Filomat 29:1 (2015), 99–106
DOI 10.2298/FIL1501099K

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Menger-Type Covering Properties of Topological Spaces

Darko Kocev[a]

[a] *Technical Faculty in Bor, University of Belgrade, Serbia*

Abstract. In this paper we find conditions under which the properties of Menger, almost Menger and weakly Menger are equivalent as well as the corresponding properties of Lindelöf-type. We give counterexamples that show the interrelations between those properties. The subject of our investigation is also the preservation of almost Menger and weakly Menger properties under subspaces and products. We also consider the weaker versions of Alster space and D-spaces.

1. Introduction and notation

The Menger property is a familiar topological notion introduced in [17] by K. Menger in 1924 and systematically studied since the paper [21] by Scheepers. Recently, the notions of almost Menger and weakly Menger properties were introduced and considered (see [8], [14], [15], [18]). In [9], Di Maio and Kočinac considered almost Menger property in hyperspaces. In [5] almost Menger and weakly Menger properties were considered in infinite games and in [4] Babinkostova, Pansera and Scheepers considered the productivity of weakly Menger property. Survey paper [7] also concerns weakly Menger property.

Recall that a topological space X is *Menger* (resp. *almost Menger*, *weakly Menger*) if for every sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X, there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for every $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\bigcup\{\mathcal{V}_n : n \in \mathbb{N}\} = X$ (resp. $\bigcup_{n\in\mathbb{N}}\{\overline{V} : V \in \mathcal{V}_n\} = X$, $\bigcup\{\mathcal{V}_n : n \in \mathbb{N}\}$ is dense in X).

On the other hand, many authors considered properties of Lindelöf type such as almost Lindelöf, weakly Lindelöf and quasi-Lindelöf (see [6], [22], [26]). Note that the notion of quasi-Lindelöf was introduced by Arhangel'skii and that recently the notion of quasi-Menger was introduced and considered by Di Maio and Kočinac in [10]. We say that a topological space X is Lindelöf (resp. *almost Lindelöf*, *weakly Lindelöf*) if for every open cover $\mathcal{U}$ of X there exists a countable subset $\mathcal{V}$ of $\mathcal{U}$ such that $\bigcup\mathcal{V} = X$ (resp. $\bigcup\{\overline{V} : V \in \mathcal{V}\} = X$, $\bigcup\mathcal{V}$ is dense in X).

We have the following implications between these notions:

2010 *Mathematics Subject Classification*. Primary 54D20 ; Secondary 54D10.
Keywords. Selection principles; Menger; almost Menger; weakly Menger; Alster space; D-space.
Received: 6 September 2014; Accepted: 14 Novembar 2014
Communicated by Dragan Djurčić
Research supported by MNRS 174025
Email address: darkosun@open.telekom.rs (Darko Kocev)

$$\begin{array}{ccccc}
\text{Menger} & \rightarrow & \text{almost Menger} & \rightarrow & \text{weakly Menger} \\
\downarrow & & \downarrow & & \downarrow \\
\text{Lindelöf} & \rightarrow & \text{almost Lindelöf} & \rightarrow & \text{weakly Lindelöf}
\end{array}$$

Neither of these implications is reversible. Pansera in [18] gave examples showing that weakly Menger property does not imply almost Menger property even in Tychonoff spaces. In [14] we showed that there exists a topological space which is almost Menger and not Lindelöf and therefore, not Menger. Recently, Sakai in [19] gave an example of a topological space which is Lindelöf and not weakly Menger. So, in the previous diagram no other implication holds between given notions.

In Section 1 we examine conditions under which some of these notions are equivalent.

In Section 2 we investigate the behavior of almost Menger and weakly Menger properties with respect to subspaces and products. We also consider the almost version of Alster property.

In [3] Aurichi proved that Menger spaces are D-spaces. In Section 4 we study the analogous assertions for the almost version of Menger property.

Let X be a topological space. We use the following notation:

$\mathcal{O}$ – the collection of all open covers of X;

$\mathcal{D}$ – the collection of all families $\mathcal{U}$ of open subsets of X such that $\bigcup \mathcal{U}$ is dense in X;

$\overline{\mathcal{O}}$ – the collection of all families $\mathcal{U}$ of open subsets of X such that $\{\overline{U} : U \in \mathcal{U}\}$ is a cover of X.

