Filomat 28:8 (2014), 1619–1627
DOI 10.2298/FIL1408619Z

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

A Global Solution Method for Semivectorial Bilevel Programming Problem

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Abstract. In this paper, we consider a class of semivectorial bilevel programming problem. An exact penalty function is proposed for such a problem. Based on this penalty function, an algorithm, which can obtain a global solution of the original problem, is presented. Finally, some numerical results illustrate its feasibility.

1. Introduction

As is well-known, bilevel programming (BP, for short) has many application fields, such as transportation, economics, ecology, and engineering, see [12]. So, it has been developed and researched by many authors. The recent monographs and surveys can refer to [4, 11, 12], for example. Most researches on algorithms of BP are limited to a specific situation that the lower level is a single objective optimization problem.

Whether from a mathematical point of view or many practical problems, however, the lower level may be a multi-objective optimization problem (MOP, for short). This situation can be interpreted as there is a follower that has several objectives. We illustrate this with an example. Let us consider a hierarchical production-distribution planning problem in a supply chain. If the core competitiveness of an enterprise is the production capacity, then the manufacturing company is the leader who aims to minimize the overall costs. The distribution company is the follower, and has maybe several competing objectives as to minimize transportation cost as well as satisfy the preferences of retailers. Clearly, this example can be modeled by

2010 *Mathematics Subject Classification*. Primary 90C26; Secondary 90C30
Keywords. semivectorial bilevel programming, multi-objective programming, linear programming, global optimization.
Received: 21 July 2013; Accepted: 09 September 2013
Communicated by Predrag Stanimirovic
Research supported by the National Science Foundation of China (71171150, 71471140, 11226226) and the Project of Huanggang Normal University to Yue Zheng (2014018503).
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a BP with a scalar optimization problem on the upper level and a MOP on the lower level, which is called semivectorial bilevel programming problem (SBP, for short) by Bonnel and Morgan [7]. Actually, such examples are very popular in practice. Thus, it is useful and significant to study the properties and algorithms of SBP.

In this paper, we will consider the following SBP in which the lower level is a linear MOP:

$$\begin{aligned} &\min_{x,y}\ f(x,y)\\ &\ \text{s.t.}\ \ x \in X,\\ &\text{where } y \text{ is an efficient solution of the following problem,}\\ &\min_{y\geq 0}\ Cy\\ &\ \text{s.t.}\ Ax + By \leq b, \end{aligned} \tag{1}$$

where $x \in R^n$, $y \in R^m$, $C \in R^{p\times m}$, $A \in R^{q\times n}$, $B \in R^{q\times m}$, $b \in R^q$, $X = \{x|x \geq 0\}$, and f is continuous.

Denote by $\Psi(x)$ the set of the efficient solutions of the lower level problem in (1). Then, problem (1) can be equivalently reformulated as follows:

$$\begin{aligned} &\min_{x,y}\ f(x,y)\\ &\ \text{s.t.}\ \ x \in X,\\ &\qquad y \in \Psi(x). \end{aligned} \tag{2}$$

Note that, there are several related papers for problem (2). For example, Bonnel [6], Dempe et al. [13] discussed the optimality condition of SBP. Moreover, Bonnel and Morgan [7] considered weakly efficient solutions, and discussed SBP in a general case that the objective functions of both levels were defined on Hausdorff topological space, but no numerical results were reported. Ankhili and Mansouri [2] also considered weakly efficient solutions, and presented an exact penalty method for (2). In order to establish the existence theorem of solutions and the well-definition of the algorithm, they assumed that the objective function of the upper level problem was concave. Calvete and Galé [9] also considered problem (2) in the case that the upper level objective function was quasiconcave. Furthermore, it was interesting that some geometrical properties of the inducible region of (2) were given. Zheng and Wan [20] put forth the idea of using a penalty method which includes two different penalty parameters for solving (2). This results in an ordinary nonlinear programming problem with an unknown objective penalty parameter and the other penalty parameter. Recently, Bonnel and Morgan [8] discussed a semivectorial bilevel optimal control problem. And it was interesting to giving sufficient conditions on the data for existence of solutions to both the optimistic and pessimistic optimal control problems.

