Filomat 29:1 (2015), 133–141
DOI 10.2298/FIL1501133S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On the Topology of Fuzzy Metric Type Spaces

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Abstract. In this paper, first we introduce the concept of fuzzy metric type space and consider the topology induced by a fuzzy metric type. Next, we consider the complete fuzzy metric type spaces and prove that any G_δ set in a complete metric type space is a complete fuzzy metrizable type space.

1. Preliminaries

The theory of fuzzy sets was introduced by Zadeh in 1965 [7]. After the pioneering work of Zadeh, there has been a great effort to obtain fuzzy analogues of classical theories. Among other fields, a progressive developments is made in the field of fuzzy topology. One of the most important problems in fuzzy topology is to obtain an appropriate concept of fuzzy metric space. This problem has been investigated by many authors from different points of view. In particular, George and Veeramani [1] have introduced and studied a notion of fuzzy metric space. Furthermore, the class of topological spaces that are fuzzy metrizable agrees with the class of metrizable topological spaces (see [1] and [2]). This result permits Gregori and Romaguera to restate some classical theorems on metric completeness and metric (pre)compactness in the realm of fuzzy metric spaces [2] ,[3] and [4].

In this paper we introduce the concept of fuzzy metric type space which is a generalization of fuzzy metric space introduced by George and Veeramani [1]. In this paper we prove that any G_δ set in a complete metric type space is a topologically complete fuzzy metrizable type space (Alexandroff Theorem).

Definition 1.1. A 3-tuple $(X, M, *)$ is called a fuzzy metric type space if X is an arbitrary (non-empty) set, $*$ is a continuous t-norm, and M is a fuzzy set on $X^2 \times (0, \infty)$, satisfying the following conditions for each $x, y, z \in X$ and $t, s > 0$,

(1) $M(x, y, t) > 0$,
(2) $M(x, y, t) = 1$ if and only if $x = y$,
(3) $M(x, y, t) = M(y, x, t)$,
(4) $M(x, y, t) * M(y, z, s) \leq M(x, z, K(t + s))$ for some constant $K \geq 1.$,
(5) $M(x, y, .) : (0, \infty) \longrightarrow [0, 1]$ is continuous.

$M(x, y, t)$ is considered as the degree of nearness of x and y with respect to t. The axiom (1) is justified because in the same way that a classical metric type cannot take the value ∞ then M cannot take the value 0. The axiom (2) is equivalent to the following:

2010 *Mathematics Subject Classification*. Primary: 54E50.
Keywords. fuzzy metric type spaces; complete fuzzy metrizable type space; Alexandroff theorem.
Received: 8 October 2014; Accepted: 15 December 2014
Communicated by Dragan Djurčić
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$$M(x,x,t) = 1 \text{ for all } x \in X \text{ and } t > 0, \text{ and } M(x,y,t) < 1 \text{ for all } x \neq y \text{ and } t > 0.$$

The axiom (2) gives the idea that when $x = y$ the degree of nearness of x and y is perfect, or simply 1, and then $M(x,x,t) = 1$ for each $x \in X$ and for each $t > 0$. Finally, in (5) we assume that the variable t be have nicely, that is assume that for fixed x and y, $t \to M(x,y,t)$ is a continuous function.

Remark 1.2. The space L_p $(0 < p < 1)$ of all real functions $f(x)$, $x \in [0,1]$ such that $\int_0^1 |f(x)|^p dx < \infty$, is a type metric space. Define

$$d(f,g) = (\int_0^1 |f(x) - g(x)|^p dx)^{\frac{1}{p}},$$

for each $f, g \in L_p$. Then d is a metric type space with $K = 2^{\frac{1}{p}}$.

Example 1.3. Let X be the set of Lebesgue measurable functions on $[0,1]$ such that $\int_0^1 |f(x)|^p dx < \infty$, where $p > 0$ is a real number. Define

$$M(f,g,t) = \frac{t}{t + (\int_0^1 |f(x) - g(x)|^p dx)^{\frac{1}{p}}}$$

for $t > 0$ and $x, y \in X$. Then by Remark 1.2, $(X, M, .)$ is fuzzy metric type space with $K = 2^{\frac{1}{p}}$.