Let $\mathcal{A}$ and $\mathcal{B}$ be collections of open subsets of a topological space X. Then the symbol $\mathsf{S}_1(\mathcal{A}, \mathcal{B})$ denotes the selection hypothesis that for each sequence $(A_n : n \in \mathbb{N})$ of elements from $\mathcal{A}$ there exists a sequence $(B_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $B_n \in A_n$ and $\{B_n : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$ and $\mathsf{S}_{fin}(\mathcal{A}, \mathcal{B})$ denotes the selection hypothesis that for each sequence $(A_n : n \in \mathbb{N})$ of elements from $\mathcal{A}$ there exists a sequence $(B_n : n \in \mathbb{N})$ so that B_n is a finite subset of A_n for each $n \in \mathbb{N}$ and $\{B_n : n \in \mathbb{N}\}$ is an element from $\mathcal{B}$.

Note that a topological space X is Menger (resp. almost Menger, weakly Menger) if it satisfies the selection hypothesis $\mathsf{S}_{fin}(\mathcal{O}, \mathcal{O})$ (resp. $\mathsf{S}_{fin}(\mathcal{O}, \overline{\mathcal{O}})$, $\mathsf{S}_{fin}(\mathcal{O}, \mathcal{D})$).

Other notation and terminology are as in [12].

2. Conditions of equivalence

In [14] it was shown that in regular spaces, Menger property and almost Menger property are equivalent. The following example shows that in Urysohn spaces the equivalence of these properties fails.

Example 2.1. A Urysohn, first countable, almost Menger space which does not have the Menger property.

Let $\mathbb{R}$ be the set of real numbers with the Euclidean topology τ and let $\mathbb{Q}$ be the set of rational numbers. We define τ', the pointed rational extension of $\mathbb{R}$ (see [23], Example 68), to be the topology generated by $\{x\} \cup (\mathbb{Q} \cap U)$, where $x \in U \in \tau$.

$(\mathbb{R}, \tau')$ is Urysohn, because $(\mathbb{R}, \tau)$ is Urysohn and closures of open sets in τ and τ' are equal. $(\mathbb{R}, \tau')$ is not Lindelöf and therefore it is not Menger. Since the closures of open sets in $(\mathbb{R}, \tau')$ are the same as in the Euclidean topology and $(\mathbb{R}, \tau)$ is Menger, then $(\mathbb{R}, \tau')$ is almost Menger.

On the other hand, weakly Menger and almost Menger properties are not equivalent in regular spaces. Pansera proved even more (see [18]): that there exists a Tychonoff, weakly Menger space that is not almost Menger. We give an example of regular, weakly Menger space which is not almost Menger.

Example 2.2. A regular, separable weakly Menger space which is not almost Menger.

Let $\mathbb{R}$ be the set of reals numbers, $\mathbb{I}$ the set of irrational numbers and $\mathbb{Q}$ the set of rational numbers and for each irrational x we choose a sequence $\{r_i : i \in \mathbb{N}\}$ of rational numbers converging to it in the Euclidean topology. The rational sequence topology τ (see [23], Example 65) is then defined by declaring each rational open and selecting the sets $U_\alpha(x) = \{x_{\alpha,i} : i \in \mathbb{N}\} \cup \{x\}$ as a basis for the irrational point x. If $r \in \mathbb{Q}$, then the closure of $\{r\}$ with respect to τ is equal $\{r\}$, and for every $x \in \mathbb{I}$, the closure of $U_\alpha(x)$ is equal $U_\alpha(x)$. For every $n \in \mathbb{N}$ $\mathcal{U}_n = \{r : r \in \mathbb{Q}\} \cup \{U_n(x) : x \in \mathbb{I}\}$ is an open cover of $(\mathbb{R}, \tau)$. $(\mathbb{R}, \tau)$ does not have the almost Menger property because every $x \in \mathbb{I}$ belongs to closure of only one element from $\mathcal{U}_n$ for each $n \in \mathbb{N}$ and since $\mathbb{I}$ is uncountable, we can not find a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ of finite subsets of $\mathcal{U}_n$ for every $n \in \mathbb{N}$ such that every $x \in \mathbb{I}$ is covered by $\bigcup_{n\in\mathbb{N}}\{\overline{V} : V \in \mathcal{V}_n\}$. On the other hand, $(\mathbb{R}, \tau)$ is weakly Menger, because $\mathbb{Q}$ is dense in $(\mathbb{R}, \tau)$. Note that this space is not Lindelöf, so this space can be used also as an example of weakly Menger and not Lindelöf space.

It is natural to ask under which conditions these properties are equivalent. In Theorem 9 from [18] it is proved that in hypocompact spaces the properties of Menger and almost Menger are equivalent. The following proposition answers Pansera's question from [18] concerning conditions under which weakly Menger and almost Menger properties are equivalent.