In this paper, we develop a global optimization algorithm based on penalty function, which is motivated from [1, 10, 19], for problem (2). Note that, our method is different from those works above. Firstly, the existence theorem of solutions can be ensured without requiring that the objective function of the upper level problem is concave or quasiconcave. Secondly, a global, rather than local, solution can be obtained.

Finally, the proposed algorithm only solves a sequences of linear/nonlinear programming problems for solving (2), and numerical results illustrate its feasibility.

The paper is organized as follows. We present a penalty function in Section 2, and establish main results in Section 3. In Section 4, we propose a global optimization algorithm, and give an example to illustrate its feasibility. Finally in Section 5, we conclude the paper.

2. Penalty Function

In order to obtain theoretical results, we first introduce the following two assumptions:
(A1) For any $x \in X$, $Y(x) = \{y \in R^m | By \leq b - Ax, y \geq 0\} \neq \emptyset$, and there exists a compact subset Z of R^m such that $Y(x) \subset Z$ for all $x \in X$.
(A2) The set X is polytope.

Next, we give the following definitions of SBP.

Definition 2.1. *For each $x \in X$, $y^* \in Y(x)$ is called an efficient solution of the lower level problem in* (1) *if there does not exist another $y \in Y(x)$ such that $Cy \leq Cy^*$ and $Cy \neq Cy^*$.*

Definition 2.2. *A point (x, y) is called a feasible point of problem (2) if $(x, y) \in IR$ where $IR = \{(x, y)|x \in X,\ y \in \Psi(x)\}$ is also referred to as the inducible region of problem (2).*

Definition 2.3. *A feasible point (x^*, y^*) is called a solution of problem (2) if $f(x^*, y^*) \leq f(x, y)$, $\forall\ (x, y) \in IR$.*

Under assumptions (A1) and (A2), we have the following results.

Lemma 2.4. *For each $x \in X$, the set $\Psi(x)$ is compact.*

Proof. See Theorem 27, Chapter 4 in White [18]. □

Lemma 2.5. *For each $x \in X$, IR is consists of the union of faces of Z_1 where $Z_1 = \{(x, y)|x \in X,\ y \in Y(x)\}$.*

Proof. The result follows from Theorem 5 of Calvete and Galé [9]. □

As is well-known, Benson [5] gave a characterization of the efficient solution set of MOP, and defined a function which is known to indicate whether a point is efficient for MOP or not. Similar to this function, we define a parametric one as follows. For each $x \in X$, define

$$g(x, y) = e^T Cy - h(x, y),$$

where $e = (1, 1, \cdots, 1)^T \in R^p$ and $h(x, y)$ represents the optimal value of the following linear programming problem $L(x, y)$:

$$\min_{w \in G(x, y)} e^T Cw, \tag{3}$$

where $G(x, y) = \{w | Cw \leq Cy, w \in Y(x)\}$. Then, we can conclude that

Lemma 2.6. *For each $x \in X$ and $y \in Y(x)$, $g(x,y) \geq 0$, and $\Psi(x) = \{y | g(x,y) = 0,\ y \in Y(x)\}$.*

Proof. For any $y \in Y(x)$, we have $y \in G(x,y)$, and then $e^T Cy \geq h(x,y)$. Hence, $g(x,y) \geq 0$.

Let $x \in X$ and $y \in \Psi(x)$. Now, suppose that $g(x,y) > 0$. Then, there exists a solution w^* of problem (3) such that $e^T Cw^* < e^T Cy$, which contradicts the definition of y. Therefore, $g(x,y) = 0$.

