infty\}.$$

The q-analogue of the exponent $(x - y)^k$ is

$$(x - y)^{(0)} = 1, \quad (x - y)^{(k)} = \prod_{j=0}^{k-1}(x - yq^j), \quad k \in \mathbb{N}, \ x, y \in \mathbb{R}.$$

The q-gamma function $\Gamma_q(y)$ is defined as

$$\Gamma_q(y) = \frac{(1 - q)^{(y-1)}}{(1 - q)^{y-1}},$$

where $y \in \mathbb{R} \setminus \{0, -1, -2, \ldots\}$. Observe that $\Gamma_q(y + 1) = [y]_q \Gamma_q(y)$. For any $x, \ y > 0$, the q-beta function $B_q(x, y)$ is given by

$$B_q(x, y) = \int_0^1 t^{(x-1)}(1 - qt)^{(y-1)} d_q t,$$

which, in terms of q-gamma function, can be expressed as

$$B_q(x, y) = \frac{\Gamma_q(x)\Gamma_q(y)}{\Gamma_q(x + y)}. \tag{4}$$

Definition 2.1. *([28]) Let f be a function defined on $[0, 1]$. The fractional q-integral of the Riemann-Liouville type of order $\beta \geq 0$ is $(I_q^0 f)(t) = f(t)$ and*

$$I_q^\beta f(t) := \int_0^t \frac{(t - qs)^{(\beta-1)}}{\Gamma_q(\beta)} f(s) d_q s = t^\beta(1 - q)^\beta \sum_{k=0}^\infty q^k \frac{(q^\beta; q)_n}{(q; q)_n} f(tq^k), \quad \beta > 0, \ t \in [0, 1].$$

Observe that $\beta = 1$ in the Definition 2.1 yields q-integral

$$I_q f(t) := \int_0^t f(s) d_q s = t(1 - q) \sum_{k=0}^\infty q^k f(tq^k).$$

For more details on q-integral and fractional q-integral, see Section 1.3 and Section 4.2 respectively in [29].

Remark 2.2. *The q-fractional integration possesses the semigroup property (Proposition 4.3 [29]):*

$$I_q^\gamma I_q^\beta f(t) = I_q^{\beta+\gamma} f(t); \ \gamma, \beta \in \mathbb{R}^+. \tag{5}$$

Further, it has been shown in Lemma 6 of [30] that

$$I_q^\beta(x)^{(\sigma)} = \frac{\Gamma_q(\sigma + 1)}{\Gamma_q(\beta + \sigma + 1)}(x)^{(\beta+\sigma)}, \ 0 < x < a, \beta \in \mathbb{R}^+, \sigma \in (-1, \infty).$$

Before giving the definition of fractional q-derivative, we recall the concept of q-derivative. We know that the q-derivative of a function $f(t)$ is defined as

$$(D_q f)(t) = \frac{f(t) - f(qt)}{t - qt}, \ t \neq 0, \ (D_q f)(0) = \lim_{t \to 0}(D_q f)(t).$$

Furthermore,

$$D_q^0 f = f, \quad D_q^n f = D_q(D_q^{n-1} f), \quad n = 1, 2, 3, \ldots \tag{6}$$

Definition 2.3. *([29]) The Caputo fractional q-derivative of order $\beta > 0$ is defined by*

$$^{c}D_{q}^{\beta}f(t) = I_{q}^{\lceil\beta\rceil-\beta}D_{q}^{\lceil\beta\rceil}f(t),$$

where $\lceil\beta\rceil$ is the smallest integer greater than or equal to β.

Next we recall some properties involving Riemann-Liouville q-fractional integral and Caputo fractional q-derivative (Theorem 5.2 [29]).

$$I_{q}^{\beta}\,{}^{c}D_{q}^{\beta}f(t) = f(t) - \sum_{k=0}^{\lceil\beta\rceil-1}\frac{t^{k}}{\Gamma_{q}(k+1)}(D_{q}^{k}f)(0^{+}), \ \forall\, t \in (0,a], \ \beta > 0; \tag{7}$$

$$^{c}D_{q}^{\beta}I_{q}^{\beta}f(t) = f(t), \quad \forall\, t \in (0,a], \ \beta > 0. \tag{8}$$

In order to define the solution for the problem (1)-(2), we need the following lemma.

Lemma 2.4. *For a given $h \in C([0,1],\mathbb{R})$ the unique solution of the boundary value problem*

$$^{c}D_{q}^{\beta}(^{c}D_{q}^{\gamma} + \lambda)x(t) = h(t), \ 0 \le t \le 1, \ 0 < q < 1, \ 0 < \beta \le 1, \ 0 < \gamma \le 1, \tag{9}$$

$$\begin{cases} x(0) = aI_{q}^{\alpha-1}x(\eta) = a\displaystyle\int_{0}^{\eta}\frac{(\eta-qs)^{(\alpha-2)}}{\Gamma_{q}(\alpha-1)}x(s)d_{q}s, \\[3mm] x(1) = bI_{q}^{\alpha-1}x(\sigma) = b\displaystyle\int_{0}^{\sigma}\frac{(\sigma-qs)^{(\alpha-2)}}{\Gamma_{q}(\alpha-1)}x(s)d_{q}s, \end{cases} \tag{10}$$

is given by

$$\begin{aligned} x(t) &= \int_{0}^{t}\frac{(t-qu)^{(\gamma-1)}}{\Gamma_{q}(\gamma)}\Big(I_{q}^{\beta}h(u) - \lambda x(u)\Big)d_{q}u \\ &\quad -\frac{[\delta_{3}t^{\gamma}-\delta_{4}]}{\Delta}\Big\{a\int_{0}^{\eta}\frac{(\eta-qs)^{(\alpha-2)}}{\Gamma_{q}(\alpha-1)}\Big(\int_{0}^{s}\frac{(s-qu)^{(\gamma-1)}}{\Gamma_{q}(\gamma)}\Big(I_{q}^{\beta}h(u) - \lambda x(u)\Big)d_{q}u\Big)d_{q}s\Big\} \\ &\quad +\frac{[\delta_{1}t^{\gamma}-\delta_{2}]}{\Delta}\Big\{b\int_{0}^{\sigma}\frac{(\sigma-qs)^{(\alpha-2)}}{\Gamma_{q}(\alpha-1)}\Big(\int_{0}^{s}\frac{(s-qu)^{(\gamma-1)}}{\Gamma_{q}(\gamma)}\Big(I_{q}^{\beta}h(u) - \lambda x(u)\Big)d_{q}u\Big)d_{q}s\Big\} \\ &\quad -\frac{[\delta_{1}t^{\gamma}-\delta_{2}]}{\Delta}\Big\{\int_{0}^{1}\frac{(1-qu)^{(\gamma-1)}}{\Gamma_{q}(\gamma)}\Big(I_{q}^{\beta}h(u) - \lambda x(u)\Big)d_{q}u\Big\}, \end{aligned} \tag{11}$$

where

$$\delta_{1} = \Big(\frac{a\eta^{(\alpha-1)}}{\Gamma_{q}(\alpha)} - 1\Big), \quad \delta_{2} = \Big(\frac{a\eta^{(\alpha+\gamma-1)}\Gamma_{q}(\gamma+1)}{\Gamma_{q}(\alpha+\gamma)}\Big),$$
$$\delta_{3} = \Big(\frac{b\sigma^{(\alpha-1)}}{\Gamma_{q}(\alpha)} - 1\Big), \quad \delta_{4} = \Big(\frac{b\sigma^{(\alpha+\gamma-1)}\Gamma_{q}(\gamma+1)}{\Gamma_{q}(\alpha+\gamma)} - 1\Big) \text{ and } \Delta = \delta_{3}\delta_{2} - \delta_{4}\delta_{1}.$$