Example 1.4. Let (X,D) be a metric type space with constant $K \geq 1$. Define

$$M(x,y,t) = \frac{t}{t + D(x,y)}$$

for $t > 0$ and $x, y \in X$. Then $(X, M, .)$ is a fuzzy metric type space with constant K. (1)–(3) and (5) are obvious and we show (3).

$$\begin{aligned}
M(x,z,t) \cdot M(z,y,s) &= \frac{t}{t + D(x,z)} . \frac{s}{s + D(z,y)} \\
&= \frac{1}{1 + \frac{D(x,z)}{t}} . \frac{1}{1 + \frac{D(z,y)}{s}} \\
&\leq \frac{1}{1 + \frac{D(x,z)}{(t+s)}} . \frac{1}{1 + \frac{D(z,y)}{(t+s)}} \\
&\leq \frac{1}{1 + \frac{(D(x,z)+D(z,y))}{(t+s)}} \\
&\leq \frac{1}{1 + \frac{D(x,z)}{K(t+s)}} \\
&= \frac{K(t+s)}{K(t+s) + D(x,y)} \\
&= M(x,y,K(t+s)).
\end{aligned}$$

Example 1.5. Let (X,D) be a metric type spaces with constant $K \geq 1$. Define

$$M(x,y,t) = e^{\frac{-(D(x,y))}{t}}$$

for $t > 0$ and $x, y \in X$. Then $(X, M, \cdot)$ is a fuzzy metric type space with constant K. (1)–(3) and (5) are obvious and we show (3).

$$\begin{aligned}
M(x,z,t) \cdot M(z,y,s) &= \frac{t}{t + D(x,z)} . \frac{s}{s + D(z,y)} \\
&= e^{\frac{-(D(x,z))}{t}} . e^{\frac{-(D(z,y))}{s}} \\
&\leq e^{-(\frac{D(x,y)}{K(t+s)})} \\
&= M(x,y,K(t+s)).
\end{aligned}$$

Remark 1.6. Let (X, d) be a metric space and $D(x, y) = (d(x, y))^p$, where $p > 1$ is a real number. Then D is a metric type space with $K = 2^{p-1}$. The triangle inequality follow easily from the convexity of the function $f(x) = x^p$ $(x > 0)$

Example 1.7. Let X be a nonempty set. Define

$$M(x, y, t) = e^{-\frac{|x-y|^p}{t}}$$

for $t > 0$ and $x, y \in X$. Then by Example 1.5 and Remark 1.6, $(X, M, \cdot)$ is a fuzzy metric type space with $K = 2^{p-1}$.

Note that the above examples show that, every fuzzy metric type is a fuzzy metric but the converse is not true generally.

Let $(X, M, *)$ be a fuzzy metric type space . For $t > 0$, the open ball $B_x(r, t)$ with center $x \in X$ and radius $0 < r < 1$ is defined by

$$B_x(r, t) = \{y \in X : M(x, y, t) > 1 - r\}.$$

Proposition 1.8. *Let $(X, M, *)$ be a fuzzy metric type space. Define*

$$\tau_M = \{A \subset M : x \in A \iff \exists\, t > 0, \; \& \; 0 < r < 1 \,, \text{ such that } B_x(r, t) \subset A\}$$

Then τ_M is a topology on X.

Proof. (i) Clearly $\emptyset$ and X belong to τ_M.
(ii) Let $A_1, A_2, ..., A_i \in \tau_M$, and put

$$U = \cup_{i \in I} A_i$$

We shall show that $U \in \tau_M$. If $a \in U$, then $a \in \cup_{i \in I} A_i$ which implies that $a \in A_i$ for some $i \in I$. Since $A_i \in \tau_M$, there exists $0 < r < 1$, $t > 0$, such that $B_a(r, t) \subset A_i$. Hence

$$B_a(r, t) \subset A_i \subset \cup_{i \in I} A_i = U.$$

This shows that $U \in \tau_F$.
(iii) Let $A_1, A_2, ..., A_n \in \tau_M$, and $U = \cap_{i=1}^n A_i$. We shall show that $U \in \tau_F$. Let $a \in U$. Then $a \in A_i$ for all $i \in I$. Hence for each $i \in I$, there exists $0 < r_i < 1$, $t_i > 0$ such that $B_a(r_i, t_i) \subset A_i$. Let

$$r = \min\{r_i \,, \; i \in I\}$$

and

$$t = \max\{t_i \,, \; i \in I\}$$

Thus $r \leq r_i$ for all $i \in I$, $1 - r \geq 1 - r_i$ for all $i \in I$. Also, $t > 0$. So, $B_a(r, t) \subseteq A_i$ for all $i \in I$. Therefore

$$B_a(r, t) \subset \cup_{i \in I} A_i = U.$$

This shows that $U \in \tau_M$. □

Please note that in the above topology X is the set of points and the fuzzy metric type is fuzzy evaluation of two points of X while, for example, in the topology introduced by Yue and Shi [6] the fuzzy metric is fuzzy evaluation of two fuzzy sets.

