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The Generalized of Selberg's Inequalities in C*-Module

Samir Kabbaj[a], Abdellatif Chahbi*[a], Ahmed Charifi[a], Nourdine Bounader[a]

[a] *Department of Mathematics - Faculty of Sciences - Ibn Tofail University, Kenitra, Morocco.*

Abstract. We obtain a Generalized of Selberg's type inequalities in Hilbert spaces and their extensions in operators algebras, in C*-modules and in algebras of adjointable $\mathbb{A}$-linear maps. Some applications for improving the Bessel type inequality result are given.

1. Introduction

Let $\mathbb{H}$ be a Hilbert space with inner product $\langle , \rangle$. The inequality of Selberg

$$\sum_{j=1}^{n} \frac{\left|\left\langle x, y_j \right\rangle\right|^2}{\sum_{k=1}^{n}\left|\left\langle y_j, y_k \right\rangle\right|} \leq \|x\|^2\,,\ x, y_1, \ldots, y_n \in \mathbb{H}, y_j \neq 0,\ 0 \leq j \leq n, \tag{1}$$

is originating from analytic theory of numbers [18]. It was discovered by A. Selberg around 1949, on account of the arguments of the distribution of primes [1,4,12,13,18,19]. Since that time it has interested many mathematicians who gave it many proofs, many extensions and refinements, see [1,5,10,11,13,17]. It is useful to recall that Schwartz's inequality

$$|\langle x, y\rangle| \leq \|x\|\|y\|,\ x, y \in \mathbb{H} \tag{2}$$

and Bessel's inequality

$$\sum_{j=1}^{n}\left|\left\langle x, y_j \right\rangle\right|^2 \leq \|x\|^2\,,\ x \in \mathbb{H}, y_1, \ldots, y_n \text{ are nonzero and orthogonal in } \mathbb{H}, \tag{3}$$

are special cases of Selberg's inequality. Let me cite also, on the occasion, following inequalities encountered to this subject in the literature. Thus, in chronological order of publication, we have.

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Email addresses: samkabbaj@yahoo.fr (Samir Kabbaj), ab-1980@live.fr (Abdellatif Chahbi*), charifi2000@yahoo.fr (Ahmed Charifi), n.bounader@live.fr (Nourdine Bounader)

In 1958 Heilbronn's inequality [13],

$$\sum_{i=1}^{n} \left|\langle x, y_i\rangle\right| \leq \|x\| (\sum_{i,j=1}^{n} \left|\left\langle y_i, y_j\right\rangle\right|)^{\frac{1}{2}}, \ x, y_1, ..., y_n \in \mathbb{H}.$$

.

In 1971, Bombieri's inequality [1],

$$\sum_{i=1}^{n} \left|\left\langle x, y_j\right\rangle\right|^2 \leq \|x\|^2 \, max_{1\leq i\leq n}\{\sum_{j=1}^{n} \left|\left\langle y_i, y_j\right\rangle\right|\}, \ x, y_1, ..., y_n \in \mathbb{H},$$

In 1992 J.E.Pečaric´´s inequality [17],

$$|\sum_{i=1}^{n} c_i \langle x, y_i\rangle|^2 \leq \|x\|^2 \sum_{i=1}^{n} |c_i|^2 max_{1\leq i\leq n}\{\sum_{j=1}^{n} \left|\left\langle y_i, y_j\right\rangle\right|\}, \ x, y_1, ..., y_n \in \mathbb{H}.$$

Moreover, in 1998, M. Fujii and R. Nakamoto [11] obtained in a Hilbert space, the following refinement for previous inequalities,

$$\left|\langle y, x\rangle\right|^2 + \sum_{j=1}^{n} \frac{\left|\left\langle x, y_j\right\rangle\right|^2}{\sum_{k=1}^{n} \left|\left\langle y_j, y_k\right\rangle\right|} \left\|y\right\|^2 \leq \|x\|^2 \left\|y\right\|^2, x, y, y_1, \ldots, y_n \in \mathbb{H}, \tag{4}$$

with the condition that $\langle y, y_i\rangle = 0$.