Proof. In view of (5) and (7), the solution of (9) can be written as

$$x(t) = \int_{0}^{t}\frac{(t-qu)^{(\gamma-1)}}{\Gamma_{q}(\gamma)}\Big[I_{q}^{\beta}h(u) - \lambda x(u)\Big]d_{q}u - c_{0}\frac{t^{\gamma}}{\Gamma_{q}(\gamma+1)} - c_{1}, \ t \in [0,1]. \tag{12}$$

Using the boundary conditions (10) in (12) and the definition of q-Beta function together with the property (4), we have

$$\frac{1}{\Gamma_q(\gamma+1)}\Big(\frac{a\eta^{(\alpha+\gamma-1)}\Gamma_q(\gamma+1)}{\Gamma_q(\alpha+\gamma)}\Big)c_0 + \Big(\frac{a\eta^{(\alpha-1)}}{\Gamma_q(\alpha)}-1\Big)c_1$$
$$= a\int_0^\eta \frac{(\eta-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\Big(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\big(I_q^\beta h(u)-\lambda x(u)\big)d_q u\Big)d_q s, \tag{13}$$

$$\frac{1}{\Gamma_q(\gamma+1)}\Big(\frac{b\sigma^{(\alpha+\gamma-1)}\Gamma_q(\gamma+1)}{\Gamma_q(\alpha+\gamma)}-1\Big)c_0 + \Big(\frac{b\sigma^{(\alpha-1)}}{\Gamma_q(\alpha)}-1\Big)c_1$$
$$= b\int_0^\sigma \frac{(\sigma-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\Big(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\big(I_q^\beta h(u)-\lambda x(u)\big)d_q u\Big)d_q s \tag{14}$$
$$-\int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\big(I_q^\beta h(u)-\lambda x(u)\big)d_q u.$$

Solving (13) and (14) for c_0, c_1, we get

$$c_0 = \frac{\Gamma_q(\gamma+1)}{\Delta}\Big\{\delta_3 a\int_0^\eta \frac{(\eta-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\Big(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\big(I_q^\beta h(u)-\lambda x(u)\big)d_q u\Big)d_q s$$
$$-\delta_1 b\int_0^\sigma \frac{(\sigma-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\Big(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\big(I_q^\beta h(u)-\lambda x(u)\big)d_q u\Big)d_q s$$
$$+\delta_1 \int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\big(I_q^\beta h(u)-\lambda x(u)\big)d_q u\Big\},$$

$$c_1 = \frac{1}{\Delta}\Big\{-\delta_4 a\int_0^\eta \frac{(\eta-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\Big(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\big(I_q^\beta h(u)-\lambda x(u)\big)d_q u\Big)d_q s$$
$$+\delta_2 b\int_0^\sigma \frac{(\sigma-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\Big(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\big(I_q^\beta h(u)-\lambda x(u)\big)d_q u\Big)d_q s$$
$$-\delta_2 \int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\big(I_q^\beta h(u)-\lambda x(u)\big)d_q u\Big\}.$$

Substituting the values of c_0, c_1 in (12), we obtain (11).   □

Let $\mathcal{P} := C([0,1], \mathbb{R})$ denote the Banach space of all continuous functions from $[0,1] \to \mathbb{R}$ endowed with the norm defined by $\|x\| = \sup\{|x(t)| : t \in [0,1]\}$.

In view of Lemma 2.4, we define an operator $\mathcal{F} : \mathcal{P} \to \mathcal{P}$ as

$$(\mathcal{F}x)(t) = \int_0^t \frac{(t-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}f(m,x(m))d_q m - \lambda x(u)\Big)d_q u$$
$$-\frac{[\delta_3 t^\gamma - \delta_4]}{\Delta}\Big\{a\int_0^\eta \frac{(\eta-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\Big(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}$$
$$\times f(m,x(m))d_q m - \lambda x(u)\Big)d_q u\Big)d_q s\Big\}$$
$$+\frac{[\delta_1 t^\gamma - \delta_2]}{\Delta}\Big\{b\int_0^\sigma \frac{(\sigma-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\Big(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} \tag{15}$$
$$\times f(m,x(m))d_q m - \lambda x(u)\Big)d_q u\Big)d_q s\Big\}$$
$$-\frac{[\delta_1 t^\gamma - \delta_2]}{\Delta}\Big\{\int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}f(m,x(m))d_q m - \lambda x(u)\Big)d_q u\Big\}.$$

Observe that problem (1)-(2) has a solution if the operator equation $\mathcal{F}x = x$ has a fixed point, where $\mathcal{F}$ is given by (15).

3. Existence of Solutions for q-Difference Equations

In the sequel, we assume that

(A_1) $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function and that there exists a q-integrable function $\zeta : [0, 1] \to \mathbb{R}$ such that $|f(t, x) - f(t, y)| \leq \zeta(t)|x - y|$, $t \in [0, 1]$, $x, y \in \mathbb{R}$.

For computational convenience, we set

$$\Omega := \kappa_1 + |\lambda|\kappa_2, \tag{16}$$

where

$$\kappa_1 := (1 + \alpha_2)(I_q^{(\beta+\gamma)}\zeta)(1) + \alpha_1|a|(I_q^{(\beta+\gamma+\alpha-1)}\zeta)(\eta) + \alpha_2|b|(I_q^{(\beta+\gamma+\alpha-1)}\zeta)(\sigma), \tag{17}$$

$$\kappa_2 := \frac{1}{\Gamma_q(\gamma + 1)}(1 + \alpha_2) + \frac{1}{\Gamma_q(\gamma + \alpha)}\left(\alpha_1|a|\eta^{(\gamma+\alpha-1)} + \alpha_2|b|\sigma^{(\gamma+\alpha-1)}\right), \tag{18}$$

and

$$\alpha_1 = \frac{|\delta_3 - \delta_4|}{|\Delta|}, \quad \alpha_2 = \frac{|\delta_1 - \delta_2|}{|\Delta|}.$$

Theorem 3.1. *Suppose that the assumption (A_1) holds and that $\Omega < 1$, where Ω is given by (16). Then the boundary value problem (1)-(2) has a unique solution.*