Recall that a topological space X is *P-space* if every intersection of countably many open subsets of X is open. The following assertion holds:

Proposition 2.3. *If a topological space $(X, \mathcal{T})$ is weakly Menger P-space, then $(X, \mathcal{T})$ is almost Menger.*

Proof. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of X. Since X is weakly Menger, there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for every $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\bigcup_{n\in\mathbb{N}} \cup\mathcal{V}_n$ is dense in X. Let $x \in X$. By the condition of theorem, the intersection of every countable family of open subsets of X is open and hence, every countable union of closed sets is closed. So, $\bigcup_{n\in\mathbb{N}}\{\overline{V} : V \in \mathcal{V}_n\}$ is the closed subset of X. Since $\overline{\bigcup_{n\in\mathbb{N}} \cup\mathcal{V}_n}$ is the least closed set that contains $\bigcup_{n\in\mathbb{N}} \cup\mathcal{V}_n$, we have that $\overline{\bigcup_{n\in\mathbb{N}} \cup\mathcal{V}_n} \subseteq \bigcup_{n\in\mathbb{N}}\{\overline{V} : V \in \mathcal{V}_n\}$, so $\bigcup_{n\in\mathbb{N}}\{\overline{V} : V \in \mathcal{V}_n\} = X$, which proves that X is almost Menger. □

We will also see that in P-spaces the properties of almost Lindelöf and almost Menger are equivalent.

Proposition 2.4. *Every almost Lindelöf P-space is almost Menger.*

Proof. Let X be an almost Lindelöf P-space and let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of X. We may assume that for each $n \in \mathbb{N}$, $\mathcal{U}_n$ is closed for finite unions. Put $\mathcal{U} = \{\bigcap_{n\in\mathbb{N}} U_n : U_n \in \mathcal{U}_n\}$. Then $\mathcal{U}$ is an open cover for X, since X is a P space. As X is almost Lindelöf, there exists a countable subset $(V_n : n \in \mathbb{N})$ of $\mathcal{U}$ so that $\bigcup_{n\in\mathbb{N}} \overline{V_n} = X$. For every $n \in \mathbb{N}$, we write $V_n = \bigcap_{k\in\mathbb{N}} U_k^n$, where $U_k^n \in \mathcal{U}_k$. But then $\bigcup_{n\in\mathbb{N}} \overline{U_n^n} = X$, since $V_n \subset U_n^n$ for every $n \in \mathbb{N}$, which shows that X is almost Menger. □

In [4] it is stated that weakly Lindelöf P-spaces are weakly Menger. We have the following corollary:

Corollary 2.5. *Let X be a regular P-space. Then the following statements are equivalent:*

(1) *X is Menger;*

(2) *X is almost Menger;*

(3) *X is weakly Menger;*

(4) *X is Lindelöf;*

(5) *X is almost Lindelöf;*

(6) *X is weakly Lindelöf.*

We conclude this section with another condition under which the properties of almost Menger and weakly Menger are equivalent. Recall that a family $\mathcal{V}$ of subsets of a topological space X is locally finite refinement of a family $\mathcal{U}$ of subsets of X if for every $V \in \mathcal{V}$ there exists $U \in \mathcal{U}$ such that $V \subset U$ and for every point $x \in X$ there exists a neighborhood of x which intersects only finitely many elements of $\mathcal{V}$. We say that a topological space X is *d-paracompact* if every dense family of subsets of X has a locally finite refinement.

Proposition 2.6. *If a topological space X is weakly Menger and d-paracompact, then X is almost Menger.*

Proof. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of X. Since X is weakly Menger, there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for every $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\bigcup_{n\in\mathbb{N}} \cup\mathcal{V}_n$ is dense in X. By the assumption, $\{\mathcal{V}_n : n \in \mathbb{N}\}$ has a locally finite refinement $\mathcal{W}$. Then $\bigcup \mathcal{W} = \bigcup_{n\in\mathbb{N}} \cup\mathcal{V}_n$ and therefore $\overline{\bigcup \mathcal{W}} = \overline{\bigcup_{n\in\mathbb{N}} \cup\mathcal{V}_n}$. As $\mathcal{W}$ is a locally finite family, we have that $\overline{\bigcup \mathcal{W}} = \bigcup_{W\in\mathcal{W}} \overline{W}$. Since for every $W \in \mathcal{W}$ there exists $n \in \mathbb{N}$ and $V_W \in \mathcal{V}_n$ so that $W \subset V_W$, we have that $\bigcup_{n\in\mathbb{N}}\{\overline{V} : V \in \mathcal{V}_n\} = X$, so we showed that X is almost Menger. □

3. Subspaces and products

It is known that Lindelöf property is preserved under closed subsets. Almost Menger and weakly Menger properties are not invariant with respect to closed subspaces as the following examples will show.