The goal of this paper is to show a generalized of selberg's inequality in Hilbert spaces and their extensions in algebras of operators, in Hilbert C^*-modules and in algebras of adjointable $\mathbb{A}$-linear maps.

2. Preliminaries in Hilbert C^*-Modules

In this section we briefly recall the definitions and examples of Hilbert C^*-modules. For information about Hilbert C^*-module, we refer to ([8,9,16]). Our reference for C^*-algebras is ([3]). Let $\mathbb{A}$ be a C^*-algebra (not necessarily unitary) and $\mathbb{X}$ be a complex linear space.

Definition 2.1. *A pre-Hilbert $\mathbb{A}$-module is a right $\mathbb{A}$-module $\mathbb{X}$ equipped with a sesquilinear map $\langle .,.\rangle : \mathbb{X}\times\mathbb{X} \to \mathbb{A}$ satisfying*

1. $\langle x, x\rangle \geq 0; \langle x, x\rangle = 0$ *if and only if* $x = 0$ *for all* $\mathbb{X}$ *in* $\mathbb{X}$,
2. $\langle x, \alpha y + \beta z\rangle = \alpha \langle x, y\rangle + \beta(\langle x, z\rangle)$ *for all* x, y, z *in* $\mathbb{X}, \alpha, \beta$ *in* $\mathbb{C}$,
3. $\langle x, y\rangle = \langle y, x\rangle^*$ *for all* x, y *in* $\mathbb{X}$,
4. $\langle x, ya\rangle = \langle x, y\rangle a$ *for all* x, y *in* $\mathbb{X}$, a *in* $\mathbb{A}$.

The map $\langle .,.\rangle$ is called an $\mathbb{A}$-valued inner product of $\mathbb{X}$, and for $x \in \mathbb{X}$, we define $\|x\| = \|\langle x, x\rangle\|^{\frac{1}{2}}$ which is a norm on $\mathbb{X}$, where the latter norm denotes that in the C^*-algebra $\mathbb{A}$. This norm makes $\mathbb{X}$ into a right normed module over $\mathbb{A}$. A pre-Hilbert module $\mathbb{X}$ is called a Hilbert $\mathbb{A}$-module if it is complete with respect to its

norm. Two typical examples of Hilbert C^*-modules are as follows:
(I) Every Hilbert space is a Hilbert C^*-module.
(II) Every C^*algebra $\mathbb{A}$ is a Hilbert $\mathbb{A}$ -module via $\langle a, b\rangle = a^*b$ $(a, b \in \mathbb{A})$.
Notice that the inner product structure of a C^*-algebra is essentially more complicated than complex numbers.
One may define an $\mathbb{A}$-valued norm $|.|$ by $|x| = \langle x, x\rangle^{\frac{1}{2}}$. Clearly, $\|x\| = \||x|\|$ for each $x \in \mathbb{X}$. It is known that $|.|$ does not satisfy the triangle inequality in general.

We recall also the definition and the following result, they will be used in the later.

Definition 2.2. *An adjointable module map $t : E \to F$ has a polar decomposition if there is a partial isometry $u : E \to F$ such that $T = u|T|, t = u|t|$ and $Ker(u) = Ker(t), Ran(u) = \overline{Ran(t)}, ker(u^* = ker(t^*))$ and $Ran(u^*) = Ran(|t|)$.*

In general bounded adjointable $\mathbb{A}$-module maps between Hilbert $\mathbb{A}$-modules do not have polar composition, but M. Joita [14] has given a necessary and sufficient condition for bounded adjointable module maps to admit polar decomposition.

Theorem 2.3. *A bounded adjointable operator t has polar decomposition if and only if $Ran(t)$ and $Ran(|t|)$ are orthogonal direct summands.*

The following lemma is useful to prove this Selberg's inequality.

