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Topologies on the Set of Borel Maps of Class α

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Abstract. Let ω_1 be the first uncountable ordinal, $\alpha < \omega_1$ an ordinal, and Y, Z two topological spaces. By $\mathbf{B}^\alpha(Y, Z)$ we denote the set of all Borel maps of class α from Y into Z and by $\mathbf{G}^Z_\alpha(Y)$ the set consisting of all subsets $f^{-1}(U)$, where $f \in \mathbf{B}^\alpha(Y, Z)$ and U is an open subset of Z. In this paper we introduce and investigate topologies on the sets $\mathbf{B}^\alpha(Y, Z)$ and $\mathbf{G}^Z_\alpha(Y)$. More precisely, we generalize the results presented by Arens, Dugundji, Aumann, and Rao (see [1], [2], [3], and [10]) for Borel maps of class α.

1. Preliminaries

In the books [7], [8], [9], and [11] the reader can find a good history and introduction on the basic notions on Borel sets and Borel maps. In what follows we give the necessary basic notions in this theory which we will use for the development of our study.

Let X, Y, and Z be three topological spaces and F a map of $X \times Y$ into Z. By F_x we denote the map of Y into Z defined by $F_x(y) = F(x, y)$ for every $y \in Y$ and by F^y the map of X into Z defined by $F^y(x) = F(x, y)$ for every $x \in X$.

By the family of *Borel sets* of a topological space X we mean the smallest family of subsets of X containing the closed subsets of X and is closed under complements and countable unions (therefore, closed also under countable intersections). We denote this family by $\mathbf{B}(X)$.

A subset Q of a space X is said to be a *G_δ-set* (respectively, an *F_σ-set*) if Q is the intersection (respectively, the union) of countable many open (respectively, closed) subsets of X.

In what follows by ω_1 we denote the first uncountable ordinal.

Let X be a space such that each closed subset of X is a G_δ-set (and, therefore, each open subset of X is an F_σ-set). We observe that the family $\mathbf{B}(X)$ can be represented as a union of some subfamilies $\mathbf{F}_\alpha$, $\alpha < \omega_1$, defined by induction as follows:

(a) $\mathbf{F}_0$ is the family of closed subsets of X,

(b) if $\alpha \neq 0$ is an odd (respectively, an even) ordinal, then $\mathbf{F}_\alpha$ consists of countable unions (respectively, of countable intersections) of elements of the family $\cup\{\mathbf{F}_\beta : \beta < \alpha\}$.

So, we have

$$\mathbf{B}(X) = \cup\{\mathbf{F}_\alpha : \alpha < \omega_1\}.$$

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Also,

$$\mathbf{B}(X) = \cup\{\mathbf{G}_\alpha : \alpha < \omega_1\},$$

where:

(a) $\mathbf{G}_0$ is the family of all open subsets of X, and

(b) if $\alpha \neq 0$ is an odd (respectively, an even) ordinal, then $\mathbf{G}_\alpha$ consists of countable intersections (respectively, of countable unions) of elements of the family $\cup\{\mathbf{G}_\beta : \beta < \alpha\}$.

The *multiplicative class* α (denoted by $\mathbf{\Pi}^0_\alpha(X)$) and the *additive class* α (denoted by $\mathbf{\Sigma}^0_\alpha(X)$), where $\alpha < \omega_1$, are subfamilies of $\mathbf{B}(X)$ defined as follows:

(a) If α is an odd ordinal, then the multiplicative class α coincides with $\mathbf{G}_\alpha$ and the additive class α with $\mathbf{F}_\alpha$.

(b) If α is an even ordinal, then the multiplicative class α coincides with $\mathbf{F}_\alpha$ and the additive class α with the $\mathbf{G}_\alpha$.

A topological space X is called a G_δ*-space* if $\mathbf{G}_0 \subseteq \mathbf{F}_1$. Note that every metric space is a G_δ-space.

Let Y and Z be two G_δ-spaces, $\alpha < \omega_1$ an ordinal, and $f : Y \to Z$ a map. The map f is called a *Borel map of class* α (or simply a *map of class* α) if for every open subset U of Z, $f^{-1}(U)$ is of additive class α in Y (equivalently, for every closed subset F of Z, $f^{-1}(F)$ is of multiplicative class α in Y). By $\mathbf{B}^\alpha(Y, Z)$ we denote the family of all Borel maps of class α from Y to Z.

