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Some Bounds for the Pseudocharacter of the Space $C_\alpha(X, Y)$

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Abstract. Let $C_\alpha(X, Y)$ be the set of all continuous functions from X into Y endowed with the set-open topology where α is a hereditarily closed, compact network on X. We obtain that:

i-) $\psi(f, C_\alpha(X, Y)) \leq wac(X) \cdot \sup_{A \in \alpha}(\psi(f(A), Y)) \cdot \sup_{A \in \alpha}(w(f(A)))$

ii-) $\psi(f, C_\alpha(X, Y)) \leq wac(X) \cdot psw_e(f(X), Y)$.

1. Introduction and Terminology

Let X and Y be topological spaces, and let $C(X, Y)$ denote the set of all continuous mappings from X into Y. Let α be a collection of subsets of X. The topology having subbase $\{[A, V] : A \in \alpha$ and V is an open subset of $Y\}$ on the set $C(X, Y)$ is denoted by $C_\alpha(X, Y)$ where
$[A, V] = \{f \in C(X, Y) : f(A) \subseteq V\}$. If α consists of all finite subsets of X, then the set $C(X, Y)$ endowed with that topology is called pointwise convergence topology and denoted by $C_p(X, Y)$.

The cardinality and the closure of a set is denoted by $|A|$ and $cl(A)$, respectively. The restriction of a mapping $f : X \to Y$ to a subset A of X is denoted by $f_{|A}$. $T(X)$ denotes the set of all non-empty open subsets of a topological space X. $ord(x, \mathcal{A})$ is the cardinality of the collection $\{A \in \mathcal{A} : x \in A\}$. Throughout this paper X and Y are regular topological spaces, and α is a hereditarily closed, compact network on the domain space X. (i.e., α is a network on X such that each member of it is compact and each closed subset of a member of it is a member of α.) Without loss of generality, we may assume that α is closed under finite unions. Recall that *the weak α-covering number* of X is defined to be $wac(X) = \min\left\{\left|\beta\right| : \beta \subseteq \alpha \text{ and } \bigcup\beta \text{ is dense in } X\right\}$. The *weight*, *density* and *character* of a space X are denoted by $w(X)$, $d(X)$ and $\chi(X)$, respectively. The *i-weight* of a topological space X, is the least of cardinals $w(Y)$ of the Tychonoff spaces Y which are continuous one-to-one images of X. *The pseudocharacter of a space X at a subset A*, denoted by $\psi(A, X)$, is defined as the smallest cardinal number of the form $|\mathcal{U}|$, where $\mathcal{U}$ is a family of open subsets of X such that $\bigcap\mathcal{U} = A$. If $A = \{x\}$ is a singleton, then we write $\psi(x, X)$ instead of $\psi(\{x\}, X)$. *The pseudocharacter of a space X* is defined to be $\psi(X) = \sup\{\psi(x, X) : x \in X\}$. *The diagonal number* $\Delta(X)$ of a space X is the pseudocharacter of its square $X \times X$ at its diagonal $\Delta_X = \{(x, x) : x \in X\}$.

The pseudocharacter of the space $C(X, Y)$ has been studied, and some remarkable equalities or inequalities was obtained between the pseudocharacter of the space $C(X, Y)$ for certain topologies and some cardinal

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functions on the spaces X and Y. For instance, in [3], the inequalities $\psi(Y) \leq \psi\left(C_p(X,Y)\right) \leq \psi(Y) \cdot d(X)$ and, in [1] and [2], the equalities $\psi\left(C_p(X,\mathbb{R})\right) = d(X) = iw\left(C_p(X,\mathbb{R})\right)$, and in [6], $\psi(C_\alpha(X,\mathbb{R})) = \Delta(C_\alpha(X,\mathbb{R})) = w\alpha c(X)$ were obtained. In this paper, when the range space Y is an arbitrary topological space instead of the space $\mathbb{R}$, we obtained some inequalities between the pseudocharacter of the space $C_\alpha(X,Y)$ at a point f and the weak α-covering number of the domain space X and some cardinal functions on the range space Y.