Lemma 2.4. *(see [6]) Let $\mathbb{A}$ be a C^*-algebra $a, b, c \in \mathbb{A}$. Then*

$$a^*cb + b^*c^*a \le \|c\| \left(|a|^2 + |b|^2\right). \tag{5}$$

3. Generalized of the Selberg's Inequality for Hilbert Space

We start our work by presenting a the generalized of the Selberg's inequality for Hilbert space.

Theorem 3.1. *Let $\mathbb{H}$ be an Hilbert space and y_{ij} be a non zero vectors in $\mathbb{H}$, such that y_{ij} is orthogonal to y_{kj} for all $i \neq k$ in $\{1, \ldots, m\}$ and $j \in \{1, \ldots, n_i\}$. If $x \in \mathbb{H}$ then*

$$\sum_{i=1}^{m} \sum_{j=1}^{n_i} \frac{\left|\left\langle y_{ij}, x\right\rangle\right|^2}{\sum_{k=1}^{n_i} \left|\left\langle y_{ij}, y_{ik}\right\rangle\right|} \le \|x\|^2. \tag{6}$$

Proof. Let $\alpha_{ij}, 1 \le i \le m,\ 1 \le j \le n_i$ be scalars elements in $\mathbb{C}$. We can write

$$0 \le \left\| x - \sum_{i=1}^{m} \sum_{j=1}^{n_i} \alpha_{ij} y_{ij} \right\|^2$$

and

$$\begin{aligned}
&\left\| x - \sum_{i=1}^{m} \sum_{j=1}^{n_i} \alpha_{ij} y_{ij} \right\|^2 \\
= \;& \left\langle x - \sum_{i=1}^{m} \sum_{j=1}^{n_i} \alpha_{ij} y_{ij}, x - \sum_{i=1}^{m} \sum_{j=1}^{n_i} \alpha_{ij} y_{ij} \right\rangle \\
= \;& \langle x, x \rangle - \left\langle x, \sum_{i=}^{m} \sum_{j=1}^{n_i} \alpha_{ij} y_{ij} \right\rangle - \left\langle \sum_{i=1}^{m} \sum_{j=1}^{n_i} \alpha_{ij} y_{ij}, x \right\rangle + \left\langle \sum_{i=1}^{m} \sum_{j=1}^{n_i} \alpha_{ij} y_{ij}, \sum_{i=1}^{m} \sum_{j=1}^{n_i} \alpha_{ij} y_{ij} \right\rangle \\
= \;& \|x\|^2 - \sum_{i=1}^{m} \sum_{j=1}^{n_i} \alpha_{ij} \left\langle x, y_{ij} \right\rangle - \sum_{i=1}^{m} \sum_{j=1}^{n_i} \overline{\alpha_{ij}} \left\langle y_{ij}, x \right\rangle + \sum_{i=1}^{m} \sum_{j,l=1}^{n_i} \overline{\alpha_{ij}} \alpha_{il} \left\langle y_{ij}, y_{il} \right\rangle
\end{aligned}$$

We choose

$$\alpha_{ij} = \frac{\left\langle y_{ij}, x \right\rangle}{\sum_{k=1}^{n_i} | \left\langle y_{ij}, y_{ik} \right\rangle |},$$

then we get

$$\begin{aligned}
\left\| x - \sum_{i=1}^{m} \sum_{j=1}^{n_i} \alpha_{ij} y_{ij} \right\|^2 &= \|x\|^2 - 2 \sum_{i=1}^{m} \sum_{j=1}^{n_i} \frac{\left| \left\langle x, y_{ij} \right\rangle \right|^2}{\sum_{k=1}^{n_i} \left| \left\langle y_{ij}, y_{ik} \right\rangle \right|} + \sum_{i=1}^{m} \sum_{j=1}^{n_i} \frac{\left| \left\langle x, y_{ij} \right\rangle \right|^2}{\sum_{k=1}^{n_i} \left| \left\langle y_{ij}, y_{ik} \right\rangle \right|} \\
&= \|x\|^2 - \sum_{i=1}^{m} \sum_{j=1}^{n_i} \frac{\left| \left\langle x, y_{ij} \right\rangle \right|^2}{\sum_{k=1}^{n_i} \left| \left\langle y_{ij}, y_{ik} \right\rangle \right|}.
\end{aligned}$$