Example 3.1. A closed subspace of an almost Menger space which is not almost Menger.

If $(\mathbb{R}, \tau')$ is the pointed rational extension of $\mathbb{R}$ (see Example 2.1), then the set of irrational points $\mathbb{I}$ with the topology τ' restricted to $\mathbb{I}$ is the closed subspace of almost Menger space and it is not almost Menger, because every $x \in \mathbb{I}$ is clopen in $\mathcal{T}'$ restricted to $\mathbb{I}$.

Example 3.2. A closed subspace of a weakly Menger space which is not weakly Menger.

If $(\mathbb{R}, \tau)$ is the same topological space as in Example 2.2, then the set of irrational points $\mathbb{I}$ with the topology τ restricted to $\mathbb{I}$ is the closed subspace of weakly Menger space and it is not weakly Menger, because it is uncountable discrete space.

In [22] it is shown that almost Lindelöf property is preserved under clopen subsets and that weakly Lindelöf property is closed under regularly closed subsets. We can ask ourselves if the analogous statements hold in the case of almost Menger and weakly Menger properties.

Proposition 3.3. *Every clopen subset of an almost Menger space is almost Menger.*

Proof. Let F be a clopen subset of almost Menger space X and let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of F. Then $\mathcal{V}_n = \{\mathcal{U}_n\} \cup \{X \setminus F\}$ is an open cover of X for every $n \in \mathbb{N}$. Since X is almost Menger, there exist finite subsets $\mathcal{V}'_n$ of $\mathcal{V}_n$ for each $n \in \mathbb{N}$ so that $\bigcup_{n\in\mathbb{N}}\{\overline{V} : V \in \mathcal{V}'_n\} = X$. But $X \setminus F$ is clopen, so $\overline{X \setminus F} = X \setminus F$ and $\bigcup_{n\in\mathbf{N}}\{\overline{V} : V \in \mathcal{V}'_n, V \neq X \setminus F\}$ covers F. □

Proposition 3.4. *Every regularly closed subset of a weakly Menger space is weakly Menger.*

Proof. Let F be a regularly closed subset of weakly Menger space X and let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of X. Then $\mathcal{V}_n = \{\mathcal{U}_n\} \cup \{X \setminus F\}$ is an open cover of X for every $n \in \mathbf{N}$. Since X is weakly Menger, there exist finite subsets $\mathcal{V}'_n$ of $\mathcal{V}_n$ for every $n \in \mathbb{N}$ so that $\overline{\bigcup_{n\in\mathbb{N}} \cup\mathcal{V}'_n}$ is dense in X. Put $\mathcal{V}' = \bigcup_{n\in\mathbb{N}}\{V : V \in \mathcal{V}'_n, V \neq X \setminus F\}$. Then $\overline{\mathcal{V}' \bigcup (X \setminus F)} = X$. Since $F = \overline{int(F)}$, we have that $int(F) \cap \overline{(X \setminus F)} = \emptyset$, so $int(F) \subset \overline{\mathcal{V}'}$ and $F = \overline{int(F)} \subset \overline{\mathcal{V}'}$. □

The productivity of the weakly Menger property is considered in [4], so we now investigate only the problem of productivity of almost Menger property. The following example shows that the product of almost Menger spaces is not always almost Menger.

Example 3.5. An almost Menger space X such that X^2 is not almost Menger.

Let $\mathbb{S}$ be the Sorgenfrey line and let $\mathbb{R}$ be the set of reals. If $i : \mathbb{S} \to \mathbb{R}$ is the identity map and $X \subset \mathbb{R}$ then by $X_{\mathbb{S}}$ we denote $i^{-1}(X)$ (see [20]). Lelek showed in [16] that for every Lusin set L in $\mathbb{R}$, $L_{\mathbb{S}}$ is Menger and, therefore, almost Menger, but if L satisfies that $(L \times L) \cap \{(x, y) : x + y = 0\}$ is uncountable, then $L_{\mathbb{S}} \times L_{\mathbb{S}}$ is not Menger and since $\mathbb{S} \times \mathbb{S}$ is regular and every subspace of a regular space is regular, we have that the square of $L_{\mathbb{S}}$ is regular and not Menger. Hence, $L^2_{\mathbb{S}}$ is not almost Menger.