We assume that all cardinal invariants are at least the first infinite cardinal $\aleph_0$.

Notations and terminology not explained above can be found in [4] and [5].

2. Main Results

First, we give an inequality between the pseudocharacter of a point f in the space $C_\alpha(X,Y)$ and some cardinal functions on spaces X and Y.

Theorem 2.1. *For each $f \in C_\alpha(X,Y)$, we have*

$$\psi(f, C_\alpha(X,Y)) \leq w\alpha c(X) \cdot \sup_{A\in\alpha}(\psi(f(A),Y)) \cdot \sup_{A\in\alpha}(w(f(A)))$$

Proof. Let $w\alpha c(X) \cdot \sup\{\psi(f(A),Y) : A \in \alpha\} \cdot \sup\{w(f(A)) : A \in \alpha\} = \kappa$. The inequality $w\alpha c(X) \leq \kappa$ gives us a subfamily $\beta = \{A_i : i \in I\}$ of α such that $|I| \leq \kappa$ and $X = cl(\bigcup\beta) = cl(\bigcup\{A_i : i \in I\})$. Since $\psi(f(A_i),Y) \leq \kappa$ for each $i \in I$, there exists a family $\mathcal{V}_i$ consisting of open subsets of the space Y such that $|\mathcal{V}_i| \leq \kappa$ and $f(A_i) = \bigcap\{V : V \in \mathcal{V}_i\}$ for each $i \in I$. Since $w(f(A_i)) \leq \kappa$ for each $i \in I$, the subspace has a base $\mathcal{B}_i$ with $|\mathcal{B}_i| \leq \kappa$. For each $i \in I$, let
$\mathcal{H}_i = \left\{\left[A_i \cap f^{-1}(cl(G)), Y\backslash cl(U)\right] : G, U \in \mathcal{B}_i \text{ and } cl(G) \cap cl(U) = \emptyset\right\}$,
$\mathcal{R}_i = \{[A_i, V] : V \in \mathcal{V}_i\}$ and $\mathcal{W} = (\bigcup_{i\in I}\mathcal{R}_i) \cup (\bigcup_{i\in I}\mathcal{H}_i)$.
It is clear that $|\mathcal{W}| \leq \kappa$ and $f \in W$ for each $W \in \mathcal{W}$. Now, we shall prove that $\bigcap\mathcal{W} = \{f\}$. Take a $g \in \bigcap\mathcal{W}$. We claim that $g_{|A_i} = f_{|A_i}$ for each $i \in I$. Assume the contrary. Suppose $g_{|A_j} \neq f_{|A_j}$ for some $j \in I$ that is, we have an $x \in A_j$ such that $f(x) \neq g(x)$. Since $g \in \bigcap\mathcal{W}$ and $f\left(A_j\right) = \bigcap\left\{V : V \in \mathcal{V}_j\right\}$, we have $g(A_j) \subseteq f\left(A_j\right)$. Therefore $g(x) \in f\left(A_j\right)$ and $f(x) \in f\left(A_j\right)$. Since $f(x) \neq g(x)$ and the space Y is regular, there exist G and U in $\mathcal{B}_j$ such that $f(x) \in cl(G)$, $g(x) \in cl(U)$ and $cl(G) \cap cl(U) = \emptyset$. On the other hand, since $\left[A_j \cap f^{-1}(cl(G)), Y\backslash cl(U)\right] \in \mathcal{H}_j$ and $g \in \bigcap\mathcal{W}$, we have $g \in \left[A_j \cap f^{-1}(cl(G)), Y\backslash cl(U)\right]$. But this contradicts to the fact that $g(x) \in cl(U)$. Hence, $g_{|A_i} = f_{|A_i}$ for each $i \in I$, or in other words $g_{|\bigcup_{i\in I}A_i} = f_{|\bigcup_{i\in I}A_i}$. Hausdorffness of the space Y and the equality $X = cl(\bigcup\beta) = cl(\bigcup\{A_i : i \in I\})$ lead us to the fact that $g = f$. Therefore $\bigcap\mathcal{W} = \{f\}$, that is $\psi(f, C_\alpha(X,Y)) \leq \kappa$. □