Proof. Let us fix $\sup_{t\in[0,1]} |f(t, 0)| = H < \infty$ and choose

$$\overline{\rho} \geq \frac{H\kappa_3}{1 - \Omega},$$

where

$$\kappa_3 = \frac{(1 + \alpha_2)}{\Gamma_q(\beta + \gamma + 1)} + \frac{1}{\Gamma_q(\beta + \gamma + \alpha)}\left(\alpha_1|a|\eta^{(\beta+\gamma+\alpha-1)} + \alpha_2|b|\sigma^{(\beta+\gamma+\alpha-1)}\right). \tag{19}$$

We define $B_{\overline{\rho}} = \{x \in \mathcal{P} : \|x\| \leq \overline{\rho}\}$. We will show that $\mathcal{F}B_{\overline{\rho}} \subset B_{\overline{\rho}}$, where $\mathcal{F}$ is defined by (15). For $x \in B_{\overline{\rho}}$, $t \in [0, 1]$, it follows by the assumption (A_1) that

$$|f(t, x(t))| \leq |f(t, x(t)) - f(t, 0)| + |f(t, 0)| \leq \zeta(t)|x(t)| + |f(t, 0)| \leq \zeta(t)\overline{\rho} + H. \tag{20}$$

Then, for $x \in B_{\overline{\rho}}$, $t \in [0,1]$, and using (16) and (19), we have

$$|(\mathcal{F}x)(t)|$$

$$\leq \int_0^t \frac{(t-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}\big(\zeta(m)\overline{\rho}+H\big)d_q m + |\lambda|\overline{\rho}\Big)d_q u$$

$$+\alpha_1|a|\int_0^\eta \frac{(\eta-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\Big(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}(\zeta(m)\overline{\rho}+H)d_q m$$

$$+|\lambda|\overline{\rho}\big)d_q u\big)d_q s$$

$$+\alpha_2|b|\int_0^\sigma \frac{(\sigma-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\Big(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}(\zeta(m)\overline{\rho}+H)d_q m$$

$$+|\lambda|\overline{\rho}\big)d_q u\big)d_q s$$

$$+\alpha_2\int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}\big(\zeta(m)\overline{\rho}+H\big)d_q m + |\lambda|\overline{\rho}\Big)d_q u$$

$$\leq \overline{\rho}\Big[\int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}\zeta(m)d_q m\Big)d_q u$$

$$+\alpha_1|a|\int_0^\eta \frac{(\eta-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\Big(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}\zeta(m)d_q m\Big)d_q u\Big)d_q s$$

$$+\alpha_2|b|\int_0^\sigma \frac{(\sigma-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\Big(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}\zeta(m)d_q m\Big)d_q u\Big)d_q s$$

$$+\alpha_2\int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}\zeta(m)d_q m\Big)d_q u\Big]$$

$$+H\Big[\int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}d_q m\Big)d_q u$$

$$+\alpha_1|a|\int_0^\eta \frac{(\eta-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\Big(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}d_q m\Big)d_q u\Big)d_q s$$

$$+\alpha_2|b|\int_0^\sigma \frac{(\sigma-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\Big(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}d_q m\Big)d_q u\Big)d_q s$$

$$+\alpha_2\int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}d_q m\Big)d_q u\Big]$$

$$+\overline{\rho}|\lambda|\Big[\int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}d_q u + \alpha_1|a|\int_0^\eta \frac{(\eta-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\Big(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}d_q u\Big)d_q s$$

$$+\alpha_2|b|\int_0^\sigma \frac{(\sigma-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\Big(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}d_q u\Big)d_q s + \alpha_2\int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}d_q u\Big]$$

$$\leq \overline{\rho}\Big[(1+\alpha_2)(I_q^{(\beta+\gamma)}\zeta)(1) + \alpha_1|a|(I_q^{(\beta+\gamma+\alpha-1)}\zeta)(\eta) + \alpha_2|b|(I_q^{(\beta+\gamma+\alpha-1)}\zeta)(\sigma)\Big]$$

$$+H\Big[\frac{(1+\alpha_2)}{\Gamma_q(\beta+\gamma+1)} + \frac{1}{\Gamma_q(\beta+\gamma+\alpha)}\big(\alpha_1|a|\eta^{(\beta+\gamma+\alpha-1)} + \alpha_2|b|\sigma^{(\beta+\gamma+\alpha-1)}\big)\Big]$$

$$+\overline{\rho}|\lambda|\Big[\frac{(1+\alpha_2)}{\Gamma_q(\gamma+1)} + \frac{1}{\Gamma_q(\gamma+\alpha)}\big(\alpha_1|a|\eta^{(\gamma+\alpha-1)} + \alpha_2|b|\sigma^{(\gamma+\alpha-1)}\big)\Big],$$

which, on taking norm, yields $\|\mathcal{F}x\| \leq H\kappa_3 + \overline{\rho}\Omega \leq \overline{\rho}$. This shows that $\mathcal{F}B_{\overline{\rho}} \subset B_{\overline{\rho}}$.

Now, for $x, y \in \mathcal{P}$, we obtain

$$\|(\mathcal{F}x) - (\mathcal{F}y)\|$$

$$\leq \sup_{t \in [0,1]} \Big\{ \int_0^t \frac{(t - qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \Big(\int_0^u \frac{(u - qm)^{(\beta-1)}}{\Gamma_q(\beta)} \Big| f(m, x(m)) - f(m, y(m)) \Big| d_q m$$

$$+ |\lambda| |x(u) - y(u)| \Big) d_q u$$

$$+ \alpha_1 |a| \int_0^\eta \frac{(\eta - qs)^{(\alpha-2)}}{\Gamma_q(\alpha - 1)} \Big(\int_0^s \frac{(s - qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \Big(\int_0^u \frac{(u - qm)^{(\beta-1)}}{\Gamma_q(\beta)}$$

$$\times \Big| f(m, x(m)) - f(m, y(m)) \Big| d_q m + |\lambda| |x(u) - y(u)| \Big) d_q u \Big) d_q s$$

$$+ \alpha_2 |b| \int_0^\sigma \frac{(\sigma - qs)^{(\alpha-2)}}{\Gamma_q(\alpha - 1)} \Big(\int_0^s \frac{(s - qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \Big(\int_0^u \frac{(u - qm)^{(\beta-1)}}{\Gamma_q(\beta)}$$

$$\times \Big| f(m, x(m)) - f(m, y(m)) \Big| d_q m + |\lambda| |x(u) - y(u)| \Big) d_q u \Big) d_q s$$

$$+ \alpha_2 \int_0^1 \frac{(1 - qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \Big(\int_0^u \frac{(u - qm)^{(\beta-1)}}{\Gamma_q(\beta)} \Big| f(m, x(m)) - f(m, y(m)) \Big| d_q m$$

$$+ |\lambda| |x(u) - y(u)| \Big) d_q u \Big\}$$

$$\leq \|x - y\| \sup_{t \in [0,1]} \Big\{ \int_0^t \frac{(t - qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \Big(\int_0^u \frac{(u - qm)^{(\beta-1)}}{\Gamma_q(\beta)} \zeta(m) d_q m \Big) d_q u$$