In [22] it is proved that the product of an almost Lindelöf space and a compact space is almost Lindelöf. Recall that a topological space X is almost compact if for every open cover $\mathcal{U}$ of X there exists a finite subset $\mathcal{V}$ of $\mathcal{U}$ such that $\bigcup\{\overline{V} : V \in \mathcal{V}\}$ covers X (Note that the notions of almost compact and weakly compact spaces are equivalent). The following statement holds:

Theorem 3.6. *If X is almost Menger, and Y is almost compact, then $X \times Y$ is almost Menger.*

Proof. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of $X \times Y$. Then for each $n \in \mathbb{N}$ there exist open covers $\mathcal{V}_n$ and $\mathcal{W}_n$ of X and Y, respectively, such that $\mathcal{U}_n = \mathcal{V}_n \times \mathcal{W}_n$. Considering that Y is almost compact there exists a sequence $(A_n : n \in \mathbb{N})$ of finite subsets of $\mathcal{W}_n$ such that $\bigcup\{\overline{A} : A \in A_n\}$ covers Y for each $n \in \mathbb{N}$. Since X is almost Menger, there exist finite subsets $\mathcal{V}'_n$ of $\mathcal{V}_n$ so that $\bigcup_{n\in\mathbb{N}}\{\overline{V} : V \in \mathcal{V}'_n\} = X$. Put $\mathcal{R}_n = \mathcal{V}'_n \times A_n$. Then for each $n \in \mathbb{N}$, $\mathcal{R}_n$ is a finite subset of $\mathcal{U}_n$ and we will show that $\bigcup_{n\in\mathbb{N}}\{\overline{R} : R \in \mathcal{R}_n\} = X \times Y$.

Let $(x, y) \in X \times Y$. Then there exist $n \in \mathbb{N}$ and $V \in \mathcal{V}'_n$ so that $x \in \overline{V}$. There is also $W \in A_n$ so that $y \in \overline{W}$. That implies $(x, y) \in \overline{V} \times \overline{W} = \overline{V \times W}$ which concludes the proof. □

Many mathematicians tried to characterize spaces that are productively Lindelöf and studied the relationship between those spaces and Menger spaces (see [24], [25]). K. Alster in [1] defined a topological space that is productively Lindelöf in the following way: Call a family $\mathcal{F}$ of G_δ subsets of a space X a G_δ *compact* cover if there is for each compact subset K of X a set $F \in \mathcal{F}$ such that $K \subseteq F$. We say that a space X is an *Alster space* if each G_δ compact cover of the space has a countable subset that covers X.

We define the almost version of Alster space.

Definition 3.7. We say that a topological space X is *almost Alster* if each G_δ compact cover of X has the countable subset $\mathcal{V}$ such that $\bigcup\{\overline{V} : V \in \mathcal{V}\} = X$.

Problem 3.8. *Find a topological space which is almost Alster and not Alster.*

The following statement holds:

Theorem 3.9. *If X and Y are almost Alster spaces, then $X \times Y$ is an almost Alster space.*

Proof. Let X and Y be almost Alster spaces and let $\mathcal{U}$ be a G_δ compact cover of $X \times Y$. Then for every compact subsets K of X and C of Y there exist, respectively, G_δ subsets $\mathsf{G}(K)$ and $\mathsf{H}(C)$ of X and Y, respectively, such that $K \subset \mathsf{G}(K)$ and $C \subset \mathsf{H}(C)$ and, since $K \times C$ is compact, we can find $U \in \mathcal{U}$ so that $\mathsf{G}(K) \times \mathsf{H}(C) \subset U$. Notice that $\{\mathsf{G}(K) : K \subset X \text{ compact}\}$ and $\{\mathsf{H}(C) : C \subset X \text{ compact}\}$ are G_δ compact covers of X and Y, respectively. Then there exist sequences $(K_n : n \in \mathbb{N})$ and $(C_m : m \in \mathbb{N})$ of compact subsets of X and Y, respectively, so that $\bigcup_{n\in\mathbb{N}} \overline{\mathsf{G}(K_n)} = X$ and $\bigcup_{m\in\mathbb{N}} \overline{\mathsf{H}(C_m)} = Y$. For every $n \in \mathbb{N}$ and every $m \in \mathbb{N}$ we can find $U_{nm} \in \mathcal{U}$ such that $\mathsf{G}(K_n) \times \mathsf{H}(C_m) \subset U_{nm}$. We claim that $\bigcup_{n,m\in\mathbb{N}} \overline{U_{nm}} = X \times Y$. Indeed, let $(x, y) \in X \times Y$. Then we can choose $n \in \mathbb{N}$ and $m \in \mathbb{N}$ such that $x \in \overline{\mathsf{G}(K_n)}$ and $y \in \overline{\mathsf{H}(C_m)}$, so $(x, y) \in \overline{\mathsf{G}(K_n)} \times \overline{\mathsf{H}(C_m)} = \overline{\mathsf{G}(K_n) \times \mathsf{H}(C_m)} \subset \overline{U_{nm}}$, so $(x, y) \in U_{nm}$. □

It is known that the Alster property implies the Menger property. The analogous statement holds for almost versions of these notions.