Recall that a cover $\mathcal{A}$ of a set X is called a *separating cover* if
$\bigcap\{A \in \mathcal{A} : x \in A\} = \{x\}$, for each $x \in X$. Also recall that *the point separating weight psw*(X) of a topological space X is the smallest infinite cardinal κ such that the space X has a separating open cover $\mathcal{V}$ with $ord(x,\mathcal{V}) \leq \kappa$ for each $x \in X$.

Definition 2.2. *Let A be a subset of a topological space (X,τ). We say that the point separating exterior weight $psw_e(A,X) \leq \kappa$, if there exists a subfamily $\mathcal{V} \subseteq \tau$ satisfying $ord(a,\mathcal{V}) \leq \kappa$ and $\bigcap\{V \in \mathcal{V} : a \in V\} = \{a\}$, for each $a \in A$.*

The following lemmas are needed for the second main theorem, and in order to prove them, let us recall the Miščenko's lemma.

Lemma 2.3 (Miščenko's lemma [5]). *Let κ be an infinite cardinal, let X be a set, and let $\mathcal{A}$ be a collection of subsets of X such that $ord(x,\mathcal{A}) \leq \kappa$ for all $x \in X$. Then the number of finite minimal covers of X by elements of $\mathcal{A}$ is at most κ.*

Lemma 2.4. *Let Z be subspace of the space X such that* $psw_e(Z, X) \leq \kappa$. *Then* $\psi(K, X) \leq \kappa$ *for each compact subset K of Z.*

Proof. Let $\mathcal{V}$ be a family of open subsets of X satisfying $ord(z, \mathcal{V}) \leq \kappa$ and $\bigcap \{V \in \mathcal{V} : z \in V\} = \{z\}$, for each $z \in Z$. Let K be any compact subspace of Z and let
$\mu = \{\mathcal{W} : \mathcal{W} \subseteq \mathcal{V}$ and $\mathcal{W}$ is a minimal finite open cover for $K\}$. By Miščenko's lemma, we have $|\mu| \leq \kappa$.

Define the family $\mathcal{O} = \{\bigcup_{W \in \mathcal{W}} W : \mathcal{W} \in \mu\}$. It is clear that $|\mathcal{O}| \leq \kappa$ and it can be easily seen that $\bigcap \mathcal{O} = K$. Hence $\psi(K, X) \leq \kappa$. □

Lemma 2.5. *Let Z be subspace of the space X such that* $psw_e(Z, X) \leq \kappa$. *Then we have* $w(K) \leq \kappa$ *for each compact subset K of Z.*

Proof. Let K be a compact subset of Z. Clearly, $psw(K) \leq psw(Z) \leq psw_e(Z, X)$. The compactness of K leads us to the fact that $w(K) = psw(K)$. [in [5], Ch. 1, Theorem 7.4]. Hence the claim. □

Now, we are ready to give another bound for the pseudocharacter of the space $C_\alpha(X, Y)$ at a point f.

Theorem 2.6. *For each* $f \in C_\alpha(X, Y)$, *we have*

$$\psi(f, C_\alpha(X, Y)) \leq w\alpha c(X) \cdot psw_e(f(X), Y).$$

Proof. Let $w\alpha c(X) \cdot psw_e(f(X), Y) = \kappa$, and let $\beta = \{A_i : i \in I\}$ be a subfamily of α such that $cl(\bigcup \beta) = X$ and $|I| \leq \kappa$. The compactness of A_i for each $i \in I$ and the inequality $psw_e(f(X), Y) \leq \kappa$ lead us to the facts that $\psi(f(A_i), Y) \leq \kappa$ and $w(A_i) \leq \kappa$ for each $i \in I$, by lemmas.2.4 and 2.5. Therefore, by Theorem 2.1, we have $\psi(f, C_\alpha(X, Y)) \leq \kappa$. □

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