



## Remarks on the Relative Tensor Degree of Finite Groups

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**Abstract.** The present paper is a note on the relative tensor degree of finite groups. This notion generalizes the tensor degree, introduced recently in literature, and allows us to adapt the concept of relative commutativity degree through the notion of nonabelian tensor square. We show two inequalities, which correlate the relative tensor degree with the relative commutativity degree of finite groups.

### 1. The Relative Tensor Degree

All the groups of the present paper are supposed to be finite. Having in mind the exponential notation for the conjugation of two elements  $x$  and  $y$  in a group  $G$ , that is, the notation  $x^y = y^{-1}xy$ , we may follow [3, 4, 17] in saying that two normal subgroups  $H$  and  $K$  of  $G$  act compatibly upon each other, if

$$\left(h_2^{k_1}\right)^{h_1} = \left(\left(h_2^{h_1^{-1}}\right)^{k_1}\right)^{h_1} \quad \text{and} \quad \left(k_2^{h_1}\right)^{k_1} = \left(\left(k_2^{k_1^{-1}}\right)^{h_1}\right)^{k_1}$$

for all  $h_1, h_2 \in H$  and  $k_1, k_2 \in K$ , and if  $H$  and  $K$  act upon themselves by conjugation. Given  $h \in H$  and  $k \in K$ , the nonabelian tensor product  $H \otimes K$  is the group generated by the symbols  $h \otimes k$  satisfying the relations  $h_1 h_2 \otimes k_1 = (h_2^{h_1} \otimes k_1^{h_1}) (h_1 \otimes k_1)$  and  $h_1 \otimes k_1 k_2 = (h_1 \otimes k_1) (h_1^{k_1} \otimes k_2^{k_1})$  for all  $h_1, h_2 \in H$  and  $k_1, k_2 \in K$ . The map

$$\kappa_{H,K} : h \otimes k \in H \otimes K \mapsto [h, k] = h^{-1}h^k \in [H, K] = \langle [h, k] \mid h \in H, k \in K \rangle$$

turns out to be an epimorphism, whose kernel  $\ker \kappa_{H,K} = J(G, H, K)$  is central in  $H \otimes K$ . The reader may find more details and a topological approach to  $J(G, H, K)$  in [4, 5, 13, 17]. The short exact sequence

$$1 \longrightarrow J(G, H, K) \longrightarrow H \otimes K \xrightarrow{\kappa_{H,K}} [H, K] \longrightarrow 1$$

is a central extension. In the special case  $G = H = K$ , we have that  $J(G) = J(G, G, G) = \ker \kappa_{G,G} = \ker \kappa$  and  $H \otimes K = G \otimes G$  is called nonabelian tensor square of  $G$ . The fundamental properties of  $G \otimes G$  have been described in the classical paper [3], in which it is noted that  $\kappa : x \otimes y \in G \otimes G \mapsto \kappa(x \otimes y) = [x, y] \in G' = [G, G]$  is an epimorphism of groups with  $\ker \kappa = J(G)$  and  $1 \rightarrow J(G) \rightarrow G \otimes G \xrightarrow{\kappa} G' \rightarrow 1$  is a central extension. The

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group  $J(G)$  is important from the perspective of the algebraic topology, in fact  $J(G) \cong \pi_3(SK(G, 1))$  is the third homotopy group of the suspension of an Eilenberg–MacLane space  $K(G, 1)$  (see [4] for more details).

As done in [14], we may consider the *tensor centralizer*

$$C_K^\otimes(H) = \{k \in K \mid h \otimes k = 1, \forall h \in H\} = \bigcap_{h \in H} C_K^\otimes(h)$$

and the *tensor center*  $C_G^\otimes(G) = Z^\otimes(G) = \bigcap_{x \in G} C_G^\otimes(x)$  and one can check that  $C_G^\otimes(x)$  and  $Z^\otimes(G)$  are subgroups of  $G$  such that  $C_G^\otimes(x) \subseteq C_G(x)$  and  $Z^\otimes(G) \subseteq Z(G)$ .