Theorem 3.10. *If X is an almost Alster space, then X is almost Menger.*

Proof. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of X. We may assume that for every $n \in \mathbb{N}$ $\mathcal{U}_n$ is closed for finite unions. Put $\mathcal{U} = \{\bigcap_{n\in\mathbb{N}} U_n : U_n \in \mathcal{U}_n\}$. Members of $\mathcal{U}$ are G_δ sets and $\mathcal{U}$ is a cover for X. If K is a compact subset of X, then there exists a finite subset $\mathcal{V}$ of $\mathcal{U}$ such that $K \subset \bigcup \mathcal{V}$. Since X is almost Alster, we can pick a countable subset $\{A_n : n \in \mathbb{N})\}$ of $\mathcal{U}$ so that $X \subset \bigcup_{n\in\mathbb{N}} \overline{A_n}$. Let $A_n = \bigcap_{k\in\mathbb{N}} U_k^n$, where $U_k^n \in \mathcal{U}_k$ for every $n \in \mathbb{N}$ and denote $B_n = U_n^n \in \mathcal{U}_n$. We shall prove that $X \subseteq \bigcup_{n\in\mathbb{N}} \overline{B_n}$. We have that $A_n \subset B_n$ for every $n \in \mathbb{N}$ and hence, $\overline{A_n} \subset \overline{B_n}$ for every $n \in \mathbb{N}$. Therefore, $X \subset \bigcup_{n\in\mathbb{N}} \overline{A_n} \subset \bigcup_{n\in\mathbb{N}} \overline{B_n}$. □

Corollary 3.11. *If X is almost Alster, then X is almost Menger in all finite powers.*

In [4] the Alster property is characterized in the terms of selection principles. In the similar way we show the connection between the notion of almost Alster and selection principles.

We need the following notation (see [1], [4]):

$\mathcal{G}_K$: The family consisting of sets $\mathcal{U}$ where X is not in $\mathcal{U}$, each element of $\mathcal{U}$ is a G_δ set, and for each compact set $C \subset X$ there is a $U \in \mathcal{U}$ such that $C \subseteq U$.

$\mathcal{G}$: The family of all covers $\mathcal{U}$ of the space X for which each element of $\mathcal{U}$ is a G_δ set.

$\mathcal{G}_\Omega$: The family of all covers $\mathcal{U} \in \mathcal{G}$ such that for every finite subset F of X there exists $U \in \mathcal{U}$ containing F.

We also define the following classes of covers:

$\overline{\mathcal{G}}$: The family consisting of sets $\mathcal{U}$ such that every $U \in \mathcal{U}$ is G_δ set and $\bigcup\{\overline{U} : U \in \mathcal{U}\} = X$.

$\overline{\mathcal{G}_\Omega}$: The family of all sets $\mathcal{U} \in \overline{\mathcal{G}}$ such that for every finite subset F of X there exists $U \in \mathcal{U}$ so that $F \subset U$.

Proposition 3.12. *For a topological space X the following statements are equivalent:*

(1) *X is almost Alster;*

(2) *X satisfies the selection hypothesis* $\mathsf{S}_1(\mathcal{G}_K, \overline{\mathcal{G}})$*;*

(3) *X satisfies the selection hypothesis* $\mathsf{S}_1(\mathcal{G}_K, \overline{\mathcal{G}_\Omega})$*.*

Proof. (1) $\Longrightarrow$ (2): Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of elements from $\mathcal{G}_K$ and put $\mathcal{U} = \{\cap U_n : U_n \in \mathcal{U}_n\}$. Then $\mathcal{U}$ is a cover of X such that his elements are G_δ sets and for every compact subset K of X there exists $U \in \mathcal{U}$ such that $K \subset U$. By assumption, we can find a countable subset $\mathcal{V}$ of $\mathcal{U}$ so that $\bigcup\{\overline{V} : V \in \mathcal{V}\} = X$. Put $V_n = \bigcap_{k\in\mathbb{N}} U_k^n$, where $U_k^n \in \mathcal{U}_k$. Then $U_n^n \in \mathcal{U}_n$ and $\bigcup_{n\in\mathbb{N}} \overline{U_n^n} = X$, so X satisfies $\mathsf{S}_1(\mathcal{G}_K, \overline{\mathcal{G}})$.