The following properties of Borel maps of class α are known (see, for example, [4] and [9]):

(1) A map f is of class 0 if and only if f is continuous.

(2) The maps of class α are also of class β, for all $\beta > \alpha$.

(3) If f is of class α, g of class β, and $g \circ f$ is defined, then the map $g \circ f$ is of class $\alpha + \beta$.

In what follows we suppose that all topological spaces are G_δ-spaces.

Let Y and Z be two fixed topological spaces and $\mathcal{O}(Z)$ the family of all open sets of Z. In section 2 we introduce and examine the notions of coordinately Borel splitting and admissible topologies of class α on the set $\mathbf{B}^\alpha(Y, Z)$. We present the corresponding notions through examples. In section 3 we define and study some relations between the topological structures on the set $\mathbf{B}^\alpha(Y, Z)$ and on the set $\mathbf{G}^Z_\alpha(Y)$ consisting of all subsets $f^{-1}(U)$, where $f \in \mathbf{B}^\alpha(Y, Z)$ and $U \in \mathcal{O}(Z)$. Furthermore, we define and present the notions of family-Borel topologies and mutually dual topologies of class α.

2. Topologies on the set $\mathbf{B}^\alpha(Y, Z)$

Let Y and Z be two fixed topological spaces and $\alpha < \omega_1$ a fixed ordinal. If t is a topology on the set $\mathbf{B}^\alpha(Y, Z)$, then the corresponding topological space is denoted by $\mathbf{B}^\alpha_t(Y, Z)$.

Definition 2.1. Let X be an arbitrary topological space. A map $F : X \times Y \to Z$ is called *coordinately Borel of class* α if for every $x \in X$ and $y \in Y$ the maps $F_x : Y \to Z$ and $F^y : X \to Z$ are Borel maps of class α.

We observe that if the map $F : X \times Y \to Z$ is a coordinately Borel map of class α, then this map is also coordinately Borel. For the notion of a coordinately Borel map see [6]. Also, if the map $F : X \times Y \to Z$ is a Borel map of class α, then this map is also coordinately Borel map of class α (see [9]).

Notations. (1) Let $F : X \times Y \to Z$ be a coordinately Borel map of class α. Then, by $\widehat{F}$ we denote the map of X into the set $\mathbf{B}^\alpha(Y, Z)$ defined by $\widehat{F}(x) = F_x$ for every $x \in X$.

(2) Let G be a map of X into $\mathbf{B}^\alpha(Y, Z)$. Then, by $\widetilde{G}$ we denote the map of $X \times Y$ into Z defined by $\widetilde{G}(x, y) = G(x)(y)$ for every $(x, y) \in X \times Y$.

Definition 2.2. Let $\mathcal{A}$ be an arbitrary fixed class of G_δ-spaces.

(1) A topology t on $\mathbf{B}^\alpha(Y, Z)$ is called *coordinately Borel* $\mathcal{A}$*-splitting of class* α if for every space $X \in \mathcal{A}$ the following implication holds: if the map $F : X \times Y \to Z$ is coordinately Borel of class α, then the map $\widehat{F} : X \to \mathbf{B}^\alpha_t(Y, Z)$ is Borel of class α.

(2) A topology t on $\mathbf{B}^{\alpha}(Y,Z)$ is called *coordinately Borel $\mathcal{A}$-admissible of class α* if for every space $X \in \mathcal{A}$ the following implication holds: if the map $G : X \to \mathbf{B}^{\alpha}(Y,Z)$ is Borel of class α, then the map $\widetilde{G} : X \times Y \to Z$ is coordinately Borel of class α.

Remark. If $\mathcal{A}$ is the class of all G_δ-spaces, then we use the notions of coordinately Borel splitting of class α and coordinately Borel admissible of class α topologies, instead of the notions of coordinately Borel $\mathcal{A}$-splitting of class α and coordinately Borel $\mathcal{A}$-admissible of class α topologies, respectively.

Definition 2.3. By e we denote the map of $\mathbf{B}^{\alpha}(Y,Z) \times Y$ into Z, defined by $e(f,y) = f(y)$ for every $(f,y) \in \mathbf{B}^{\alpha}(Y,Z) \times Y$. The map e is called *evaluation map of class α*.