Conversely, for each $x \in X$ and $y \in Y(x)$, let $g(x,y) = 0$. Suppose that $y \notin \Psi(x)$. Then, there exists another $\hat{y} \in Y(x)$ such that $C\hat{y} \leq Cy$ and $C\hat{y} \neq Cy$. Thus, it follows that $e^T Cy > e^T C\hat{y}$. Since $\hat{y} \in G(x,y)$, we have $e^T C\hat{y} \geq h(x,y)$, and then $g(x,y) = e^T Cy - h(x,y) \geq e^T Cy - e^T C\hat{y} > 0$. This contradicts $g(x,y) = 0$. □

Lemma 2.7. *$g(x,y)$ is continuous over the set Z_1.*

Proof. The result follows immediately from Theorem 4.3.3 of Bank et al. [3]. □

Lemma 2.8. *$g(x,y)$ is a concave function over the set Z_1.*

Proof. Let $(x_1,y_1),(x_2,y_2) \in Z_1$, $r \in [0,1]$, $x_{12} = rx_1+(1-r)x_2$ and $y_{12} = ry_1+(1-r)y_2$. Moreover, suppose that w_i is a solution of problem $L(x_i,y_i)$ $(i = 1,2)$, respectively. Then, it is easy to check that $rw_1+(1-r)w_2 \in G(x_{12},y_{12})$, and we have

$$\begin{aligned} g(x_{12},y_{12}) &= e^T Cy_{12} - h(x_{12},y_{12}) \\ &\geq e^T Cy_{12} - e^T C[rw_1 + (1-r)w_2] \\ &= rg(x_1,y_1) + (1-r)g(x_2,y_2) \end{aligned}$$

which implies that $g(x,y)$ is a concave function over Z_1. □

Besides, we have the following theorem which shows that problem (2) admits at least one solution.

Theorem 2.9. *Suppose that (A1) and (A2) are satisfied, then problem (2) has at least one solution.*

Proof. From (A2) and Lemma 2.1, IR is compact. Then, the result follows immediately from Weierstrass's theorem. □

For each $x \in X$, the dual of problem (3) is given by

$$\begin{aligned} &\max_{v,\mu}\ -v^T Cy - (b-Ax)^T\mu \\ &\text{s.t.}\ -C^T v - B^T\mu \leq C^T e, \\ &\qquad v \geq 0,\ \mu \geq 0. \end{aligned} \tag{4}$$

Let $Z_2 = \{(v,\mu) | -C^T v - B^T\mu \leq C^T e,\ v \geq 0,\ \mu \geq 0\}$ and $\pi(x,y,v,\mu) = e^T Cy + v^T Cy + (b-Ax)^T\mu$. Then, we have the following results with respect to the values of $g(x,y)$ and $\pi(x,y,v,\mu)$.

Lemma 2.10. *For any $(x,y) \in Z_1$ and $(v,\mu) \in Z_2$, we have $g(x,y) \leq \pi(x,y,v,\mu)$.*

Proof. It follows immediately from $h(x,y) \geq -v^T Cy - (b-Ax)^T\mu$. □

Lemma 2.11. *For any* $(x, y) \in Z_1$ *and* $(v, \mu) \in Z_2$, *if* $\pi(x, y, v, \mu) = 0$, *then* $g(x, y) = 0$. *Conversely, if* $g(x, y) = 0$, *then there exists* $(v^*, \mu^*) \in Z_2$ *such that* $\pi(x, y, v^*, \mu^*) = 0$.

Proof. The first result follows immediately from Lemmas 2.3 and 2.7. Now, we prove the second result. For each $(x, y) \in Z_1$, there exists $(v^*, \mu^*) \in Z_2$ such that $h(x, y) = -v^{*T}Cy - (b - Ax)^T\mu^*$. Therefore, we have

$$\begin{aligned} g(x, y) = 0 &\Leftrightarrow e^TCy - h(x, y) = 0 \\ &\Rightarrow e^TCy + v^{*T}Cy + (b - Ax)^T\mu^* = 0 \\ &\Leftrightarrow \pi(x, y, v^*, \mu^*) = 0. \end{aligned}$$

This completes the proof. □

Now, we consider the following problem:

$$\begin{aligned} \min_{x,y,v,\mu} \ & f(x, y) \\ \text{s.t.} \ & \pi(x, y, v, \mu) = 0, \\ & -C^Tv - B^T\mu \le C^Te, \\ & y \in Y(x), \ x \in X, \\ & v \ge 0, \ \mu \ge 0. \end{aligned} \tag{5}$$

Using the dual theory, we can easily obtain the following lemma which relates the solutions of problems (2) and (5).