(2) $\Longrightarrow$ (1): Let U be a cover of X consisting of G_δ sets such that for every compact subset K of X there exists $U \in \mathcal{U}$ so that $K \subset U$. Let $\mathcal{U}_n = \mathcal{U}$ for every $n \in \mathbb{N}$. By (2), we can choose $U_n \in \mathcal{U}$ for every $n \in \mathbb{N}$ such that $\bigcup_{n\in\mathbb{N}} \overline{U_n} = X$,so X is almost Alster.

(2) $\Longrightarrow$ (3): Since the finite power of almost Alster spaces is almost Alster, we have that if X satisfies $\mathsf{S}_1(\mathcal{G}_K, \overline{\mathcal{G}})$, then X^k also satisfies $\mathsf{S}_1(\mathcal{G}_K, \overline{\mathcal{G}})$ for every $k \in \mathbb{N}$. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of covers from $\mathcal{G}_K$. Then $\{\overline{(U)^k} : U \in \mathcal{U}_n\}$ is a sequence of $\mathcal{G}_K$ covers of X^k, so we can choose $U_n \in \mathcal{U}_n$ for every $n \in \mathbb{N}$ such that $\bigcup_{n\in\mathbb{N}} \overline{(U_n)^k} = X^k$. Put $F = (x_1, x_2, ..., x_k) \in X^k$. Then there exists $n \in \mathbb{N}$ such that $(x_1, x_2, ..., x_k) \in \overline{(U_n)^k}$ and therefore, for every i from 1 to k, $x_i \in \overline{U_n}$. That implies $F \subset \overline{U_n}$, so X satisfies $\mathsf{S}_1(\mathcal{G}_K, \overline{\mathcal{G}_\Omega})$. □

In [4] the following class of covers was defined:

$\mathcal{G}_\Gamma$: The set of covers $\mathcal{U} \in \mathcal{G}$ which are infinite, and each infinite subset of $\mathcal{U}$ is a cover of X.

We define the following class of covers:

$\mathcal{G}_{\overline{\Gamma}}$: The set of all covers $\mathcal{U} \in \mathcal{G}$ which are infinite, and for every $x \in X$ the set $\{U \in \mathcal{U} : x \notin \overline{U}\}$ is finite.

In [4] it is proved that a topological space that satisfies the selection hypothesis $\mathsf{S}_1(\mathcal{G}_K, \mathcal{G}_\Gamma)$ is productively Menger. In the same manner we prove the following statement:

Theorem 3.13. *If X satisfies the selection hypothesis $\mathsf{S}_1(\mathcal{G}_K, \mathcal{G}_{\overline{\Gamma}})$ and Y is almost Menger, then $X \times Y$ is almost Menger.*

Proof. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of $X \times Y$. We may assume that for each $n \in \mathbb{N}$, $\mathcal{U}_n$ is closed for finite unions. Then for every compact subset K of X and every $n \in \mathbb{N}$ there is G_δ set $\Phi_n(K)$ such that $K \subset \Phi_n(K)$. Then for every $y \in Y$ and every $n \in \mathbb{N}$ there exists $U \in \mathcal{U}_n$ so that $\Phi_n(K) \times \{y\} \subset U$. Let $\mathcal{W}_n = \{\Phi_n(K) : K \subset X\text{compact}\}$. Then $(\mathcal{W}_n : n \in \mathbb{N})$ is a sequence of G_δ compact covers, and since X satisfies the selection hypothesis $\mathsf{S}_1(\mathcal{G}_K, \mathcal{G}_{\overline{\Gamma}})$, we can find a sequence $(K_n : n \in \mathbb{N})$ of compact subspaces of X so that every element $x \in X$ belongs to $\overline{\Phi_n(K_n)}$ for all but finitely many $n \in \mathbb{N}$. Put $\mathcal{S}_n = \{V \subset Y : V$ open and there is $U \in \mathcal{U}_n$ such that $\Phi_n(K_n) \times V \subset U\}$. Then $(\mathcal{S}_n : n \in \mathbb{N})$ is a sequence of open covers of Y. As Y is almost Menger, we can choose for each $n \in \mathbb{N}$ finite subsets $\mathcal{V}_n$ of $\mathcal{S}_n$, so that $\bigcup_{n\in\mathbb{N}}\{\overline{V} : V \in \mathcal{V}_n\} = Y$. For every $n \in \mathbb{N}$ and every $V \in \mathcal{V}_n$ we pick $U_V \in \mathcal{U}_n$ so that $\Phi_n(K_n) \times V \subset U_V$. We claim that $\bigcup_{n\in\mathbb{N}}\{\overline{U_V} : V \in \mathcal{V}_n\} = X \times Y$. Indeed, let $(x, y) \in X \times Y$. Then there is $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$, $x \in \overline{\Phi_n(K_n)}$ and there exist $n_1 \geq n_0$ and $V \in \mathcal{V}_{n_1}$ so that $y \in \overline{V}$. That implies $(x, y) \in \overline{\Phi_{n_1}(K_{n_1})} \times \overline{V} \subset \overline{U_V}$, where $U_V \in \mathcal{U}_{n_1}$. □