Let $(X, M, *)$ be a fuzzy metric type space. A sequence $\{x_n\}$ in X converges to x if and only if $M(x_n, x, t) \to 1$ as $n \to \infty$, for each $t > 0$. It is called a Cauchy sequence if for each $0 < \varepsilon < 1$ and $t > 0$, there exits $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for each $n, m \geq n_0$. The fuzzy metric type space $(X, M, *)$ is said to be complete if every Cauchy sequence is convergent. A subset A of X is said to be F-bounded if there exists $t > 0$ and $0 < r < 1$ such that $M(x, y, t) > 1 - r$ for all $x, y \in A$.

Proposition 1.9. *Every fuzzy metric type space with constant K is Hausdorff.*

Proof. Let $(X, M, *)$ be a fuzzy metric type space. Let x, y be two distinct points of X. Then $0 < M(x, y, t) < 1$. Let $M(x, y, t) = r$, for some r, $0 < r < 1$. For each r_0, $r < r_0 < 1$, we can find an r_1 such that $r_1 * r_1 \geq r_0$. Now consider the open balls $B_x(1 - r_1, \frac{t}{2K})$ and $B_y\left(1 - r_1, \frac{t}{2K}\right)$. Clearly

$$B_x\left(1 - r_1, \frac{t}{2K}\right) \cap B_y\left(1 - r_1, \frac{t}{2K}\right) = \emptyset$$

Otherwise, if there exists $z \in B_x\left(1 - r_1, \frac{t}{2K}\right) \cap B_y\left(1 - r_1, \frac{t}{2K}\right)$. Then

$$\begin{aligned} r &= M(x, y, t) \\ &\geq M\left(x, z, \frac{t}{2K}\right) * M\left(z, y, \frac{t}{2K}\right) \\ &\geq r_1 * r_1 \geq r_0 \\ &> r \end{aligned}$$

which is a contradiction. Therefore $(X, M, *)$ is Hausdorff. □

Proposition 1.10. *Let (X, D) be a metric type space and $M(x, y, t) = \frac{t}{t+D(x,y)}$ be the corresponding standard fuzzy metric type on X. Then the topology τ_D induced by the metric D and the topology τ_M induced by the M are the same. That is, $\tau_D = \tau_M$.*

Proof. Suppose that $A \in \tau_D$. Then there exists $\epsilon > 0$ such that $B(x, \epsilon) \subset A$, for every $x \in A$. For a fixed $t > 0$, we obtain that

$$M(x, y, t) = \frac{t}{t + D(x, y)} > \frac{t}{t + \epsilon}.$$

Let

$$1 - r = \frac{t}{t + \epsilon}.$$

Then

$$M(x, y, t) > 1 - r.$$

It follows that, $B_x(r, t) \subset A$. Hence $A \in \tau_M$. This shows that $\tau_D \subseteq \tau_M$. Conversely, suppose that $A \in \tau_M$. Then there exists $0 < r < 1$ and $t > 0$ such that $B_x(r, t) \subset A$ for every $x \in A$. We obtain that

$$M(x, y, t) = \frac{t}{t + D(x, y)} > 1 - r$$

$$t > (1 - r)t + (1 - r)D(x, y)$$

$$D(x, y) < \frac{rt}{1 - r}$$

Let $\epsilon = \frac{rt}{1-r}$ where $0 < \epsilon < 1$. Then $D(x, y) < \epsilon$, and therefore $B(x, \epsilon) \subset A$. Hence $A \in \tau_D$. This implies that $\tau_M \subseteq \tau_D$. Therefore $\tau_D = \tau_M$. □

Proposition 1.11. *Every compact subset S of a fuzzy metric type space $(X, M, *)$ is F-bounded.*

Proof. Given S a compact subset of X. Fix $t > 0$ and $0 < r < 1$. Consider an open *cover*$\{B_x(r, t) : x \in X\}$ of S. Since S is compact, there exists $x_1, x_2, ..., x_n \in X$ such that

$$S \subseteq \cup_{i=1}^{n} B_{x_i}(r, t).$$

Let $x, y \in X$. Then $x \in B_{x_i}(r, t)$ and $y \in B_{x_j}(r, t)$ for some i, j. Therefore $M(x, x_i, t) > 1 - r$ and $M(y, x_j, t) > 1 - r$. Now, let $\alpha = \min\{M(x_i, x_j, t) : 1 \leq i, j \leq n\}$. Then $\alpha > 0$. Now

$$M(x, y, K(2Kt + t)) \geq M(x, x_i, t) * M(x_i, x_j, t) * M(x_j, y, t) \geq (1 - r) * (1 - r) * \alpha,$$

where K is the constant to the condition (3). Taking $t' = K(2Kt + t)$ and $(1 - r) * (1 - r) * \alpha > 1 - s$, $0 < s < 1$, we have $M(x, y, t') > 1 - s$ for all $x, y \in X$. Hence S is F-bounded. □