Then

$$\|x\|^2 - \sum_{i=1}^{m} \sum_{j=1}^{n_i} \frac{| \left\langle x, y_{ij} \right\rangle |^2}{\sum_{k=1}^{n_i} | \left\langle y_{ij}, y_{ik} \right\rangle |} \geq 0, \tag{7}$$

which ends the proof. □

The following Selbergs inequality in [10] can be obtained by taking $m = 1$ in Theorem 3.1.

Theorem 3.2 (See [10]). *Let* $\mathbb{H}$ *be a Hilbert space and* $y_1 \ldots y_n$ *non zero vectors in* $\mathbb{H}$. *If* $x \in \mathbb{H}$, *then*

$$\sum_{j=1}^{n} \frac{\left| \left\langle x, y_j \right\rangle \right|^2}{\sum_{k=1}^{n} \left| \left\langle y_j, y_k \right\rangle \right|} \leq |x|^2 . \tag{8}$$

The following refinement of Selbergs inequality in [11] can be obtained by taking $m = 2$ and $n_2 = 1$ in Theorem 3.1.

Theorem 3.3 (See [11]). *Let* $\mathbb{H}$ *be a Hilbert space,* y *and* $y_1 \ldots y_n$ *non zero vectors in* $\mathbb{H}$ *such that* $\left\langle y, y_j \right\rangle = 0$ *for* $j = 1 \ldots n$. *If* $x \in \mathbb{H}$ *then*

$$\left| \langle y, x \rangle \right|^2 + \sum_{j=1}^{n} \frac{\left| \left\langle x, y_j \right\rangle \right|^2}{\sum_{k=1}^{n} \left| \left\langle y_j, y_k \right\rangle \right|} \left\| y \right\|^2 \leq |x|^2 \left\| y \right\|^2 . \tag{9}$$

The following theorems give the generalized Selberg's inequality in an algebra of operators in Hilbert space.

Theorem 3.4. *Let $\mathbb{H}$ be a Hilbert space and $T = U|T|$ be the polar decomposition of an operator T on $\mathbb{H}$, $\{y_{ij};\ i = 1, 2, \cdots, m$ and $j = 1 \cdots n_i\} \not\subset ker(T^*)$ and $\alpha \in [0, 1]$.*
If $(|T|^{1-\alpha}U^*y_{ij}, |T|^{1-\alpha}U^*y_{kj}) = 0$ for all j and $k \neq i$, then

$$\sum_{i=1}^{m}\sum_{j=1}^{n_i} \frac{|\left\langle Tx, y_{ij}\right\rangle|^2}{\sum_{k=1}^{n_i} |\left\langle |T^*|^{2(1-\alpha)}y_{ik}, y_{ij}\right\rangle|} \leq \||T|^{\alpha}x\|^2 \tag{10}$$

holds for all $x \in \mathbb{H}$.

Proof. We replace, in Theorem 3.1 respectively x and y_{ij} by $|T|^{\alpha}x$ and $|T|^{1-\alpha}U^*y_{ij}$, for all $i \in \{1, 2, \cdots, m\}$ and $j \in \{1 \cdots n_i\}$. Then we have the result. □