4. *D*-spaces

The properties of *D*-spaces are explored by several authors (see survey [13]), but there are problems which remained unsolved for many years. It is still unknown whether Lindelöf spaces are *D*-spaces. However, recently Aurichi in [3] showed that Menger spaces are *D*-spaces. We shall prove the analogous statement for almost Menger spaces.

We say that a topological space (X, τ) is a *D-space* (see [11]) if for every function $f : X \to \tau$ such that $x \in f(x)$ which is called neighborhood assignment, there exists a closed and discrete subspace D of X such that $\bigcup_{x\in D} f(x) = X$.

We introduce the following notion:

Definition 4.1. We say that a topological space (X, τ) is an *almost D-space* if for every function $f : X \to \tau$ such that $x \in f(x)$, there exists a closed and discrete subspace D of X such that $\bigcup_{x\in D} \overline{f(x)} = X$.

Problem 4.2. *Find a topological space which is an almost D-space and not a D-space.*

In the proof of the following theorem, we use the notion of game corresponding to the almost Menger property, so we need to explain the notation.

Let $\mathcal{A}$ and $\mathcal{B}$ be classes of open covers of a topological space X. The symbol $\mathsf{G}^{\omega}_{fin}(\mathcal{A}, \mathcal{B})$ denotes the following infinite game: For every $n \in \mathbb{N}$, players *ONE* and *TWO* play an inning. In inning n, *ONE* chooses $A_n \in \mathcal{A}$ and then *TWO* responds with a finite subset B_n of A_n. The play $A_0, B_0, ..., A_n, B_n, ...$ is won by *TWO* if $\{B_n : n \in \mathbb{N}\} \in \mathcal{B}$. Otherwise, *ONE* wins.

In [5] it was proved that in Lindelöf spaces, the almost Menger property is equivalent to the statement that *ONE* does not have a winning strategy in the game $\mathsf{G}^{\omega}_{fin}(\mathcal{O}, \overline{\mathcal{O}})$, so we will use that in order to prove the following statement.

Theorem 4.3. *Let X be a Lindelöf space. If X is almost Menger, then X is an almost D-space.*

Proof. Let $(\mathcal{N}(x) : x \in X)$ be a neighborhood assignment. Then $\mathcal{U} = \{\mathcal{N}(F) : F \subset X$ is finite$\}$, where $\mathcal{N}(F)$ is an open subset of X which contains F, is an open cover of X. By theorem 28 from [5], *ONE* does not have the winning strategy in the game $\mathsf{G}^{\omega}_{fin}(\mathcal{O}, \overline{\mathcal{O}})$. Let the first move of *ONE* be $\mathcal{U} = \{\mathcal{N}(F_0)$: F_0 is a finite subset of $X\}$ and let *TWO* responds with $\mathcal{N}(F_0)$. In the second inning, *ONE* chooses a cover $\mathcal{U}_1 = \{\mathcal{N}(F_1) : F_1 = F \cup F_0, F \cap \mathcal{N}(F_0) = \emptyset, F$ is a finite subset of $X\}$ and *ONE* responds with $\mathcal{N}(F_1)$. In the inning n, *ONE* selects $\mathcal{U}_n = \{\mathcal{N}(F_n) : F_n = F_0 \cup F_1 \cup ... \cup F_{n-1} \cup F, F \cap \bigcup_{i=0}^{n-1} \mathcal{N}(F_i) = \emptyset, F$ is a finite subset of $X\}$ and *TWO* responds with $\mathcal{N}(F_n)$. Considering that *ONE* does not have a winning strategy in this game, there exists a sequence $(F_n : n \in \mathbb{N})$ of finite subsets of X such that $\bigcup_{n\in\mathbb{N}} \overline{\mathcal{N}(F_n)} = X$. Put $D = \bigcup_{n\in\mathbb{N}} F_n$. First note that for every finite subset F of X, $\overline{\mathcal{N}(F)} = \bigcup_{x\in F} \overline{\mathcal{N}(x)}$. By construction, it is obvious that D is closed and discrete and hence we proved our statement. □

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