Proposition 2.4. *Let $\mathcal{A}$ be a class of G_δ-spaces and t a topology on $\mathbf{B}^{\alpha}(Y,Z)$ such that $\mathbf{B}^{\alpha}_t(Y,Z) \in \mathcal{A}$. If the topology t on $\mathbf{B}^{\alpha}(Y,Z)$ is coordinately Borel $\mathcal{A}$-admissible of class α, then the evaluation map $e : \mathbf{B}^{\alpha}_t(Y,Z) \times Y \to Z$ is coordinately Borel map of class α.*

Proof. We consider as X the space $\mathbf{B}^{\alpha}_t(Y,Z)$ and as G the identity map

$$id : \mathbf{B}^{\alpha}_t(Y,Z) \to \mathbf{B}^{\alpha}_t(Y,Z).$$

Since the map G is a Borel map of class α and the topology t on $\mathbf{B}^{\alpha}(Y,Z)$ is coordinately Borel $\mathcal{A}$-admissible of class α, we have that the map

$$\widetilde{G} : \mathbf{B}^{\alpha}_t(Y,Z) \times Y \to Z$$

is coordinately Borel map of class α. We observe that $\widetilde{G} = e$. Thus, the map e is coordinately Borel map of class α. □

Corollary 2.5. *Let t be a coordinately Borel admissible of class α topology on $\mathbf{B}^{\alpha}(Y,Z)$ such that the space $\mathbf{B}^{\alpha}_t(Y,Z)$ is a G_δ-space. Then, the evaluation map $e : \mathbf{B}^{\alpha}_t(Y,Z) \times Y \to Z$ is coordinately Borel map of class α.*

Proposition 2.6. *Let $\mathcal{A}$ be a class of G_δ-spaces and t a topology on $\mathbf{B}^{\alpha}(Y,Z)$. If the map $e^y : \mathbf{B}^{\alpha}_t(Y,Z) \to Z$ is continuous, for every $y \in Y$ (and, hence, the evaluation map $e : \mathbf{B}^{\alpha}_t(Y,Z) \times Y \to Z$ is coordinately Borel map of class α), then the topology t on $\mathbf{B}^{\alpha}(Y,Z)$ is coordinately Borel $\mathcal{A}$-admissible of class α.*

Proof. Let $X \in \mathcal{A}$ and $G : X \to \mathbf{B}^{\alpha}_t(Y,Z)$ be a Borel map of class α. It suffices to prove that the map $\widetilde{G} : X \times Y \to Z$ is coordinately Borel of class α. Let $x \in X$. Then, for every $y \in Y$ we have

$$\widetilde{G}(x,y) = G(x)(y).$$

Thus, $\widetilde{G}_x(y) = G(x)(y)$ for every $y \in Y$. Therefore, $\widetilde{G}_x = G(x)$ which is a Borel map of class α by the assumption.

Now, we prove that the map $\widetilde{G}^y$, $y \in Y$ is a Borel map of class α. Let $y \in Y$. Then, for every $x \in X$ we have

$$\widetilde{G}^y(x) = \widetilde{G}(x,y) = G(x)(y) = e(G(x),y) = e^y(G(x)) = (e^y \circ G)(x).$$

Since the map e^y is continuous and the map G is a Borel map of class α we have that the map $e^y \circ G$ is a Borel map of class α. Thus, the map $\widetilde{G}^y$ is a Borel map of class α and, therefore, the map $\widetilde{G}$ is coordinately Borel map of class α. □

Corollary 2.7. *If the map $e^y : \mathbf{B}^{\alpha}_t(Y,Z) \to Z$ is continuous, for every $y \in Y$, then the topology t on $\mathbf{B}^{\alpha}(Y,Z)$ is coordinately Borel admissible of class α.*

Proposition 2.8. *Let $\mathcal{A}$ be a class of G_δ-spaces. Then, the following statements are true:*
(1) If t is a topology smaller than a coordinately Borel $\mathcal{A}$-splitting topology u of class α on $\mathbf{B}^{\alpha}(Y,Z)$, then t is also coordinately Borel $\mathcal{A}$-splitting topology of class α.
(2) If t is a topology larger than a coordinately Borel $\mathcal{A}$-admissible topology u of class α on $\mathbf{B}^{\alpha}(Y,Z)$ and the map $e^y : \mathbf{B}^{\alpha}_u(Y,Z) \to Z$ is continuous for every $y \in Y$, then t is also coordinately Borel $\mathcal{A}$-admissible topology of class α.