$$+ \alpha_1 |a| \int_0^\eta \frac{(\eta - qs)^{(\alpha-2)}}{\Gamma_q(\alpha - 1)} \Big(\int_0^s \frac{(s - qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \Big(\int_0^u \frac{(u - qm)^{(\beta-1)}}{\Gamma_q(\beta)} \zeta(m) d_q m \Big) d_q u \Big) d_q s$$

$$+ \alpha_2 |b| \int_0^\sigma \frac{(\sigma - qs)^{(\alpha-2)}}{\Gamma_q(\alpha - 1)} \Big(\int_0^s \frac{(s - qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \Big(\int_0^u \frac{(u - qm)^{(\beta-1)}}{\Gamma_q(\beta)} \zeta(m) d_q m \Big) d_q u \Big) d_q s$$

$$+ \alpha_2 \int_0^1 \frac{(1 - qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \Big(\int_0^u \frac{(u - qm)^{(\beta-1)}}{\Gamma_q(\beta)} \zeta(m) d_q m \Big) d_q u \Big\}$$

$$+ \|x - y\| |\lambda| \sup_{t \in [0,1]} \Big\{ \int_0^1 \frac{(1 - qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} d_q u + \alpha_1 |a| \int_0^\eta \frac{(\eta - qs)^{(\alpha-2)}}{\Gamma_q(\alpha - 1)}$$

$$\times \Big(\int_0^s \frac{(s - qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} d_q u \Big) d_q s + \alpha_2 |b| \int_0^\sigma \frac{(\sigma - qs)^{(\alpha-2)}}{\Gamma_q(\alpha - 1)} \Big(\int_0^s \frac{(s - qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} d_q u \Big) d_q s$$

$$+ \alpha_2 \int_0^1 \frac{(1 - qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} d_q u \Big\}$$

$$\leq \Omega \|x - y\|,$$

where we have used (16).

Since $\Omega \in (0, 1)$ by the given assumption, therefore $\mathcal{F}$ is a contraction. Hence it follows by Banach's contraction principle that the problem (1)-(2) has a unique solution. □

In case $\zeta(t) = L$ (L is a constant), the condition $\Omega < 1$ becomes $L\kappa_3 + |\lambda|\kappa_2 < 1$ and Theorem 3.1 takes the form of the following result.

Corollary 3.2. *Assume that $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function and that*

there exists a constant $L \in \left(0, \frac{1-|\lambda|\kappa_2}{\kappa_3}\right)$ with $|f(t,x) - f(t,y)| \le L|x-y|, t \in [0,1], x, y \in \mathbb{R}$, where κ_2 and κ_3 are given by (18) and (19).

Then the boundary value problem (1)-(2) has a unique solution.

Our next existence results is based on Krasnoselskii's fixed point theorem [31].

Lemma 3.3. *(Krasnoselskii). Let Y be a closed, bounded, convex and nonempty subset of a Banach space X. Let Q_1, Q_2 be two operators such that:*

(i) *$Q_1 x + Q_2 y \in Y$ whenever $x, y \in Y$;*

(ii) *Q_1 is compact and continuous;*

(iii) *Q_2 is a contraction mapping.*

Then there exists $z \in Y$ such that $z = Q_1 z + Q_2 z$.

Theorem 3.4. *Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying (A_1). In addition we assume that*

(A_2) *there exists a function $\mu \in C([0,1], \mathbb{R}^+)$ and a nondecreasing function $\varphi \in C([0,1], \mathbb{R}^+)$ with*

$$|f(t,x)| \le \mu(t)\varphi(|x|), \quad (t,x) \in [0,1] \times \mathbb{R};$$

(A_3) *there exists a constant $\bar{r}$ with*

$$\bar{r} \ge \varphi(\bar{r}) \|\mu\| \kappa_4, \tag{21}$$

where $\kappa_4 = \frac{\kappa_3}{1-|\lambda|\kappa_2}$, $1 - |\lambda|\kappa_2 > 0$ and $\|\mu\| = \sup_{t\in[0,1]} |\mu(t)|$.

If

$$\alpha_1|a|(I_q^{(\beta+\gamma+\alpha-1)}\zeta)(\eta) + \alpha_2|b|(I_q^{(\beta+\gamma+\alpha-1)}\zeta)(\sigma) + \alpha_2(I_q^{(\beta+\gamma)}\zeta)(1)$$

$$+|\lambda|\left[\frac{1}{\Gamma_q(\gamma+\alpha)}\left(\alpha_1|a|\eta^{(\gamma+\alpha-1)} + \alpha_2|b|\sigma^{(\gamma+\alpha-1)}\right) + \frac{\alpha_2}{\Gamma_q(\gamma+1)}\right] < 1, \tag{22}$$

then the boundary value problem (1)-(2) has at least one solution on $[0,1]$.

Proof. Consider the set $B_{\bar{r}} = \{x \in \mathcal{P} : \|x\| \leq \bar{r}\}$, where $\bar{r}$ is given in (21) and define operators $\mathcal{S}_1$ and $\mathcal{S}_2$ on $B_{\bar{r}}$ as

$$(\mathcal{S}_1 x)(t) = \int_0^t \frac{(t-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \Big(\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} f(m, x(m)) d_q m - \lambda x(u) \Big) d_q u, \quad t \in [0,1],$$

$$\begin{aligned}
(\mathcal{S}_2 x)(t) = & -\frac{[\delta_3 t^\gamma - \delta_4]}{\Delta} \Big\{ a \int_0^\eta \frac{(\eta-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} \Big(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \Big(\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} \\
& \times f(m, x(m)) d_q m - \lambda x(u) \Big) d_q u \Big) d_q s \Big\} \\
& +\frac{[\delta_1 t^\gamma - \delta_2]}{\Delta} \Big\{ b \int_0^\sigma \frac{(\sigma-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} \Big(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \Big(\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} \\
& \times f(m, x(m)) d_q m - \lambda x(u) \Big) d_q u \Big) d_q s \Big\} \\
& -\frac{[\delta_1 t^\gamma - \delta_2]}{\Delta} \Big\{ \int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \Big(\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} f(m, x(m)) d_q m \\
& -\lambda x(u) \Big) d_q u \Big\}, \quad t \in [0,1].
\end{aligned}$$