Proposition 1.12. *Let $(X, M, *)$ be a fuzzy metric type space and τ_M be the topology induced by fuzzy metric type. Then for any nonempty subset $S \subset X$ we have*

1. *S is closed if and only if for any sequence $\{x_n\}$ in X which converges to x, we have $x \in S$;*
2. *if we define $\bar{S}$ to be the intersection of all closed subset of X which contain S, then for any $x \in \bar{S}$ and for any $0 < r < 1$ and $t > 0$, we have $B_x(r, t) \cap S \neq \emptyset$.*

Proof. 1. Assume that S is closed and let $\{x_n\}$ be a sequence in S such that $\lim_{n\to\infty} x_n = x$. Let us prove that $x \in S$. Assume not, i.e. $x \notin S$. Since S is closed, then there exists $0 < r < 1$ and $t > 0$ such that $B_x(r, t) \cap X = \emptyset$. Since $\{x_n\}$ converges to x, then there exists $N \geq 1$ such that for any $n \geq N$ we have $x_n \in B_x(r, t)$. Hence $x_n \in B_x(r, t) \cap S$, which leads to a contradiction. Conversely assume that for any sequence $\{x_n\}$ in S which converges to x, we have $x \in S$. Let us prove that S is closed. Let $x \notin S$. We need to prove that there exists $0 < r < 1$ and $t > 0$ such that $B_x(r, t) \cap S = \emptyset$. Assume not, i.e. for any $0 < r < 1$ and $t > 0$, we have $B_x(r, t) \cap S \notin \emptyset$. So for any $n \geq 1$, choose $x_n \in B_x(\frac{1}{n}, t) \cap S$. Clearly we have $\{x_n\}$ converges to x. Our assumption on S implies $x \in S$, a contradiction.

2. Clearly $\bar{S}$ is the smallest closed subset which contains S. Set

$$S^* = \{x \in X;\ for\, any\, \varepsilon > 0,\ there\, exists\, a \in S\, such\, that\ :\ M(x, a, t) > 1 - r\}$$

We have $S \subset S^*$. Next we prove that S^* is closed. For this we use property 1. Let $\{x_n\}$ be a sequence in S^* such that $\{x_n\}$ converges to x. Let $0 < r < 1$ and $t > 0$. Since $\{x_n\}$ converges to x, there exists $N \geq 1$ such that for any $n \geq N$ we have

$$M\left(x, x_n, \frac{t}{2K}\right) > 1 - r\,,$$

where K is the constant. Let $r_0 = M(x, x_n, \frac{t}{2K}) > 1 - r$. Since $r_0 > 1 - r$, we can find an s, $0 < s < 1$, such that $r_0 > 1 - s > 1 - r_0$. Now for a given r_0 and s such that $r_0 > 1 - s$ we can find r_1, $0 < r_1 < 1$, such that

$$r_0 * (1 - r_1) \geq 1 - s.$$

Now since $x_n \in S^*$, there exists $a \in X$ such that

$$M\left(x_n, a, \frac{t}{2K}\right) > 1 - r_1.$$

Hence

$$M(x, a, t) \geq M\left(x, x_n, \frac{t}{2K}\right) * M\left(x_n, a, \frac{t}{2K}\right) > r_0 * (1 - r_1) \geq 1 - s > 1 - r,$$

which implies $x \in S^*$. Therefore S^* is closed and contains S. The definition of $\bar{S} \subset S^*$, which implies the conclusion of 2. □

Note that, every compact subset of a Hausdorff topological space is closed.

Proposition 1.13. *Let $(X, M, *)$ be a fuzzy metric type space and τ_M be the topology induced by fuzzy metric type. Let S be a nonempty subset of X. The following properties are equivalent*

(a) *S is compact.*

(b) *For any sequence $\{x_n\}$ in S, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges, and if $\{x_{n_k}\}$ converges to x then $x \in S$.*

Proof. Assume that S is a nonempty compact subset of X. It is easy to see that any decreasing sequence of nonempty closed subsets of S have a nonempty intersection. Let $\{x_n\}$ be a sequence in S. Set $C_n = \{x_m : m \geq n\}$. Then we have $\bigcap_{n\geq 1} \bar{C}_n \neq \emptyset$. Let $x \in \bigcap_{n\geq 1} \bar{C}_n$. Then for $0 < r < 1$, $t > 0$ and for any $n \geq 1$, there exists $m_n \geq n$ such that $M(x, x_{m_n}, t) > 1 - r$. This clearly implies the existence of a subsequence of $\{x_n\}$ which converges to x. Since S is closed, then we must have $x \in S$.