Generalizing what has been done in [14], we may define the *relative tensor degree*

$$d^\otimes(H, K) = \frac{|\{(h, k) \in H \times K \mid h \otimes k = 1\}|}{|H||K|} = \frac{1}{|H||K|} \sum_{h \in H} |C_K^\otimes(h)|$$

of  $H$  and  $K$ . Notice that  $d^\otimes(G) = d^\otimes(G, G)$  is the *tensor degree* of  $G$  in [14] and  $d^\otimes(G) = 1$  if and only if  $Z^\otimes(G) = G$ . Unfortunately, few results are available on the relative tensor degree at the moment and these are contained mainly in [14]. On the other hand, there is a rich literature (see for instance [1, 2, 6, 9–11, 16]) on the *relative commutativity degree*

$$d(H, K) = \frac{|\{(h, k) \in H \times K \mid [h, k] = 1\}|}{|H||K|} = \frac{1}{|H||K|} \sum_{h \in H} |C_K(h)| = \frac{k_K(H)}{|H|}$$

of  $H$  and  $K$  (not necessarily normal this time) of  $G$ . Here  $k_K(H)$  is the number of  $K$ -conjugacy classes that constitute  $H$ . In particular, if  $G = H = K$ , we find the well known *commutativity degree*  $d(G) = d(G, G) = k_G(G)/|G|$ . Our first result is the following.

**Theorem 1.1.** *Let  $H, K$  be two normal subgroups of a group  $G$ . Then*

$$\frac{d(H, K)}{|J(G, H, K)|} \leq d^\otimes(H, K) \leq d(H, K).$$

*In particular, if  $J(G, H, K)$  is trivial, then  $d^\otimes(H, K) = d(H, K)$ .*

On the other hand, we may correlate the relative tensor degree, the relative commutativity degree and another notion, studied recently in [13]. In order to proceed in this direction, we recall from [3, 5, 12] that the *nonabelian exterior product*  $H \wedge K$  of  $H$  and  $K$  is the quotient of the nonabelian tensor product  $H \otimes K$ , defined by  $H \wedge K = (H \otimes K) / \nabla(H \cap K) = \langle (x \otimes y) \nabla(H \cap K) \mid x, y \in H \cap K \rangle = \langle x \wedge y \mid x, y \in H \cap K \rangle$ , where  $\nabla(H \cap K) = \langle x \otimes x \mid x \in H \cap K \rangle$ . From [3, 4], we may note that

$$\kappa'_{H,K} : h \wedge k \in H \wedge K \mapsto \kappa'_{H,K}(h \wedge k) = [h, k] \in [H, K]$$

is an epimorphism of groups such that

$$1 \longrightarrow M(G, H, K) \longrightarrow H \wedge K \xrightarrow{\kappa'_{H,K}} [H, K] \longrightarrow 1$$

is a central extension, where  $M(G, H, K) = \ker \kappa'_{H,K}$  is the so-called *Schur multiplier of the triple*  $(G, H, K)$ . We inform the reader that several references on the theory of the Schur multipliers of triples can be found in [4, 13]. In particular,  $M(G, G, G) = M(G) = H_2(G, \mathbb{Z})$  is the *Schur multiplier* of  $G$ , that is, the second integral homology group of  $G$ .

In our situation, it is possible to consider the set

$$C_K^\wedge(H) = \{k \in K \mid h \wedge k = 1, \forall h \in H\} = \bigcap_{h \in H} C_K^\wedge(h),$$

called *exterior centralizer* of  $H$  with respect to  $K$  and it is actually a subgroup of  $K$  (see [12] for details). In particular,  $C_G^\wedge(G) = Z^\wedge(G) = \bigcap_{x \in G} C_G^\wedge(x)$  is called *exterior center* of  $G$ . It is easy to check that  $C_G^\wedge(x) \subseteq C_G(x)$  and  $Z^\wedge(G) \subseteq Z(G)$ .