Theorem 3.5. *Let $\mathbb{H}$ be a Hilbert space and $T = U|T|$ be the polar decomposition of an operator T on $\mathbb{H}$, $\{y_{ij};\ i = 1, 2, \cdots, m$ and $j = 1 \cdots n_i\} \not\subset ker(T^*)$ and $\alpha, \alpha_1, \ldots, \alpha_m$, such that $\alpha + \alpha_i \geq 1$, for all $i = 1, \ldots, m$. If $\left\langle |T|^{\alpha_i}U^*y_{ij}, |T|^{\alpha_k}U^*y_{kj}\right\rangle = 0$ for all j and $k \neq i$, then*

$$\sum_{i=1}^{m}\sum_{j=1}^{n_i} \frac{|\left\langle T|T|^{\alpha_i+\alpha-1}x, y_{ij}\right\rangle|^2}{\sum_{k=1}^{n_i} |(|T^*|^{2\alpha_i}y_{ik}, y_{ij})|} \leq \||T|^{\alpha}x\|^2 \tag{11}$$

holds for all $x \in \mathbb{H}$.

Proof. We replace x and y_{ij} by $|T|^{\alpha}x$ and $|T|^{\alpha_i}U^*y_{ij}$ in Theorem 3.1 respectively. Then we have the result. □

This tow previous theorems are the generalized of the following result obtained in [11]:

Theorem 3.6 (See [11]). *Let $\mathbb{H}$ be a Hilbert space and $T = U|T|$ be the polar decomposition of an operator T on $\mathbb{H}$, $\{y_j; j = 1, 2, \cdots, n\} \not\subset ker(T^*)$ and $\alpha \in [0, 1]$. If $\left\langle U|T|^{1-\alpha}y, y_j\right\rangle = 0$ for all j and $i \neq k$, then*

$$|(|T|^{\alpha}x, y)|^2 + \sum_{j=1}^{n} \frac{|\left\langle Tx, y_j\right\rangle|^2}{\sum_{k=1}^{n} |\left\langle |T^*|^{2(1-\alpha)}y_j, y_k\right\rangle|}\|y\|^2 \leq \||T|^{\alpha}x\|^2\|y\|^2$$

holds for all $x \in \mathbb{H}$.

Theorem 3.7 (See [11]). *Let $\mathbb{H}$ be a Hilbert space. Suppose that $\{y_j; j = 1, 2, \cdots, n\} \not\subset ker(T^*)$ and $\alpha, \beta \geq 0$ with $\alpha + \beta \geq 1 \geq \alpha$. If $(|T^*|^{\beta+1-\alpha}y, y_j) = 0$ for all j , then*

$$|\left\langle T|T|^{\alpha+\beta-1}x, y\right\rangle|^2 + \sum_{j=1}^{n} \frac{|\left\langle Tx, y_j\right\rangle|^2\||T^*|^{\beta}y\|^2}{\sum_{k=1}^{n} |\left\langle |T^*|^{2(1-\alpha)}y_j, y_k\right\rangle|} \leq \||T|^{\alpha}x\|^2\||T^*|^{\beta}y\|^2$$

holds for all $x \in \mathbb{H}$. In particular, if $\left\langle |T^*|^{2(1-\alpha)}y, y_j\right\rangle = 0$ for $\alpha \in [0, 1]$, then

$$|\langle Tx, y\rangle|^2 + \sum_{j=1}^{n} \frac{|\langle Tx, y_i\rangle|^2\||T^*|^{1-\alpha}y\|^2}{\sum_{k=1}^{n} |\left\langle |T^*|^{2(1-\alpha)}, y_j\right\rangle|} \leq \||T|^{\alpha}x\|^2\||T^*|^{1-\alpha}y\|^2$$

holds for all $x \in \mathbb{H}$.

4. Generalized of Selberg's Inequality in C^*-Module

The following theorem gives an extension of generalized Selberg's inequality in a Hilbert $\mathbb{A}$-module.