Conversely let S be a nonempty subset of X such that the conclusion of (b) is true. Let us prove that S is compact. First note that for any $0 < r < 1$, $t > 0$, there exists $x_1, x_2, ..., x_n \in A$ such that

$$S \subseteq \bigcup_{i=1}^{n} B_{x_i}(r, t).$$

Assume not, then there exists $0 < r_0 < 1$, such that for any finite number of points $x_1, x_2, ..., x_n \in X$, we have

$$S \nsubseteq \bigcup_{i=1}^{n} B_{x_i}(r_0, t).$$

Fix $x_1 \in X$. Since $S \nsubseteq B_{x_1}(r_0, t)$, there exists $x_2 \in S \setminus B_{x_1}(r_0, t)$. By induction we build a sequence $\{x_n\}$ such that

$$x_{n+1} \in S \setminus (B_{x_1}(r_0, t) \cup ... \cup B_{x_n}(r_0, t))$$

for all $n \geq 1$. Clearly we have $M(x_n, x_m, t) < 1 - r_0$, for all $n, m \geq 1$, with $n \neq m$. This condition implies that no subsequence of $\{x_n\}$ will be Cauchy or convergent. This contradicts our assumption on X. Next let $\{O_\alpha\}_{\alpha \in J}$ be an open cover of S. Let us prove that only finitely many O_α cover S. Fix $t > 0$, First note that there exists $0 < r_0 < 1$ such that for any $x \in S$, there exists $\alpha \in J$ such that $B_x(r_0, t) \subset O_\alpha$. Assume not, then for any $0 < r < 1$, there exists $x_r \in X$ such that for any $\alpha \in J$, we have$B_{x_r}(r, t) \nsubseteq O_\alpha$. In particular, for any $n \geq 1$, there exists $x_n \in X$ such that for any $\alpha \in J$, we have $B_{x_n}\left(\frac{1}{n}, t\right) \nsubseteq O_\alpha$. By our assumption on S, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges to some point $x \in X$. Since the family $\{O_\alpha\}_{\alpha \in J}$ covers X, there exists $\alpha_0 \in J$ such that $x \in O_{\alpha_0}$. Since O_{α_0} is open, there exists $0 < r_0 < 1$, and $t_0 > 0$ such that $B_x(r_0, t) \subset O_{\alpha_0}$. Fix $t > 0$ and let $t_1 = tK$, for any $n_K \geq 1$ and $a \in B_{x_{n_K}}(\frac{1}{n_K}, t) = B_{x_{n_K}}(\frac{1}{n_K}, \frac{t_1}{K})$, we have

$$M(x, a, t_0) \geq M\left(x, x_{n_k}, \frac{t_0 - t_1}{K}\right) * M\left(x_{n_k}, a, \frac{t_1}{K}\right) > M\left(x, x_{n_k}, \frac{t_0 - t_1}{K}), 1 - \frac{1}{n_k}\right)$$

for n_k large enough, we will get $FM(x, a, t) > 1 - r_0$ for any $a \in B_{x_{n_k}}(\frac{1}{n_k}, t)$. In the other words, we have $B_{x_{n_k}}(\frac{1}{n_k}, t) \subset B_x(r_0, t_0)$, which implies

$$B_{x_{n_k}}\left(\frac{1}{n_k}, t\right) \subset O_{\alpha_0}.$$

This is in clear contradiction with the way the sequence $\{x_n\}$ was constructed. Therefore there exists $0 < r_0 < 1$ such that for any $x \in S$, there exists $\alpha \in J$ such that $B_x(r_0, t) \subset O_\alpha$. For such r_0, there exist $x_1, x_2, ..., x_n \in X$ such that

$$S \subset B_{x_1}(r_0, t) \cup ... \cup B_{x_n}(r_0, t)$$

But for any $i = 1, ..., n$, there exists $\alpha \in J$ such that $B_{x_i}(r_0, t) \subset O_{\alpha_i}$, i.e. $S \subset O_{\alpha_1} \cup ... \cup O_{\alpha_n}$. This completes the proof that S is compact. □

Definition 1.14. Let $(X, M, *)$ be a fuzzy metric type space. The subset S of X is called sequentially compact if and only if for any sequence $\{x_n\}$ in S, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges, and $\lim_{n_k \to \infty} \in X$. Also S is called totally bounded if for any $0 < r < 1$ and $t > 0$, there exist $x_1, x_2, ..., x_n \in X$ such that

$$S \subset B_{x_1}(r, t) \cup ... \cup B_{x_n}(r, t).$$

Theorem 1.15. *In a fuzzy metric type space every compact set is closed and F-bounded.*

Proof. From Propositions 1.9 and 1.11 we get the result. □

Two next corollaries get from last results about fuzzy metric type spaces.