Some recent papers as [7, 13] show that it is possible to have a combinatorial approach for measuring how far a group  $G$  is from  $Z^\wedge(G)$  and this is interesting, because a result of Ellis [5] characterizes a capable group by the triviality of its exterior center (i.e.: a group  $G$  is *capable* if  $G \simeq E/Z(E)$  for a given group  $E$ ). This aspect has motivated the notion of *relative exterior degree*

$$d^\wedge(H, K) = \frac{|\{(h, k) \in H \times K \mid h \wedge k = 1\}|}{|H||K|} = \frac{1}{|H||K|} \sum_{h \in H} |C_K^\wedge(h)|$$

of  $H$  and  $K$ . When  $G = H = K$ , we find the *exterior degree*  $d^\wedge(G, G) = d^\wedge(G)$  of  $G$  in [13]. It is easy to prove that  $d^\wedge(G) = 1$  if and only if  $G = Z^\wedge(G)$ . Hence the exterior degree represents the probability that two randomly chosen elements commute with respect to the operator  $\wedge$ . Roughly speaking, this means that there are many chances of finding capable groups for small values of exterior degree.

From [14, Theorem 2.8], we may correlate the above notions via the inequality

$$d^\otimes(G) \leq d^\wedge(G) \leq d(G)$$

and our second result shows that something of similar holds.

**Theorem 1.2.** *Let  $H, K$  be normal subgroups of a group  $G$ . Then  $d^\otimes(H, K) \leq d^\wedge(H, K) \leq d(H, K)$ . Moreover, if  $J(G, H, K)$  is trivial, then  $d^\otimes(H, K) = d^\wedge(H, K) = d(H, K)$ .*

The reader will note that we do not use explicitly the notions of exterior center and of tensor center, but we have mentioned these concepts for understanding the interest in the relative tensor degree. Some recent papers (see for instance [15]) deal with the size of these important subgroups. In the present paper, we get information on the size of the same subgroups from the perspective of the probability.

## 2. Proofs of the Results

We begin with a technical lemma, whose proof uses an argument which appears in [13, Lemma 2.1] and [14, Lemma 2.2] in different ways.

**Lemma 2.1.** *Let  $H, K$  be normal subgroups of a group  $G$ . Then*

$$d^\otimes(H, K) = \frac{1}{|H|} \sum_{i=1}^{k_K(H)} \frac{|C_K^\otimes(h_i)|}{|C_K(h_i)|}.$$

*In particular, if  $G = HK$ , then  $C_K(h_i)/C_K^\otimes(h_i)$  is isomorphic to a subgroup of  $J(G, H, K)$  and  $|C_K(h_i) : C_K^\otimes(h_i)| \leq |J(G, H, K)|$  for all  $i = 1, 2, \dots, k_K(H)$ .*

*Proof.* Since  $H$  is normal in  $G$ , we consider the  $K$ -conjugacy classes  $C_1, \dots, C_{k_K(H)}$  that constitute  $H$ . It follows that

$$|H| |K| d^\otimes(H, K) = \sum_{h \in H} |C_K^\otimes(h)| = \sum_{i=1}^{k_K(H)} \sum_{h \in C_i} |C_K^\otimes(h)| = \sum_{i=1}^{k_K(H)} |K : C_K(h_i)| |C_K^\otimes(h_i)| = |K| \sum_{i=1}^{k_K(H)} \frac{|C_K^\otimes(h_i)|}{|C_K(h_i)|}.$$

Now assume that  $G = HK$ . For all  $i = 1, \dots, k_K(H)$ , the map

$$\varphi : kC_K^\otimes(h_i) \in C_K(h_i)/C_K^\otimes(h_i) \mapsto k \otimes h_i \in J(G, H, K)$$

satisfies the condition

$$\varphi(k_1 k_2 C_K^\otimes(h_i)) = k_1 k_2 \otimes h_i = (k_1 \otimes h_i)^{k_2} (k_2 \otimes h_i) = (k_1 \otimes h_i) (k_2 \otimes h_i) = \varphi(k_1 C_K^\otimes(h_i)) \varphi(k_2 C_K^\otimes(h_i))$$

for all  $k_1, k_2 \in C_K(h_i)$ . This means that  $\varphi$  is a homomorphism of groups (the reader may find a variation on this theme in [14, Proof of Lemma 2.1] and [12, Proof of Proposition 2.7]). Furthermore,  $\ker \varphi = \{kC_K^\otimes(h_i) \mid k \otimes h_i = 1\} = C_K^\otimes(h_i)$ . Then  $\varphi$  is a monomorphism and  $C_K(h_i)/C_K^\otimes(h_i)$  is isomorphic to a subgroup of  $J(G, H, K)$ . We conclude that  $|C_K(h_i) : C_K^\otimes(h_i)| \leq |J(G, H, K)|$ .  $\square$