Lemma 2.12. *If* $(\tilde{x}, \tilde{y})$ *solves problem (2), then there exists* $(\tilde{v}, \tilde{\mu}) \in Z_2$ *such that* $(\tilde{x}, \tilde{y}, \tilde{v}, \tilde{\mu})$ *solves problem (5); Conversely, if* $(\tilde{x}, \tilde{y}, \tilde{v}, \tilde{\mu})$ *solves problem (5), then* $(\tilde{x}, \tilde{y})$ *solves problem (2). In addition, these two problems have the same optimal value.*

Proof. Since problems (2) and (5) have the same objective function, we only need to prove that the feasible regions of both problems are equal.

Let $(\hat{x}, \hat{y})$ be a feasible point of problem (2). Then, $g(\hat{x}, \hat{y}) = 0$, and it follows from Lemma 2.11 that, there exists $(\hat{v}, \hat{\mu}) \in Z_2$ such that $\pi(\hat{x}, \hat{y}, \hat{v}, \hat{\mu}) = 0$. Hence, $(\hat{x}, \hat{y}, \hat{v}, \hat{\mu})$ is a feasible point of problem (5). Similar arguments demonstrate the converse. This completes the proof. □

For $k > 0$, we consider the following penalized problem of (5):

$$\begin{aligned} \min_{x,y,v,\mu} \ & f(x, y) + k\pi(x, y, v, \mu) \\ \text{s.t.} \ & -C^Tv - B^T\mu \le C^Te, \\ & y \in Y(x), \ x \in X, \\ & v \ge 0, \ \mu \ge 0. \end{aligned} \tag{6}$$

With respect to the relation between problems (5) and (6), we can easily obtain the following result.

Lemma 2.13. *Under assumptions (A1) and (A2), suppose that* (x_k, y_k, v_k, μ_k) *is a solution of problem (6) for fixed* $k > 0$*. Then,* (x_k, y_k, v_k, μ_k) *is also a solution to problem (5) if it is a feasible point to problem (5).*

3. Main Results

In the sequel, denote the set of vertices of Z_2 by $E(Z_2)$.

For fixed $k > 0$, define a function on Z_2 as follows:

$$\theta(v, \mu) = \min_{(x,y) \in Z_1} [f(x, y) + k\pi(x, y, v, \mu)]$$

Then, we have the following result.

Theorem 3.1. *Under assumptions (A1) and (A2), for fixed* $k > 0$*, there exists a point* $(v^*, \mu^*) \in E(Z_2)$ *which solves the following problem*

$$\min_{(v,\mu) \in Z_2} \theta(v, \mu). \tag{7}$$

Proof. It is easy to check that $\theta(v, \mu)$ is a concave function. As problem (6) is also defined by (7), we have

$$\begin{aligned} \inf_{(v,\mu) \in Z_2} \theta(v, \mu) &= \inf_{\substack{(v,\mu) \in Z_2, \\ (x,y) \in Z_1}} [f(x, y) + k\pi(x, y, v, \mu)] \\ &\geq \min_{(x,y) \in Z_1} f(x, y), \end{aligned}$$

where the last inequality follows from $\pi(x, y, v, \mu) \geq 0$.

It follows from Weierstrass's theorem that $\min\limits_{(x,y) \in Z_1} f(x, y)$ has at least one solution, and then the function $\theta(v, \mu)$ is bounded from below on Z_2 . Hence, the result follows from Corollary 32.3.4 of Rockafellar [17]. This completes the proof. □

Then, we can deduce the existence theorem of solution of problem (6).

Theorem 3.2. *Let assumptions (A1) and (A2) be satisfied. For fixed* $k > 0$*, there exists* $(x^*, y^*, v^*, \mu^*) \in Z_1 \times E(Z_2)$ *which solves problem (6).*

Proof. Let $(v^*, \mu^*) \in E(Z_2)$ be a solution to problem (7). Then, we consider the following problem

$$\min_{(x,y) \in Z_1} [f(x, y) + k\pi(x, y, v^*, \mu^*)]. \tag{8}$$

It follows from Weierstrass's theorem that problem (8) has at least one solution $(x^*, y^*) \in Z_1$. Therefore, $(x^*, y^*, v^*, \mu^*) \in Z_1 \times E(Z_2)$ is a solution of problem (6). □

Now, we will establish the following theorem which shows that our penalty method is exact.