Theorem 1.16. *In a fuzzy metric type space every compact set is complete.*

Corollary 1.17. *Every closed subset of a complete fuzzy metric type space is complete.*

2. Main Result

Lemma 2.1. *Let $(X, M, *)$ be a fuzzy metric type space with $K \geq 1$ and let $\lambda \in [0, 1)$ then there exists a fuzzy metric type m on X such that $m(x, y, t) \geq \lambda$, for each $x, y \in X$ and $t > 0$ and m and M induce the same topology on X.*

Proof. We define $m(x, y, t) = max\{\lambda, M(x, y, t)\}$. We claim that m is fuzzy metric type on X. The properties of (1),(2),(3) and (5)are immediate from the definition. For triangle inequality, suppose that $x, y, z \in X$ and $t, s > 0$. Then $m(x, z, K(t + s)) \geq \lambda$ and so $m(x, z, K(t + s)) \geq m(x, y, t) * m(y, z, s)$ when either $m(x, y, t) = \lambda$ or $m(y, z, s) = \lambda$. The only remaining case is when $m(x, y, t) = M(x, y, t) > \lambda$ and $m(y, z, s) = M(y, z, s) > \lambda$. But $M(x, z, K(t+s)) \geq M(x, y, t)*M(y, z, s)$ and $m(x, z, t+s) \geq M(x, z, t+s)$ and so $m(x, z, K(t+s)) \geq m(x, y, t)*m(y, z, s)$. Thus m is a fuzzy metric type on X. It only remains to show that the topology induced by m is the same as that induced by M. But we have $m(x_n, x, t) \longrightarrow 1$ if and only if $\{\lambda, M(x_n, x, t)\} \longrightarrow 1$ if and only if $M(x_n, x, t) \longrightarrow 1$, for each $t > 0$, and we are done. □

The fuzzy metric type m in above lemma is said to be bounded by λ.

Definition 2.2. Let $(X, M, *)$ be a fuzzy metric type space, $x \in X$ and $\phi \neq A \subseteq X$. We define

$$D(x, A, t) = \sup\{M(x, y, t) : y \in A\} \quad (t > 0).$$

Note that $D(x, A, t)$ is a degree of closeness of x to A at t.

Definition 2.3. A topological space is called a (topologically complete) fuzzy metrizable type space if there exists a (topologically complete) fuzzy metric type inducing the given topology on it.

Example 2.4. Let $X = (0, 1]$. The fuzzy metric type space $(X, M, \cdot)$ where $M(x, y, t) = \frac{t}{t+(x-y)^2}$ is not complete because the Cauchy sequence $\{1/n\}$ in this space is not convergent. Now, if we consider triple $(X, m, .)$ where $m(x, y, t) = \frac{t}{t+(x-y)^2+(\frac{1}{x}-\frac{1}{y})^2}$. It is straightforward to show that $(X, m, \cdot)$ is a fuzzy metric type space, and that is complete. Since, x_n tend to x with respect to fuzzy metric type M if and only if $(x_n - x)^2 \longrightarrow 0$ if and only if x_n tend to x with respect to fuzzy metric type m, then M and m are equivalent fuzzy metrics types. Hence the fuzzy metric type space $(X, M, \cdot)$ is topologically complete fuzzy metrizable type.

Lemma 2.5. *Fuzzy metrizability type is preserved under countable Cartesian product.*

Proof. Without loss of generality we may assume that the index set is $\mathbb{N}$. Let $\{(X_n, m_n, *) : n \in \mathbb{N}\}$ be a collection of fuzzy metrizable type spaces. Let τ_n be the topology induced by m_n on X_n for $n \in \mathbb{N}$ and let (X, τ) be the Cartesian product of $\{(X_n, \tau_n) : n \in \mathbb{N}\}$ with product topology. We have to prove that there is a fuzzy metric type m on X which induces the topology τ. By the above lemma, we may suppose that m_n is bounded by $1 - \varepsilon^{(n)}$, $(\varepsilon^{(n)} = \overbrace{\varepsilon * \varepsilon * \cdots * \varepsilon}^{n}, \varepsilon \in (0, 1))$ i.e. $m_n(x_n, y_n, t) = max\{1 - \varepsilon^{(n)}, M(x_n, y_n, t)\}$. Points of $X = \prod_{n \in N} X_n$ are denoted as sequences $x = \{x_n\}$ with $x_n \in X_n$ for $n \in \mathbb{N}$. Define $m(x, y, t) = \prod_{n=1}^{\infty} m_n(x_n, y_n, t)$, for each $x, y \in X$ and $t > 0$, $(\prod_{n=1}^{m} a_n = a_1 * a_2 * \cdots * a_m)$. First note that m is well defined since $a_i = \prod_{n=1}^{i}(1 - \varepsilon^{(n)})$ is decreasing and bounded then converges to $\alpha \in (0, 1)$. Also m is a fuzzy metric type on X because each m_n is a fuzzy metric type. Let $\mathcal{U}$ be the topology induced by fuzzy metric type m. We claim that $\mathcal{U}$ coincides with τ. If $G \in \mathcal{U}$ and $x = \{x_n\} \in G$, then there exists $0 < r < 1$ and $t > 0$ such that $B(x, r, t) \subset G$. For each $0 < r < 1$, we can find a sequence $\{\delta_n\}$ in $(0, 1)$ and a positive integer N_0 such that