Proof. (1) Let $X \in \mathcal{A}$ and $F : X \times Y \to Z$ be a coordinately Borel map of class α. Since the topology u is coordinately Borel $\mathcal{A}$-splitting topology of class α, the map $\widehat{F} : X \to \mathbf{B}^{\alpha}_{u}(Y, Z)$ is a Borel map of class α. Since $t \subseteq u$ the identity map

$$id : \mathbf{B}^{\alpha}_{u}(Y, Z) \to \mathbf{B}^{\alpha}_{t}(Y, Z)$$

is continuous. Thus, the map $\widehat{F} = id \circ \widehat{F} : X \to \mathbf{B}^{\alpha}_{t}(Y, Z)$ is a Borel map of class α and, therefore, t is coordinately Borel $\mathcal{A}$-splitting topology of class α.

(2) Let $y \in Y$. Since the map $e^y : \mathbf{B}^{\alpha}_{u}(Y, Z) \to Z$ is continuous and $u \subseteq t$, the map $e^y : \mathbf{B}^{\alpha}_{t}(Y, Z) \to Z$ is also continuous. Therefore, by Proposition 2.6, the topology t is coordinately Borel $\mathcal{A}$-admissible of class α. □

Corollary 2.9. *The following statements are true:*
(1) If t is a topology smaller than a coordinately Borel splitting topology u of class α on $\mathbf{B}^{\alpha}(Y, Z)$, then t is also coordinately Borel splitting topology of class α.
(2) If t is a topology larger than a coordinately Borel admissible topology u of class α on $\mathbf{B}^{\alpha}(Y, Z)$ and the map $e^y : \mathbf{B}^{\alpha}_{u}(Y, Z) \to Z$ is continuous for every $y \in Y$, then t is also coordinately Borel admissible topology of class α.

Remark. Let $\mathcal{A}$ be the class of all G_δ-spaces. It is clear that in general in the set $\mathbf{B}^{\alpha}(Y, Z)$ there not exists the greatest coordinately Borel $\mathcal{A}$-splitting topology. This fact gives a different result from the classical theory of function topological spaces (see [1] and [5]).

Proposition 2.10. *Let $\mathcal{A}$ be a class of G_δ-spaces. The following statements are true:*
(1) The anti-discrete topology t_{tr} on $\mathbf{B}^{\alpha}(Y, Z)$ is the smallest coordinately Borel $\mathcal{A}$-splitting topology of class α.
(2) The discrete topology t_d on $\mathbf{B}^{\alpha}(Y, Z)$ is the greatest coordinately Borel $\mathcal{A}$-admissible topology of class α.

Proof. (1) Let $X \in \mathcal{A}$ be a G_δ-space and $F : X \times Y \to Z$ be a coordinately Borel map of class α. It is suffices to prove that $\widehat{F} : X \to \mathbf{B}^{\alpha}_{t_{tr}}(Y, Z)$ is a Borel map of class α.

We observe that

$$\widehat{F}^{-1}(\mathbf{B}^{\alpha}(Y, Z)) = X \in \mathbf{G}_0 \subseteq \mathbf{G}_\alpha.$$

Thus, the map $\widehat{F} : X \to \mathbf{B}^{\alpha}_{t_{tr}}(Y, Z)$ is a Borel map of class α.

(2) We observe that the map $e^y : \mathbf{B}^{\alpha}_{t}(Y, Z) \to Z$ is continuous. Therefore, by Proposition 2.6, the discrete topology on $\mathbf{B}^{\alpha}(Y, Z)$ is coordinately Borel $\mathcal{A}$-admissible of class α. □

Definition 2.11. Let $y \in Y$. The topology on $\mathbf{B}^{\alpha}(Y, Z)$ consisting of all sets

$$(\{y\}, U) = \{f \in \mathbf{B}^{\alpha}(Y, Z) : f(y) \in U\},$$

where U is an open subset of Z, is called the *y-topology of class α* and denoted by t^{α}_{y}.