Theorem 3.3. *Under assumptions (A1) and (A2), if* $\{(x_k, y_k, v_k, \mu_k)\}$ *is a sequence of solutions of problem (6), then there exists* $k^* > 0$ *such that for all* $k > k^*$, (x_k, y_k) *is a solution of problem (2).*

Proof. We can prove this result by using a similar reasoning as in Theorem 3.3 in Liu et al. [15]. □

4. Numerical Results

Based on Theorems 3.2 and 3.3 in section 3, we can obtain that the solution to problem (5) occurs at a point $(x^*, y^*, v^*, \mu^*) \in Z_1 \times E(Z_2)$. Then, we present a simple algorithm for solving problem (5) as follows.

Algorithm£

Step 0. Choose $k > 0$, $\tau > 1$ and set $i = 1$.

Step 1. Generate all vertices $(v^1, \mu^1), (v^2, \mu^2), \cdots, (v^t, \mu^t)$ of Z_2.

Step 2. If $i \le t$, then go to Step 3. Otherwise, go to Step 5.

Step 3. Solve the following problem $P(v^i, \mu^i)$:

$$\min_{(x,y) \in Z_1} [f(x, y) + k\pi(x, y, v^i, \mu^i)],$$

and get a solution (x^i, y^i).

Step 4. If $\pi(x^i, y^i, v^i, \mu^i) = 0$, then $i = i + 1$, and go to Step 2. Otherwise, $k = k\tau$, and go to Step 3.

Step 5. Set $f(x^*, y^*) = \min\{f(x^i, y^i)|1 \le i \le t\}$, and then (x^*, y^*, v^*, μ^*) is a solution of problem (5).

Remark Many authors propose algorithms that obtain all vertices of a polyhedron. The reader can refer to Matheiss and Rubin [16], Fukuda et al. [14]. Note that, the above algorithm can obtain a global solution of the original bilevel programming problem by only solving a sequence of linear/nonlinear programming problems after obtaining all vertices of Z_2.

To illustrate the feasibility of the proposed algorithm, we consider the following examples.

Example 1 [2]

$$\begin{aligned}
&\min_{x,y}\ x - 4y,\\
&\text{ s.t. } 0 \le x \le 3,\\
&\text{where } y \text{ solves},\\
&\min_{y \ge 0}\ (y, 2y)^T\\
&\text{ s.t. } -x - y \le -3, \qquad -x + 2y \le 0,\\
&\qquad 2x + y \le 12, \qquad -3x + 2y \le -4.
\end{aligned}$$

Example 2 [9]

$$\begin{aligned}
&\min_{x,y,z}\ x + 2y + z,\\
&\text{ s.t. } x \ge 0,\\
&\text{where } (y, z) \text{ solves},\\
&\min_{y,z} \begin{pmatrix} y - 2z \\ -y + z \end{pmatrix}\\
&\text{ s.t. } x + y \ge 1,\\
&\quad x + y \le 3,\\
&\quad x + y + 2z \le 5,\\
&\quad y, z \ge 0.
\end{aligned}$$

Table 1: The results by the proposed algorithm for example 1

i	(v^i, μ^i)	(x^i, y^i)	$\theta(v^i, \mu^i)$	$\pi(x^i, y^i, v^i, \mu^i)$
1	$(0,0,0,0,0,0)$	$(3,0)$	3	0
2	$(0,0,3,0,0,0)$	$(2,1)$	-2	0

Table 2: The results by the proposed algorithm for example 2

i	(v^i, μ^i)	(x^i, y^i, z^i)	$\theta(v^i, \mu^i)$	$\pi(x^i, y^i, z^i, v^i, \mu^i)$
1	$(0,0,0,0,\frac{1}{2})$	$(1,0,2)$	3	0
2	$(0,1,0,1,0)$	$(3,0,0)$	3	0
3	$(0,\frac{1}{3},0,0,\frac{1}{3})$	$(1,0,2)$	3	0
4	$(0,0,\frac{1}{2},0,\frac{1}{2})$	$(1,0,2)$	3	0

In our experiment, we first choose $k = 10$ and $\tau = 10$. Using the proposed algorithm, we find that the solution for example 1 is $(2, 1)$ with the optimal value -2 from Table 1. Moreover, the results can be obtained by solving only 2 linear programming problems after obtaining all vertices of Z_2. From Table 2, the solution of example 2 occurs at the points $(1, 0, 2)$ and $(3, 0, 0)$, and the optimal value is 3. Our algorithm obtains the solution by solving only 4 linear programming problems. In fact, the results are the same as that in the references [2, 9].