For $x, y \in B_{\bar{r}}$, we find that

$$\begin{aligned}
& |(\mathcal{S}_1 x + \mathcal{S}_2 y)(t)| \\
\leq & \int_0^t \frac{(t-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \Big(\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} \mu(m)\varphi(|x(m)|) d_q m + |\lambda| |x(u)| \Big) d_q u \\
& +\alpha_1 |a| \int_0^\eta \frac{(\eta-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} \Big(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \Big(\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} \mu(m)\varphi(|x(m)|) d_q m \\
& +|\lambda| |x(u)| \Big) d_q u \Big) d_q s \\
& +\alpha_2 |b| \int_0^\sigma \frac{(\sigma-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} \Big(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \Big(\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} \mu(m)\varphi(|x(m)|) d_q m \\
& +|\lambda| |x(u)| \Big) d_q u \Big) d_q s \\
& +\alpha_2 \int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \Big(\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} \mu(m)\varphi(|x(m)|) d_q m + |\lambda| |x(u)| \Big) d_q u \\
\leq & \|\mu\|\varphi(\bar{r}) \Big[\frac{(1+\alpha_2)}{\Gamma_q(\beta+\gamma+1)} + \frac{1}{\Gamma_q(\beta+\gamma+\alpha)} \Big(\alpha_1 |a| \eta^{(\beta+\gamma+\alpha-1)} \alpha_2 |b| \sigma^{(\beta+\gamma+\alpha-1)} \Big) \Big] \\
& +\bar{r} |\lambda| \Big[\frac{(1+\alpha_2)}{\Gamma_q(\gamma+1)} + \frac{1}{\Gamma_q(\gamma+\alpha)} \Big(\alpha_1 |a| \eta^{(\gamma+\alpha-1)} + \alpha_2 |b| \sigma^{(\gamma+\alpha-1)} \Big) \Big] \\
\leq & \|\mu\|\varphi(\bar{r})\kappa_3 + \bar{r}|\lambda|\kappa_2 \\
\leq & \bar{r}.
\end{aligned}$$

Thus, $\mathcal{S}_1 x + \mathcal{S}_2 y \in B_{\bar{r}}$. From (A_1) and (22), it follows that $\mathcal{S}_2$ is a contraction mapping. Continuity of f implies that the operator $\mathcal{S}_1$ is continuous. Also, $\mathcal{S}_1$ is uniformly bounded on $B_{\bar{r}}$ as

$$\|\mathcal{S}_1 x\| \leq \frac{\varphi(\bar{r}) \|\mu\|}{\Gamma_q(\beta+\gamma+1)} + \frac{|\lambda|\bar{r}}{\Gamma_q(\gamma+1)}.$$

Now, for any $x \in B_{\bar{r}}$, and $t_1, t_2 \in [0, 1]$ with $t_1 < t_2$, we have

$$
\begin{aligned}
&|(\mathcal{S}_1 x)(t_2) - (\mathcal{S}_1 x)(t_1)| \\
&= \left| \int_0^{t_2} \frac{(t_2 - qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \left(\int_0^u \frac{(u - qm)^{(\beta-1)}}{\Gamma_q(\beta)} f(m, x(m)) d_q m - \lambda x(u) \right) d_q u \right. \\
&\quad \left. - \int_0^{t_1} \frac{(t_1 - qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \left(\int_0^u \frac{(u - qm)^{(\beta-1)}}{\Gamma_q(\beta)} f(m, x(m)) d_q m - \lambda x(u) \right) d_q u \right| \\
&\leq \varphi(\bar{r}) \|\mu\| \left| \int_0^{t_2} \frac{(t_2 - qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \frac{u^\beta}{\Gamma_q(\beta+1)} d_q u - \int_0^{t_1} \frac{(t_1 - qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \frac{u^\beta}{\Gamma_q(\beta+1)} d_q u \right| \\
&\quad + \bar{r} |\lambda| \left| \int_0^{t_2} \frac{(t_2 - qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} d_q u - \int_0^{t_1} \frac{(t_1 - qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} d_q u \right| \\
&\leq \frac{\varphi(\bar{r}) \|\mu\|}{\Gamma_q(\beta + \gamma + 1)} |t_2^{(\beta+\gamma)} - t_1^{(\beta+\gamma)}| + \frac{|\lambda| \bar{r}}{\Gamma_q(\gamma+1)} |t_2^{(\gamma)} - t_1^{(\gamma)}|,
\end{aligned}
$$

which is independent of x and tends to zero as $t_2 \to t_1$. Thus, $\mathcal{S}_1$ is equicontinuous. So $\mathcal{S}_1$ is relatively compact on $B_{\bar{r}}$. Hence, by the Arzelá-Ascoli Theorem, $\mathcal{S}_1$ is compact on $B_{\bar{r}}$. Thus all the assumptions of Lemma 3.3 are satisfied. So the conclusion of Lemma 3.3 implies that the boundary value problem (1)-(2) has at least one solution on $[0, 1]$. □

In the special case when $\varphi(x) \equiv 1$ we see that there always exist a positive $\bar{r}$ so that (21) holds true. Thus we have the following corollary.

Corollary 3.5. *Let $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying (A_1). In addition we assume that*

$$|f(t, x)| \leq \mu(t), \quad \forall (t, x) \in [0, 1] \times \mathbb{R}, \text{ and } \mu \in C([0, 1], \mathbb{R}^+).$$

If (22) holds, then the boundary value problem (1)-(2) has at least one solution on $[0, 1]$.

The next existence result is based on Leray-Schauder Nonlinear alternative.

Lemma 3.6. *(Nonlinear alternative for single valued maps)[32]. Let E be a Banach space, C a closed, convex subset of E, W an open subset of C with $0 \in W$. Suppose that $\mathcal{F} : \overline{W} \to C$ is a continuous, compact (that is, $\mathcal{F}(\overline{W})$ is a relatively compact subset of C) map. Then either*

  (i) *$\mathcal{F}$ has a fixed point in $\overline{W}$, or*

  (ii) *there is a $x \in \partial W$ (the boundary of W in C) and $\lambda \in (0, 1)$ with $x = \lambda \mathcal{F}(x)$.*

Theorem 3.7. *Let $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that:*

(A_4) *there exist functions $b_1, b_2 \in L^1([0, 1], \mathbb{R}^+)$, and a nondecreasing function $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $|f(t, x)| \leq b_1(t)\Psi(|x|) + b_2(t)$, for $(t, x) \in [0, 1] \times \mathbb{R}$;*

(A_5) *there exists a number $N > 0$ such that*

$$N > \frac{\Psi(N)\omega_1 + \omega_2}{1 - |\lambda|\kappa_2}, \quad 1 - |\lambda|\kappa_2 > 0, \tag{23}$$

where $\omega_i := (1 + \alpha_2)(I_q^{\beta+\gamma} b_i)(1) + \alpha_1|a|(I_q^{\beta+\gamma+\alpha-1} b_i)(\eta) + \alpha_2|b|(I_q^{\beta+\gamma+\alpha-1} b_i)(\sigma), \quad i = 1, 2.$

Then the boundary value problem (1)-(2) has at least one solution on $[0, 1]$.