$$\prod_{n=1}^{N_0}(1 - \delta_n) * \prod_{n=N_0+1}^{\infty}(1 - \varepsilon^{(n)}) > 1 - r.$$

For each $n = 1, 2, \cdots, N_0$, let $V_n = B(x_n, \delta_n, t)$, where the ball is with respect to fuzzy metric type m_n. Let $V_n = X_n$ for $n > N_0$. Put $V = \prod_{n \in N} V_n$, then $x \in V$ and V is an open set in the product topology τ on X.

Furthermore $V \subset B(x, r, t)$, since for each $y \in V$

$$\begin{aligned} m(x,y,t) &= \prod_{n=1}^{\infty} m_n(x_n, y_n, t) \\ &= \prod_{n=1}^{N_0} m_n(x_n, y_n, t) * \prod_{n=N_0+1}^{\infty} m_n(x_n, y_n, t) \\ &\geq \prod_{n=1}^{N_0}(1-\delta_n) * \prod_{n=N_0+1}^{\infty}(1-\varepsilon^{(n)}) \\ &> 1-r. \end{aligned}$$

Hence $V \subset B(x,r,t) \subset G$. Therefore G is open in the product topology. Conversely suppose G is open in the product topology and let $x = \{x_n\} \in G$. Choose a standard basic open set V such that $x \in V$ and $V \subset G$. Let $V = \prod_{n \in N} V_n$, where each V_n is open in X_n and $V_n = X_n$ for all $n > N_0$. For $n = 1, 2, \cdots, N_0$, let $r_n = D_n(x_n, X_n - V_n, t)$, if $X_n \neq V_n$, and $r_n = \varepsilon^{(n)}$, otherwise. Let $r = min\{r_1, r_2, \cdots, r_{N_0}\}$. We claim that $B(x,r,t) \subset V$. If $y = \{y_n\} \in B(x,r,t)$, then $m(x,y,t) = \prod_{n=1}^{\infty} m_n(x_n, y_n, t) > 1-r$ and so $m_n(x_n, y_n, t) > 1-r \geq 1-r_n$ for each $n = 1, 2, \cdots, N_0$. Then $y_n \in V_n$, for $n = 1, 2, \cdots, N_0$. Also for $n > N_0$, $y_n \in V_n = X_n$. Hence $y \in V$ and so $B(x,r,t) \subset V \subset G$. Therefore G is open with respect to the fuzzy metric type topology and $\tau \subset \mathcal{U}$. Hence τ and $\mathcal{U}$ coincide. □

Theorem 2.6. *An open subspace of a complete fuzzy metrizable type space is a complete fuzzy metrizable type space.*

Proof. Let $(X, M, *)$ be a complete fuzzy metric type space and G an open subspace of X. If the restriction of M to G is not complete we can replace M on G by other fuzzy metric type as follows. Define $f : G \times (0, \infty) \longrightarrow \mathbb{R}^+$ by $f(x,t) = \frac{1}{1-D(x,X-G,t)}$ (f is undefined if $X - G$ is empty, but then there is nothing to prove.) Fix an arbitrary $s > 0$ and for $x, y \in G$ define

$$m(x,y,t) = M(x,y,t) * M(f(x,s), f(y,s), t),$$

for each $t > 0$. We claim that m is fuzzy metric type on G. The properties (1),(2),(3) and (5) are immediate from the definition. For triangle inequality, suppose that $x, y, z \in G$ and $t, s, u > 0$, then

$$\begin{aligned} & m(x,y,t) * m(y,z,u) = \\ & (M(x,y,t) * M(f(x,s), f(y,s), t)) * (M(y,z,u) * M(f(y,s), f(z,s), u)) \\ = \ & (M(x,y,t) * M(y,z,u)) * (M(f(x,s), f(y,s), t) * M(f(y,s), f(z,s), u)) \\ \leq \ & M(x,z,K(t+u)) * M(f(x,s), f(z,s), K(t+u)) = m(x,z,K(t+u)). \end{aligned}$$