Theorem 4.1. *Let $\mathbb{X}$ be a Hilbert $\mathbb{A}$-module and and y_{ij} be a non zero vectors in $\mathbb{X}$, such that y_{ij} is orthogonal to y_{kj} for all $i \neq k$ in $\{1, \ldots, m\}$ and $j \in \{1, \ldots, n_i\}$. If $x \in \mathbb{X}$ then*

$$\sum_{i=1}^{m} \sum_{j=1}^{n_i} \frac{\left|\left\langle y_{ij}, x \right\rangle\right|^2}{\sum_{k=1}^{n_i} \left\|\left\langle y_{ij}, y_{ik} \right\rangle\right\|} \leq |x|^2 . \tag{12}$$

Proof. For any $\alpha_{ij} \in \mathbb{A}$, we have

$$\begin{aligned}
0 \ &\leq \ \left| x - \sum_{i=1}^{m} \sum_{j=1}^{n_i} y_{ij}\alpha_{ij} \right|^2 = \left\langle x - \sum_{i=1}^{m} \sum_{j=1}^{n_i} y_{ij}\alpha_{ij}, x - \sum_{i=1}^{m} \sum_{j=1}^{n_i} y_{ij}\alpha_{ij} \right\rangle \\
&= \ \langle x, x\rangle - \left\langle \sum_{i=1}^{m} \sum_{j=1}^{n_i} y_{ij}\alpha_{ij}, x \right\rangle - \left\langle x, \sum_{i=1}^{m} \sum_{j=1}^{n_i} y_{ij}\alpha_{ij} \right\rangle + \left\langle \sum_{i=1}^{m} \sum_{j=1}^{n_i} y_{ij}\alpha_{ij}, \sum_{i=1}^{m} \sum_{j=1}^{n_i} y_{ij}\alpha_{ij} \right\rangle \\
&= \ |x|^2 - \sum_{i=1}^{m} \sum_{j=1}^{n_i} \alpha_{ij}^* \left\langle y_{ij}, x \right\rangle - \sum_{i=1}^{m} \sum_{j=1}^{n_i} \left\langle x, y_{ij} \right\rangle \alpha_{ij} + \sum_{i=1}^{m} \sum_{j,l=1}^{n_i} \alpha_{ij}^* \left\langle y_{ij}, y_{il} \right\rangle \alpha_{il} \\
&= \ |x|^2 - \sum_{i=1}^{m} \sum_{j=1}^{n_i} \alpha_{ij}^* \left\langle y_{ij}, x \right\rangle - \sum_{i=1}^{m} \sum_{j=1}^{n_i} \left\langle x, y_{ij} \right\rangle \alpha_{ij} + \frac{1}{2} \sum_{i=1}^{m} \sum_{j,l=1}^{n_i} (\alpha_{ij}^* \left\langle y_{ij}, y_{il} \right\rangle \alpha_{il} \\
&+ \ \alpha_{il}^* \left\langle y_{il}, y_{ij} \right\rangle \alpha_{ij}).
\end{aligned}$$

By Lemma (2.4) we get

$$\alpha_{ij}^* \left\langle y_{ij}, y_{il} \right\rangle \alpha_{il} + \alpha_{il}^* \left\langle y_{il}, y_{ij} \right\rangle \alpha_{ij} \leq \quad \left|\alpha_{ij}\right|^2 \left\|\left\langle y_{ij}, y_{il} \right\rangle\right\| + |\alpha_{il}|^2 \left\|\left\langle y_{ij}, y_{il} \right\rangle\right\|$$

then

$$\begin{gathered}
\left| x - \textstyle\sum_{i=1}^{m} \sum_{j=1}^{n_i} y_{ij}\alpha_{ij} \right|^2 \leq |x|^2 - \textstyle\sum_{i=1}^{m} \sum_{j=1}^{n_i} \alpha_{ij}^* \left\langle y_{ij}, x \right\rangle \\
- \textstyle\sum_{i=1}^{m} \sum_{j=1}^{n_i} \left\langle x, y_{ij} \right\rangle \alpha_{ij} + \frac{1}{2} \sum_{i=1}^{m} \sum_{j,l=1}^{n_i} \left|\alpha_{ij}\right|^2 \left\|\left\langle y_{ij}, y_{il} \right\rangle\right\| + |\alpha_{il}|^2 \left\|\left\langle y_{il}, y_{ij} \right\rangle\right\| .
\end{gathered} \tag{13}$$