Proof. Consider the operator $\mathcal{F} : \mathcal{P} \to \mathcal{P}$ defined by (15). It is easy to show that $\mathcal{F}$ is continuous. Next, we show that $\mathcal{F}$ maps bounded sets into bounded sets in $\mathcal{P}$. For a positive number $\overline{\rho}$, let $B_{\overline{\rho}} = \{x \in \mathcal{P} : \|x\| \le \overline{\rho}\}$ be a bounded set in $C([0, 1], \mathbb{R})$. Then, we have

$$
\begin{aligned}
|(\mathcal{F}x)(t)| \le{} & \int_0^t \frac{(t - qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \left(\int_0^u \frac{(u - qm)^{(\beta-1)}}{\Gamma_q(\beta)} |f(m, x(m)| d_q m + |\lambda| |x(u)|\right) d_q u \\
& + \alpha_1|a| \int_0^\eta \frac{(\eta - qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} \left(\int_0^s \frac{(s - qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \right. \\
& \times \left(\int_0^u \frac{(u - qm)^{(\beta-1)}}{\Gamma_q(\beta)} |f(m, x(m)| d_q m + |\lambda| |x(u)|\right) d_q u \Big) d_q s \\
& + \alpha_2|b| \int_0^\sigma \frac{(\sigma - qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} \left(\int_0^s \frac{(s - qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \right. \\
& \times \left(\int_0^u \frac{(u - qm)^{(\beta-1)}}{\Gamma_q(\beta)} |f(m, x(m)| d_q m + |\lambda| |x(u)|\right) d_q u \Big) d_q s \\
& + \alpha_2 \int_0^1 \frac{(1 - qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \left(\int_0^u \frac{(u - qm)^{(\beta-1)}}{\Gamma_q(\beta)} |f(m, x(m)| d_q m + |\lambda| |x(u)|\right) d_q u \\
\le{} & \int_0^t \frac{(t - qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \left(\int_0^u \frac{(u - qm)^{(\beta-1)}}{\Gamma_q(\beta)} [b_1(m)\Psi(\|x\|) + b_2(m)] d_q m + |\lambda| |x(u)|\right) d_q u \\
& + \alpha_1|a| \int_0^\eta \frac{(\eta - qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} \left(\int_0^s \frac{(s - qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \right. \\
& \times \left(\int_0^u \frac{(u - qm)^{(\beta-1)}}{\Gamma_q(\beta)} [b_1(m)\Psi(\|x\|) + b_2(m)] d_q m + |\lambda| |x(u)|\right) d_q u \Big) d_q s \\
& + \alpha_2|b| \int_0^\sigma \frac{(\sigma - qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} \left(\int_0^s \frac{(s - qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \right. \\
& \times \left(\int_0^u \frac{(u - qm)^{(\beta-1)}}{\Gamma_q(\beta)} [b_1(m)\Psi(\|x\|) + b_2(m)] d_q m + |\lambda| |x(u)|\right) d_q u \Big) d_q s \\
& + \alpha_2 \int_0^1 \frac{(1 - qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \left(\int_0^u \frac{(u - qm)^{(\beta-1)}}{\Gamma_q(\beta)} [b_1(m)\Psi(\|x\|) + b_2(m)] d_q m + |\lambda| |x(u)|\right) d_q u \\
\le{} & \Psi(\overline{\rho}) \Big\{ \int_0^1 \frac{(1 - qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \left(\int_0^u \frac{(u - qm)^{(\beta-1)}}{\Gamma_q(\beta)} b_1(m) d_q m \right) d_q u \\
& + \alpha_1|a| \int_0^\eta \frac{(\eta - qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} \left(\int_0^s \frac{(s - qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \left(\int_0^u \frac{(u - qm)^{(\beta-1)}}{\Gamma_q(\beta)} b_1(m) d_q m \right) d_q u \right) d_q s \\
& + \alpha_2|b| \int_0^\sigma \frac{(\sigma - qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} \left(\int_0^s \frac{(s - qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \left(\int_0^u \frac{(u - qm)^{(\beta-1)}}{\Gamma_q(\beta)} b_1(m) d_q m \right) d_q u \right) d_q s
\end{aligned}
$$

$$+\alpha_2 \int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \Big(\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} b_1(m) d_q m \Big) d_q u \Big\}$$

$$+\Big\{ \int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \Big(\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} b_2(m) d_q m \Big) d_q u$$

$$+\alpha_1 |a| \int_0^\eta \frac{(\eta-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} \Big(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \Big(\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} b_2(m) d_q m \Big) d_q u \Big) d_q s$$

$$+\alpha_2 |b| \int_0^\sigma \frac{(\sigma-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} \Big(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \Big(\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} b_2(m) d_q m \Big) d_q u \Big) d_q s$$

$$+\alpha_2 \int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \Big(\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} b_2(m) d_q m \Big) d_q u \Big\}$$

$$+\overline{\rho}|\lambda| \Big\{ \int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} d_q u + \alpha_1 |a| \int_0^\eta \frac{(\eta-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} \Big(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} d_q u \Big) d_q s$$

$$+\alpha_2 |b| \int_0^\sigma \frac{(\sigma-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} \Big(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} d_q u \Big) d_q s + \alpha_2 \int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} d_q u \Big\}$$

$$\leq \quad \Psi(\overline{\rho})\omega_1 + \omega_2 + \overline{\rho}|\lambda|\kappa_2.$$

This proves our assertion.

Now we show that $\mathcal{F}$ maps bounded sets into equicontinuous sets of $\mathcal{P}$. Let $t_1, t_2 \in [0,1]$ with $t_1 < t_2$ and $x \in B_{\overline{\rho}}$, where $B_{\overline{\rho}}$ is a bounded set of $\mathcal{P}$. Taking into account the inequality: $(t_2-qu)^{(\alpha-1)}-(t_1-qu)^{(\alpha-1)} \leq (t_2-t_1)$ for $0 < t_1 < t_2 < 1$ (see, [21] p. 4) we have

$$|(\mathcal{F}x)(t_2) - (\mathcal{F}x)(t_1)|$$

$$\leq \quad \Big| \int_0^{t_1} \frac{(t_2-t_1)}{\Gamma_q(\gamma)} \Big(\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} [b_1(m)\Psi(\overline{\rho}) + b_2(m)] d_q m + |\lambda|\overline{\rho} \Big) d_q u$$

$$+ \int_{t_1}^{t_2} \frac{(t_2-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \Big(\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} [b_1(m)\Psi(\overline{\rho}) + b_2(m)] d_q m + |\lambda|\overline{\rho} \Big) d_q u \Big|$$

$$+ \frac{1}{|\Delta|} \Big\{ |a||\delta_3(t_2^\gamma - t_1^\gamma)| \int_0^\eta \frac{(\eta-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} \Big(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}$$

$$\times \Big(\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} [b_1(m)\Psi(\overline{\rho}) + b_2(m)] d_q m + |\lambda|\overline{\rho} \Big) d_q u \Big) d_q s$$

$$+ |b||\delta_1(t_2^\gamma - t_1^\gamma)| \int_0^\sigma \frac{(\sigma-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} \Big(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}$$

$$\times \Big(\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} [b_1(m)\Psi(\overline{\rho}) + b_2(m)] d_q m + |\lambda|\overline{\rho} \Big) d_q u \Big) d_q s$$

$$+ |\delta_1(t_2^\gamma - t_1^\gamma)| \int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \Big(\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} [b_1(m)\Psi(\overline{\rho}) + b_2(m)] d_q m$$

$$+ |\lambda|\overline{\rho} \Big) d_q u \Big\}.$$

Obviously the right hand side of the above inequality tends to zero independently of $x \in B_{\overline{\rho}}$ as $t_2 - t_1 \to 0$. Therefore it follows by the Arzelá-Ascoli theorem that $\mathcal{F} : \mathcal{P} \to \mathcal{P}$ is completely continuous.