Proposition 2.12. *Let $\mathcal{A}$ be a class of G_δ-spaces. The topology t^{α}_{y} on $\mathbf{B}^{\alpha}(Y, Z)$ is coordinately Borel $\mathcal{A}$-splitting topology of class α.*

Proof. Let $X \in \mathcal{A}$ be a G_δ-space and $F : X \times Y \to Z$ be a coordinately Borel map of class α. It suffices to prove that the map $\widehat{F} : X \to \mathbf{B}^{\alpha}_{t^{\alpha}_{y}}(Y, Z)$ is a Borel map of class α. Let $(\{y\}, U) \in t_y$. We have

$$\widehat{F}^{-1}((\{y\}, U)) = \{x \in X : \widehat{F}(x)(y) = F^y(x) \in U\} = (F^y)^{-1}(U).$$

Since F^y is a Borel map of class α and U is an open subset of Z, we have that the set $(F^y)^{-1}(U) \in \mathbf{\Sigma}^0_\alpha(X)$. Thus, the map $\widehat{F}$ is a Borel map of class α. □

Definition 2.13. The t^{α}_{p} topology on $\mathbf{B}^{\alpha}(Y, Z)$ is that having as subbasis all sets

$$(\{y\}, U) = \{f \in \mathbf{B}^{\alpha}(Y, Z) : f(y) \in U\},$$

where $y \in Y$ and U an open subset of Z. The topology t^{α}_{p} is called *point-open topology of class α*.

Proposition 2.14. *Let $\mathcal{A}$ be a class of G_δ-spaces. The following statements are true:*
(1) The topology t_p^α on $\mathbf{B}^\alpha(Y,Z)$ is a coordinately Borel $\mathcal{A}$-admissible topology of class α.
(2) If the subbasis mentioned in Definition 2.13 is countable, then the topology t_p^α on $\mathbf{B}^\alpha(Y,Z)$ is coordinately Borel $\mathcal{A}$-splitting topology of class α.

Proof. (1) By Proposition 2.6 it suffices to prove that the map $e^y : \mathbf{B}_t^\alpha(Y,Z) \to Z$ is continuous. Indeed, let U be an open subset of Z. We have

$$(e^y)^{-1}(U) = \{f \in \mathbf{B}^\alpha(Y,Z) : e^y(f) = e(f,y) = f(y) \in U\} = (\{y\},U).$$

Therefore, e^y is continuous.

(2) Let $X \in \mathcal{A}$ be a G_δ-space and $F : X \times Y \to Z$ be a coordinately Borel map of class α. We prove that the map $\widehat{F} : X \to \mathbf{B}^\alpha_{t_p^\alpha}(Y,Z)$ is a Borel map of class α. Let

$$\bigcup_{i\in I}(\bigcap_{j\in J_i}(\{y_j\},W_j)) \in t_p,$$

where $|I| \leq \aleph_0$, $|J_i| < \aleph_0$, $y_j \in Y$, and W_j is an open subset of Z, for every $j \in J_i$. Then,

$$\begin{aligned}
&\widehat{F}^{-1}(\bigcup_{i\in I}(\bigcap_{j\in J_i}(\{y_j\},W_j))) = \bigcup_{i\in I}(\bigcap_{j\in J_i}\widehat{F}^{-1}(\{y_j\},W_j))\\
&= \bigcup_{i\in I}(\bigcap_{j\in J_i}\{x \in X : \widehat{F}(x)(y_j) = F(x,y_j) = F^{y_j}(x) \in W_j\})\\
&= \bigcup_{i\in I}(\bigcap_{j\in J_i}(F^{y_j})^{-1}(W_j)).
\end{aligned}$$

Since F^{y_i} is a Borel map of class α and $|I| \leq \aleph_0$, we have that

$$\bigcup_{i\in I}(\bigcap_{j\in J_i}(F^{y_j})^{-1}(W_j)) \in \mathbf{\Sigma}^0_\alpha(X).$$

This means that the map $\widehat{F}$ is a Borel map of class α. □

Definition 2.15. The $t^\alpha_{p\mathbf{B}}$ topology on $\mathbf{B}^\alpha(Y,Z)$ is that having as subbasis all sets

$$(\{y\},B) = \{f \in \mathbf{B}^\alpha(Y,Z) : f(y) \in B\},$$

where $y \in Y$ and $B \in \mathbf{\Sigma}^0_\alpha(Z)$. The topology $t^\alpha_{p\mathbf{B}}$ is called *point-Borel topology of class α.*

We observe that $t_p^\alpha \subseteq t^\alpha_{p\mathbf{B}}$.