We show that m and M are equivalent fuzzy metrics type on G. We do this by showing that $m(x_n, x, t) \longrightarrow 1$ if and only if $M(x_n, x, t) \longrightarrow 1$. Since $m(x,y,t) \leq M(x,y,t)$ for all $x, y \in G$ and $t > 0$, $M(x_n, x, t) \longrightarrow 1$ whenever $m(x_n, x, t) \longrightarrow 1$. To prove the converse, let $M(x_n, x, t) \longrightarrow 1$, since M is continuous function on $X \times X \times (0, \infty)$, then

$$\begin{aligned} \lim_n D(x_n, X-G, s) &= \lim_n(\sup\{M(x_n, y, s) : y \in G\}) \\ &\geq \lim_n M(x_n, y, s) \\ &= M(x, y, s). \end{aligned}$$

Therefore $\lim_n D(x_n, X-G, s) \geq D(x, X-G, s)$. On the other hand, there exists a $y_0 \in X - G$ and $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ we have

$$D(x_n, X-G, s) * (1 - \frac{1}{n}) \leq M(x_n, y_0, s).$$

Then $\lim_n D(x_n, X-G, s) \leq M(x, y_0, s) \leq \sup\{M(x,y,s) : y \in X-G\} = D(x, X-G, s)$. Therefore $\lim_n D(x_n, X - G, s) = D(x, X-G, s)$. This implies $M(f(x_n, s), f(x,s), t) \longrightarrow 1$. Hence $m(x_n, x, t) \longrightarrow 1$. Therefore m and M are equivalent. Next we show that m is a complete fuzzy metric type. Suppose that $\{x_n\}$ is a Cauchy sequence in G with respect to m. Since for each $m, n \in \mathbb{N}$, and $t > 0$ $m(x_m, x_n, t) \leq M(x_m, x_n, t)$, therefore $\{x_n\}$ is also a Cauchy sequence with respect to M. By completeness of $(X, M, *)$, $\{x_n\}$ converges to point p in X. We claim that $p \in G$. Assume otherwise, then for each $n \in \mathbb{N}$, if $p \in X - G$ and $M(x_n, p, t) \leq D(x_n, X - G, t)$, then

$$1 - M(x_n, p, t) \geq 1 - D(x_n, X - G, t) > 0,$$

Therefore

$$\frac{1}{1 - D(x_n, X - G, t)} \geq \frac{1}{1 - M(x_n, p, t)},$$

That is

$$f(x_n, t) \geq \frac{1}{1 - M(x_n, p, t)},$$

for each $t > 0$. Therefore as $n \longrightarrow \infty$, for every $t > 0$ we get $f(x_n, t) \longrightarrow \infty$. In particular, $f(x_n, s) \longrightarrow \infty$. On the other hand, $M(f(x_n, s), f(x_m, s), t) \geq m(x_m, x_n, t)$, for every $m, n \in \mathbb{N}$, that is $\{f(x_n, s)\}$ is an F-bounded sequence. This contradiction shows that $p \in G$. Hence $\{x_n\}$ converges to p with respect to m and $(G, m, *)$ is a complete fuzzy metrizable type space. □

Theorem 2.7. (Alexandroff) *A G_δ set in a complete fuzzy metric type space is a topologically complete fuzzy metrizable type space.*

Proof. Let $(X, M, *)$ be a complete fuzzy metric type space and G be a G_δ set in X, that is $G = \cap_{n=1}^{\infty} G_n$, where each G_n is open in X. By the above theorem, there exists a complete fuzzy metric type m_n on G_n and we may assume that m_n is bounded by $1 - \varepsilon^{(n)}$. Let $\mathcal{H}$ be the Cartesian product $\prod_{n=1}^{\infty} G_n$ with the product topology. Then $\mathcal{H}$ is a complete fuzzy metrizable type space. Now, for each $n \in \mathbb{N}$ let $f_n : G \longrightarrow G_n$ be the inclusion map. So the evaluation map $e : G \longrightarrow \mathcal{H}$ is an embedding. Image of e is the diagonal ΔG which is a closed subset of $\mathcal{H}$ and by Corollary 1.6 , ΔG is complete. Thus ΔG is a complete fuzzy metrizable type space and so is G which is homeomorphic to it. □

Acknowledgements

The author is thankful to the referee for giving valuable comments and suggestions which helped to improve the final version of this paper.

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