Now we may prove Theorems 1.1 and 1.2. The first one is an interesting bound, which connects the notion of relative tensor degree with that of relative commutativity degree.

*Proof.* [Proof of Theorem 1.1] We begin to prove the lower bound. From Lemma 2.1,

$$|C_K^\otimes(h_i)|/|C_K(h_i)| \geq 1/|J(G, H, K)|$$

for all  $i = 1, 2, \dots, k_K(H)$  and  $h_i \in H$ . Together with the equality

$$d(H, K) = \frac{k_K(H)}{|H|},$$

we deduce

$$d^\otimes(H, K) = \frac{1}{|H|} \sum_{i=1}^{k_K(H)} \left| \frac{C_K^\otimes(h_i)}{C_K(h_i)} \right| \geq \frac{1}{|H|} \cdot \underbrace{\left( \frac{1}{|J(G, H, K)|} + \dots + \frac{1}{|J(G, H, K)|} \right)}_{k_K(H)\text{-times}} = \frac{k_K(H)}{|H| |J(G, H, K)|} = \frac{d(H, K)}{|J(G, H, K)|}.$$

Conversely, we apply again Lemma 2.1, but in the following form:

$$d^\otimes(H, K) = \frac{1}{|H|} \sum_{i=1}^{k_K(H)} \left| \frac{C_K^\otimes(h_i)}{C_K(h_i)} \right| \leq \frac{1}{|H|} \cdot \underbrace{(1 + \dots + 1)}_{k_K(H)\text{-times}} = \frac{k_K(H)}{|H|} = d(H, K).$$

We used the fact that  $|C_K(h_i)|/|C_K^\otimes(h_i)|$  is a positive integer; then  $|C_K^\otimes(h_i)|/|C_K(h_i)|$  is a number in  $]0, 1]$ . The remaining part of the statement follows easily.  $\square$

We may note a strong connection among [14, Theorem 2.3] and Theorem 1.1. Our second main theorem is a result of comparison. Its proof is the following.

*Proof.* [Proof of Theorem 1.2] We have

$$d^\wedge(H, K) = \frac{1}{|H|} \sum_{i=1}^{k_K(H)} \left| \frac{C_K^\wedge(h_i)}{C_K(h_i)} \right| \leq \frac{1}{|H|} \sum_{i=1}^{k_K(H)} \left| \frac{C_K(h_i)}{C_K(h_i)} \right| = d(H, K)$$

and the upper bound follows.

Now  $k \in C_K^\wedge(H)$  if and only if  $k \wedge h = 1$  for all  $h \in H$  if and only if  $(k \otimes h)\nabla(H \cap K) = \nabla(H \cap K)$  if and only if  $k \otimes h \in \nabla(H \cap K)$ . This condition is weaker than the condition  $k \otimes h = 1$ , characterizing the elements of  $C_K^\otimes(H)$ . Then  $C_K^\otimes(H) \subseteq C_K^\wedge(H) \subseteq C_K(H)$ . This and Lemma 2.1 imply the lower bound

$$d^\otimes(H, K) = \frac{1}{|H|} \sum_{i=1}^{k_K(H)} \left| \frac{C_K^\otimes(h_i)}{C_K(h_i)} \right| \leq \frac{1}{|H|} \sum_{i=1}^{k_K(H)} \left| \frac{C_K^\wedge(h_i)}{C_K(h_i)} \right| = d^\wedge(H, K).$$

The rest follows from Theorem 1.1.  $\square$

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