Thus the operator $\mathcal{F}$ satisfies all the conditions of Lemma 3.6 and hence, by its conclusion, either condition (i) or condition (ii) holds. We show that the conclusion (ii) is not possible.

Let $W = \{x \in \mathcal{P} : \|x\| < N\}$ with N given by (23). Then it can be shown that $\|\mathcal{F}x\| < N$. Indeed, in view of (A_4), we have

$$\|\mathcal{F}x\|$$
$$\leq \Psi(\|x\|)\Bigg\{ \int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}b_1(m)d_q m\Big)d_q u$$
$$+\alpha_1|a| \int_0^\eta \frac{(\eta-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\Big(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}b_1(m)d_q m\Big)d_q u\Big)d_q s$$

$$+\alpha_2|b| \int_0^\sigma \frac{(\sigma-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\Big(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}b_1(m)d_q m\Big)d_q u\Big)d_q s$$
$$+\alpha_2 \int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}b_1(m)d_q m\Big)d_q u\Bigg\}$$
$$+\Bigg\{ \int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}b_2(m)d_q m\Big)d_q u$$
$$+\alpha_1|a| \int_0^\eta \frac{(\eta-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\Big(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}b_2(m)d_q m\Big)d_q u\Big)d_q s$$
$$+\alpha_2|b| \int_0^\sigma \frac{(\sigma-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\Big(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}b_2(m)d_q m\Big)d_q u\Big)d_q s$$
$$+\alpha_2 \int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}b_2(m)d_q m\Big)d_q u\Bigg\}$$
$$+\|x\||\lambda|\Bigg\{ \int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}d_q u + \alpha_1|a| \int_0^\eta \frac{(\eta-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\Big(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}d_q u\Big)d_q s$$
$$+\alpha_2|b| \int_0^\sigma \frac{(\sigma-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\Big(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}d_q u\Big)d_q s + \alpha_2 \int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}d_q u\Bigg\}$$
$$\leq \quad \Psi(N)\omega_1 + \omega_2 + N|\lambda|\kappa_2 < N.$$

Suppose there exists a $x \in \partial W$ and a $\lambda \in (0,1)$ such that $x = \lambda\mathcal{F}x$. Then for such a choice of x and λ, we have

$$N = \|x\| = \lambda\|\mathcal{F}x\| < \Psi(\|x\|)\omega_1 + \omega_2 + N|\lambda|\kappa_2 = \Psi(N)\omega_1 + \omega_2 + N|\lambda|\kappa_2 < N,$$

which is a contradiction. Consequently, by the Leray-Schauder alternative (Lemma 3.6), we deduce that $\mathcal{F}$ has a fixed point $x \in \overline{W}$ which is a solution of the problem (1)-(2). This completes the proof. $\square$

Remark 3.8. *If b_1, b_2 in (A_4) are continuous, then $\omega_i = \kappa_3\|b_i\|$, $i = 1, 2$, where κ_3 is defined by (19).*

Example 3.9. *Consider the fractional q-difference nonlocal boundary value problem*

$$^cD_q^{1/2}(^cD_q^{1/2} + \frac{1}{6})x(t) = L\Big(\frac{1}{2}(|x| + \tan^{-1}x) + \sin t + 1\Big), \quad 0 \leq t \leq 1, \tag{24}$$

$$x(0) = I_q^2 x(1/3), \qquad x(1) = \frac{1}{2} I_q^2 x(2/3). \tag{25}$$

Here $\beta = \gamma = q = b = 1/2$, $a = 1$, $\alpha = 3$, $\eta = 1/3$, $\sigma = 2/3$, $\lambda = 1/6$, and L is a constant to be fixed later on. Moreover, $\delta_1 = -0.925926$, $\delta_2 = 0.030136$, $\delta_3 = -0.851852$, $\delta_4 = -0.914762$, $\Delta = -0.872674$, $\alpha_1 = 0.072089$, $\alpha_2 = 1.095555$, $\kappa_2 = 2.37938$, $\kappa_3 = 2.158402$, $|f(t,x) - f(t,y)| \le L|x - y|$ and $\Omega = L\kappa_3 + |\lambda|\kappa_2 < 1$. Choosing $L < 0.279576$, all the assumptions of Theorem 3.1 are satisfied. Therefore, by Theorem 3.1, problem (24)-(25) has a unique solution.

Example 3.10. *Consider the problem*

$$^cD_q^{1/2}(^cD_q^{1/2} + \frac{1}{6})x(t) = \frac{1}{4}\cos t^2 \sin(|x|/2) + \frac{e^{-x^2}(t^2+1)}{1+(t^2+1)} + \frac{1}{3}, \quad 0 \le t \le 1, \tag{26}$$

$$x(0) = I_q^2 x(1/3), \qquad x(1) = \frac{1}{2}I_q^2 x(2/3), \tag{27}$$

where $\beta = \gamma = q = b = 1/2$, $a = 1$, $\alpha = 3$, $\eta = 1/3$, $\sigma = 2/3$, $\lambda = 1/6$. The values of δ_1, δ_2, δ_3, δ_4, Δ, α_1, α_2, κ_2 and κ_3 are the same as found in Example 3.9 and

$$|f(t,x)| = \left|\frac{1}{4}\cos t^2 \sin(|x|/2) + \frac{e^{-x^2}(t^2+1)}{1+(t^2+1)} + \frac{1}{3}\right| \le \frac{1}{8}|x| + 1.$$

Clearly $b_1 = 1/8$, $b_2 = 1$, $\Psi(N) = N$. In consequence, $\omega_1 = 0.269800$, $\omega_2 = 2.158402$, and the condition (23) implies that $N > 6.469320$. Thus, all the assumptions of Theorem 3.7 are satisfied. Therefore, the conclusion of Theorem 3.7 applies to the problem (26)-(27).

4. Existence Results for q-Difference Integral Equations

This section is devoted to some existence results for q-difference integral equations (3) with nonlocal integral boundary conditions (2). As in case of q-difference equations, we define an integral operator $\mathcal{G}: \mathcal{P} \to \mathcal{P}$ related to the integral boundary value problem (3)-(2) as follows

$$
\begin{aligned}
(\mathcal{G}x)(t) = & \int_0^t \frac{(t-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \Big(p \int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} f(m, x(m)) d_q m \\
& + k \int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)} g(m, x(m)) d_q m - \lambda x(u) \Big) d_q u \\
& - \frac{[\delta_3 t^\gamma - \delta_4]}{\Delta} \Big\{ a \int_0^\eta \frac{(\eta-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} \Big(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \Big(p \int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} \\
& \times f(m, x(m)) d_q m + k \int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)} g(m, x(m)) d_q m - \lambda x(u) \Big) d_q u \Big) d_q s \Big\} \\
& + \frac{[\delta_1 t^\gamma - \delta_2]}{\Delta} \Big\{ b \int_0^\sigma \frac{(\sigma-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} \Big(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \Big(p \int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} \\
& \times f(m, x(m)) d_q m + k \int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)} g(m, x(m)) d_q m - \lambda x(u) \Big) d_q u \Big) d_q s \Big\} \\
& - \frac{[\delta_1 t^\gamma - \delta_2]}{\Delta} \Big\{ \int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \Big(p \int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} f(m, x(m)) d_q m \\
& + k \int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)} g(m, x(m)) d_q m - \lambda x(u) \Big) d_q u \Big\}.
\end{aligned}
\tag{28}
$$

Observe that the problem (3)-(2) has solutions only if the operator equation $x = \mathcal{G}x$ has fixed points.