We choose

$$\alpha_{ij} = \frac{\left\langle y_{ij}, x \right\rangle}{\sum_{k=1}^{n_i} \left\|\left\langle y_{ij}, y_{ik} \right\rangle\right\|},$$

then we obtain

$$\begin{aligned}
\left|x-\sum_{i=1}^{m}\sum_{j=1}^{n_i} y_{ij}\alpha_{ij}\right|^2 &\leq |x|^2-\sum_{i=1}^{m}\sum_{j=1}^{n_i}\frac{\left|\left\langle y_{ij},x\right\rangle\right|^2}{\sum_{k=1}^{n_i}\left\|\left\langle y_{ij},y_{ik}\right\rangle\right\|}-\sum_{i=1}^{m}\sum_{j=1}^{n_i}\frac{\left|\left\langle y_{ij},x\right\rangle\right|^2}{\sum_{k=1}^{n_i}\left\|\left\langle y_{ij},y_{ik}\right\rangle\right\|}\\
&+\frac{1}{2}\sum_{i=1}^{m}\sum_{j,l=1}^{n_i}\frac{\left|\left\langle y_{ij},x\right\rangle\right|^2\left\|\left\langle y_{ij},y_{il}\right\rangle\right\|}{(\sum_{k=1}^{n_i}\left\|\left\langle y_{ij},y_{ik}\right\rangle\right\|)^2}+\frac{1}{2}\sum_{i=1}^{m}\sum_{j,l=1}^{n_i}\frac{\left|\left\langle y_{ij},x\right\rangle\right|^2\left\|\left\langle y_{ij},y_{il}\right\rangle\right\|}{(\sum_{k=1}^{n_i}\left\|\left\langle y_{ij},y_{ik}\right\rangle\right\|)^2}\\
&= |x|^2-2\sum_{i=1}^{m}\sum_{j=1}^{n_i}\frac{\left|\left\langle y_{ij},x\right\rangle\right|^2}{\sum_{k=1}^{n_i}\left\|\left\langle y_{ij},y_{ik}\right\rangle\right\|}+\sum_{i=1}^{m}\sum_{j=1}^{n_i}\frac{\left|\left\langle y_{ij},x\right\rangle\right|^2}{\sum_{k=1}^{n_i}\left\|\left\langle y_{ij},y_{ik}\right\rangle\right\|}\\
&= |x|^2-\sum_{i=1}^{m}\sum_{j=1}^{n_i}\frac{\left|\left\langle y_{ij},x\right\rangle\right|^2}{\sum_{k=1}^{n_i}\left\|\left\langle y_{ij},y_{ik}\right\rangle\right\|}.
\end{aligned}$$

Then

$$|x|^2-\sum_{i=1}^{m}\sum_{j=1}^{n_i}\frac{\left|\left\langle y_{ij},x\right\rangle\right|^2}{\sum_{k=1}^{n_i}\left\|\left\langle y_{ij},y_{ik}\right\rangle\right\|}\geq 0, \tag{14}$$

which completes the proof. □

The previous theorem is a generalized of the following Selberg's and refinement of selberg's inequalities :

Theorem 4.2 (See [2]). *Let $\mathbb{X}$ be a Hilbert $\mathbb{A}$-module, $y_1\ldots y_n$ non zero vectors in $\mathbb{X}$. If $x\in\mathbb{X}$ then*

$$\sum_{j=1}^{n}\frac{\left|\left\langle x,y_j\right\rangle\right|^2}{\sum_{k=1}^{n}\left|\left\langle y_j,y_k\right\rangle\right|}\leq |x|^2. \tag{15}$$