5. Conclusion

In this paper, we present a global optimization algorithm based on penalty function for semivectorial bilevel programming problem. Two examples show that this algorithm is feasible. For the further research, we may discuss a general case that the functions and constraints of the lower level problem are convex.

References

[1] G. Anandalingam, D. J. White, A Solution Method for the Linear Static Stackelberg Problem using Penalty Functions, IEEE Transactions on Automatic Control 35 (1990) 1170-1173.

[2] Z. Ankhili, A. Mansouri, An exact penalty on bilevel programs with linear vector optimization lower level, European Journal of Operational Research 197 (2009) 36-41.

[3] B. Bank, J. Guddat, D. Klatte, B. Kummer, K. Tammer, Nonlinear parametric optimization, Academic Verlag, Berlin, Germany, 1983.

[4] J. F. Bard, Practical Bilevel Optimization: Algorithms and Applications, Kluwer Academic, Dordrecht, 1998.

[5] H. P. Benson, Optimization over the eficient set, Journal of Mathematical Analysis and Applications 98 (1984) 562-580.

[6] H. Bonnel, Optimality condition for the semivectorial bilevel optimization problem, Pacific Journal of Optimization 2 (2006) 447-468.

[7] H. Bonnel, J. Morgan, Semivectorial bilevel optimization problem: Penalty approach, Journal of Optimization Theory and Applications 131 (2006) 365-382.

[8] H. Bonnel, J. Morgan, Semivectorial Bilevel Convex Optimal Control Problems: Existence Results, SIAM Journal on Control and Optimization 50 (2012) 3224-3241.

[9] H. I. Calvete, C. Galé, On linear bilevel problems with multiple objectives at the lower level, Omega 39 (2011) 33-40.

[10] M. Campelo, S. Dantas, S. Scheimberg, A note on a penalty function approach for solving bilevel linear programs, Journal of Global Optimization 16 (2000) 245-255.

[11] S. Dempe, Foundations of Bilevel Programming, Nonconvex Optimization and its Applications Series, Kluwer Academic, Dordrecht, 2002.

[12] S. Dempe, Annotated bibliography on bilevel programming and mathematical problems with equilibrium constraints, Optimization 52 (2003) 333-359.

[13] S. Dempe, N. Gadhi, A. B. Zemkoho, New optimality conditions for the semivectorial bilevel optimization problem, Journal of Optimization Theory and Applications 157 (2013) 54-74.
[14] K. Fukuda, T. M. Liebling, F. Margot, Analysis of backtrack algorithms for listing all vertices and all faces of a convex polyhedron, Computational Geometry 8 (1997) l-12.
[15] G. Liu, J. Han, J. Zhang, Exact Penalty Functions for Convex Bilevel Programming Problems, Journal of Optimization Theory and Applications 110 (2001) 621-643.
[16] T. H. Matheiss, D. S. Rubin, A survey and comparison of methods for finding all vertices of convex polyhedral sets, Mathematics of Operations Research 5 (1980) 167-185.
[17] R. T. Rockafellar, Convex Analysis, Princeton University Press, Princeton, 1970.
[18] D. J. White, Optimality and Efficiency, Willey, Chichester, 1982.
[19] D. J. White, G. Anandalingam, A Penalty Function Approach for Solving Bi-Level Linear Programs, Journal of Global Optimization 3 (1993) 397-419.
[20] Y. Zheng, Z. Wan, A solution method for semivectorial bilevel programming problem via penalty method, Journal of Applied Mathematics and Computing 37 (2011) 207-219.