In the sequel, we assume that

(B_1) $f, g : [0,1] \times \mathbb{R} \to \mathbb{R}$ are continuous functions such that $|f(t, x) - f(t, y)| \le L_1|x - y|$ and $|g(t, x) - g(t, y)| \le L_2|x - y|$, $\forall t \in [0,1]$, $L_1, L_2 > 0$, $x, y \in \mathbb{R}$.

For computational convenience, we set

$$\overline{\Omega} = L\Lambda + |\lambda|\kappa_2, \quad \Lambda = |p|\kappa_3 + |k|\kappa_4, \tag{29}$$

where κ_2, κ_3 are given by (18), (19) respectively,

$$\kappa_4 = \frac{(1 + \alpha_2)}{\Gamma_q(\beta + \xi + \gamma + 1)} + \frac{1}{\Gamma_q(\beta + \xi + \gamma + \alpha)}\Big(\alpha_1|a|\eta^{(\beta+\xi+\gamma+\alpha-1)} + \alpha_2|b|\sigma^{(\beta+\xi+\gamma+\alpha-1)}\Big),$$

and

$$\alpha_1 = \frac{|\delta_3 - \delta_4|}{|\Delta|}, \quad \alpha_2 = \frac{|\delta_1 - \delta_2|}{|\Delta|}.$$

Now we present some existence results for the problem (3)-(2). Since these results are analogue to the ones established in Section 3, so we omit the proofs.

Theorem 4.1. *Suppose that the assumption (B_1) holds and that $\overline{\Omega} < 1$, where $\overline{\Omega}$ is given by (29) and $L = \max\{L_1, L_2\}$. Then the boundary value problem (3)-(2) has a unique solution on $[0,1]$.*

Theorem 4.2. *Let $f, g : [0,1] \times \mathbb{R} \to \mathbb{R}$ be continuous functions satisfying the assumption (B_1) and*

(B_2) *there exist $\mu_1, \mu_2 \in C([0,1], \mathbb{R}^+)$ with $|f(t, x)| \le \mu_1(t)$, $|g(t, x)| \le \mu_2(t)$, $\forall (t, x) \in [0,1] \times \mathbb{R}$, where $\sup_{t \in [0,1]} |\mu_i(t)| = \|\mu_i\|$, $i = 1, 2$.*

Then the problem (3)-(2) has at least one solution on $[0,1]$ provided that

$$L\Big[|p|\Big(\frac{1}{\Gamma_q(\beta + \gamma + \alpha)}\big(\alpha_1|a|\eta^{(\beta+\gamma+\alpha-1)} + \alpha_2|b|\sigma^{(\beta+\gamma+\alpha-1)}\big) + \frac{\alpha_2}{\Gamma_q(\beta + \gamma + 1)}\Big)$$
$$+ |k|\Big(\frac{1}{\Gamma_q(\beta + \xi + \gamma + \alpha)}\big(\alpha_1|a|\eta^{(\beta+\xi+\gamma+\alpha-1)} + \alpha_2|b|\sigma^{(\beta+\xi+\gamma+\alpha-1)}\big) + \frac{\alpha_2}{\Gamma_q(\beta + \xi + \gamma + 1)}\Big)\Big]$$
$$+ |\lambda|\Big[\frac{1}{\Gamma_q(\gamma + \alpha)}\big(\alpha_1|a|\eta^{(\gamma+\alpha-1)} + \alpha_2|b|\sigma^{(\gamma+\alpha-1)}\big) + \frac{\alpha_2}{\Gamma_q(\gamma + 1)}\Big] < 1.$$

Theorem 4.3. *Let $f, g : [0,1] \times \mathbb{R} \to \mathbb{R}$ be continuous functions and the following assumptions hold:*

(B_3) *there exist functions $v_1, v_2 \in C([0,1], \mathbb{R}^+)$, and nondecreasing functions $\Psi_1, \Psi_2 : \mathbb{R}^+ \to \mathbb{R}^+$ such that $|f(t, x)| \le v_1(t)\Psi_1(\|x\|)$, $|g(t, x)| \le v_2(t)\Psi_2(\|x\|)$, $\forall (t, x) \in [0,1] \times \mathbb{R}$.*

(B_4) *There exists a constant $N > 0$ such that*

$$N > \frac{|p|\|v_1\|\Psi_1(N)\kappa_3 + |k|\|v_2\|\Psi_2(N)\kappa_4}{1 - |\lambda|\kappa_2}, \quad 1 - |\lambda|\kappa_2 > 0.$$

Then the boundary value problem (3)-(2) has at least one solution on $[0, 1]$.

Example 4.4. *Consider a boundary value problem of integro-differential equations of fractional order given by*

$$\begin{cases} {}^cD_q^{1/4}({}^cD_q^{1/4} + \frac{1}{8})x(t) = \frac{1}{2}f(t, x(t)) + I_q^{1/2}g(t, x(t)), \ \ 0 < t < 1, \ 0 < q < 1, \\ \\ \qquad\qquad x(0) = I_q^2 x(\frac{1}{3}), \ \ x(1) = \frac{1}{2}I_q^2 x(\frac{2}{3}), \end{cases} \tag{30}$$

where $\beta = \gamma = 1/4$, $a = k = 1$, $b = p = q = \xi = 1/2$, $\alpha = 3$, $\eta = 1/3$, $\sigma = 2/3$, $\lambda = 1/8$, *and*

$$f(t, x) = \frac{1}{(4 + t^2)^2}\Big(\sin t + \frac{|x|}{1 + |x|} + |x|\Big), \ g(t, x) = \frac{1}{(t + 3)^2}\frac{|x|}{1 + |x|}.$$

With the given data, it is found that $L_1 = 1/8$, $L_2 = 1/9$ *as* $|f(t, x) - f(t, y)| \le \frac{1}{8}|x - y|$, $|g(t, x) - g(t, y)| \le \frac{1}{9}|x - y|$, *Moreover,* $\delta_1 = -0.925926$, $\delta_2 = 0.0461128$, $\delta_3 = -0.851852$, $\delta_4 = -0.890325$, $\Delta = -0.863656$, $\alpha_1 = 0.0445466$, $\alpha_2 = 1.12549$, $\kappa_2 = 2.41227$, $\kappa_3 = 2.41376$, $\kappa_4 = 2.18964$ *and* $\overline{\Omega} = 0.726099 < 1$. *Clearly* $L = \max\{L_1, L_2\} = 1/8$. *Thus all the assumptions of Theorem 4.1 are satisfied. Hence, by the conclusion of Theorem 4.1, the problem (30) has a unique solution.*

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