Theorem 4.3 (See [2]). *Let $\mathbb{X}$ be a Hilbert $\mathbb{A}$-module, y and $y_1\ldots y_n$ non zero vectors in $\mathbb{X}$ such that $\left\langle y,y_j\right\rangle=0$ for $j=1\ldots n$. If $x\in\mathbb{X}$ then*

$$\left|\langle y,x\rangle\right|^2+\sum_{j=1}^{n}\frac{\left|\left\langle x,y_j\right\rangle\right|^2}{\sum_{k=1}^{n}\left|\left\langle y_j,y_k\right\rangle\right|}\left\|y\right\|^2\leq |x|^2\left\|y\right\|^2. \tag{16}$$

By Theorem 4.1 we can obtained a generalized of Bombieri's inequality in C^*-module ,

Theorem 4.4. *Let $\mathbb{X}$ be a Hilbert $\mathbb{A}-$module and y_{ij} be a non zero vectors in $\mathbb{X}$, such that y_{ij} is orthogonal to y_{kj} for all $i\neq k$ in $\{1,\ldots,m\}$ and $j\in\{1,\ldots,n_i\}$. If $x\in\mathbb{X}$ then*

$$\sum_{i=1}^{m}\sum_{j=1}^{n_i}\left|\left\langle y_{ij},x\right\rangle\right|^2\leq |x|^2\max_{1\leq i\leq m,1\leq j\leq n_i}\sum_{k=1}^{n_i}\left\|\left\langle y_{ij},y_{ik}\right\rangle\right\| \tag{17}$$

The following theorems extend previous results to algebra of operators in Hilbert C^*-module.

Theorem 4.5. *Let $\mathbb{X}$ be a Hilbert $\mathbb{A}$-module and $t = u|t|$ be the polar decomposition of adjointable module map t on $\mathbb{X}$, $\{y_{ij};\ i = 1, 2, \cdots, m$ and $j = 1 \cdots n_i\} \not\subset ker(t^*)$ and $\alpha \in [0, 1]$.*
If $\left\langle |t|^{1-\alpha} u^* y_{ij}, |t|^{1-\alpha} u^* y_{kj} \right\rangle = 0$ for all j and $k \neq i$, then

$$\sum_{i=1}^{m} \sum_{j=1}^{n_i} \frac{|\left\langle y_{ij}, tx \right\rangle|^2}{\sum_{k=1}^{n_i} |\left\langle |t^*|^{2(1-\alpha)} y_{ik}, y_{ij} \right\rangle|} \leq \||t|^{\alpha} x\|^2 \tag{18}$$

holds for all $x \in \mathbb{X}$.

Proof. We replace x and y_{ij} by $|t|^{\alpha} x$ and $|t|^{1-\alpha} u^* y_{ij}$ in Theorem 4.1 respectively. Then we have the result. □

Theorem 4.6. *Let $\mathbb{X}$ be a Hilbert $\mathbb{A}$-module and $t = u|t|$ be the polar decomposition adjointable module map t on $\mathbb{X}$, $\{y_{ij};\ i = 1, 2, \cdots, m$ and $j = 1 \cdots n_i\} \not\subset ker(t^*)$ and $\alpha, \alpha_1, \ldots, \alpha_m$, such that $\alpha + \alpha_i \geq 1$, for all $i = 1, \ldots, m$. If $\left\langle |t|^{\alpha_i} u^* y_{ij}, |t|^{\alpha_k} u^* y_{kj} \right\rangle = 0$ for all j and $k \neq i$, then*

$$\sum_{i=1}^{m} \sum_{j=1}^{n_i} \frac{|\left\langle y_{ij}, t|t|^{\alpha_i+\alpha-1} x \right\rangle|^2}{\sum_{k=1}^{n_i} \|\left\langle |t^*|^{2\alpha_i} y_{ik}, y_{ij} \right\rangle\|} \leq \||t|^{\alpha} x\|^2 \tag{19}$$

holds for all $x \in \mathbb{X}$.

Proof. We replace x and y_{ij} by $|t|^{\alpha} x$ and $|t|^{\alpha_i} u^* y_{ij}$ in Theorem 4.1 respectively. Then we have the result . □

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