Proposition 2.16. *Let $\mathcal{A}$ be a class of G_δ-spaces. If the map $e^y : \mathbf{B}^\alpha_{t_p^\alpha}(Y,Z) \to Z$ is continuous, for every $y \in Y$, then the topology $t^\alpha_{p\mathbf{B}}$ on $\mathbf{B}^\alpha(Y,Z)$ is coordinately Borel $\mathcal{A}$-admissible topology of class α.*

Proof. The proof of this follows by the fact that $t_p \subseteq t_{p\mathbf{B}}$ and by Proposition 2.8(2). □

Definition 2.17. The $t^\alpha_{pG_\delta}$ topology on $\mathbf{B}^\alpha(Y,Z)$ is that having as subbasis all sets

$$(\{y\},A) = \{f \in \mathbf{B}^\alpha(Y,Z) : f(y) \in A\},$$

where $y \in Y$ and A is a G_δ-set of Z. The topology $t^\alpha_{pG_\delta}$ is called *point-G_δ topology of class α.*

We observe that $t_p^\alpha \subseteq t^\alpha_{pG_\delta} \subseteq t^\alpha_{p\mathbf{B}}$.

Proposition 2.18. *Let $\mathcal{A}$ be a class of G_δ-spaces. The following statements are true:*
(1) If the map $e^y : \mathbf{B}^\alpha_{t^\alpha_p}(Y,Z) \to Z$ is continuous for every $y \in Y$, then the topology $t^\alpha_{pG_\delta}$ on $\mathbf{B}^\alpha(Y,Z)$ is coordinately Borel $\mathcal{A}$-admissible topology of class α.
(2) If the subbasis mentioned in Definition 2.17 is countable, then the topology $t^\alpha_{pG_\delta}$ on $\mathbf{B}^\alpha(Y,Z)$ is coordinately Borel $\mathcal{A}$-splitting topology of class α.

Proof. (1) It follows by the fact that $t_p \subseteq t^\alpha_{pG_\delta}$ and by Proposition 2.8(2).
(2) The proof is similar to the proof of Proposition 2.14(2). □

3. Dual topologies of class α

Let Y and Z be two fixed G_δ-spaces and $\mathcal{O}(Z)$ the family of all open sets of Z. We consider the set:

$$\mathbf{G}^Z_\alpha(Y) = \{f^{-1}(U) : f \in \mathbf{B}^\alpha(Y,Z),\ U \in \mathcal{O}(Z)\}.$$

We define and study some relations between the topologies on the set $\mathbf{B}^\alpha(Y,Z)$ and the topologies on the set $\mathbf{G}^Z_\alpha(Y)$ concerning the notions of coordinately Borel $\mathcal{A}$-splitting of class α and coordinately Borel $\mathcal{A}$-admissible of class α topologies.

Notations. Let $\mathbb{H}^\alpha \subseteq \mathbf{G}^Z_\alpha(Y)$, $\mathcal{H}^\alpha \subseteq \mathbf{B}^\alpha(Y,Z)$, and $U \in \mathcal{O}(Z)$. We set

$$(\mathbb{H}^\alpha, U) = \{f \in \mathbf{B}^\alpha(Y,Z) : f^{-1}(U) \in \mathbb{H}^\alpha\}$$

and

$$(\mathcal{H}^\alpha, U) = \{f^{-1}(U) : f \in \mathcal{H}^\alpha\}.$$

Definition 3.1. Let τ be a topology on $\mathbf{G}^Z_\alpha(Y)$. The $t(\tau)$ topology on $\mathbf{B}^\alpha(Y,Z)$ is that having as subbasis all the sets $(\mathbb{H}^\alpha, U)$, where $\mathbb{H}^\alpha \in \tau$ and $U \in \mathcal{O}(Z)$. The $t(\tau)$ topology is called *dual of class α to τ*.

Definition 3.2. Let t be a topology on $\mathbf{B}^\alpha(Y,Z)$. The $\tau(t)$ topology on $\mathbf{G}^Z_\alpha(Y)$, is that having as subbasis all the sets $(\mathcal{H}^\alpha, U)$, where $\mathcal{H}^\alpha \in t$ and $U \in \mathcal{O}(Z)$. The $\tau(t)$ topology is called *dual of class α to t*.

Proposition 3.3. *The following statements are true:*
(1) Let τ_1 and τ_2 be two topologies on the set $\mathbf{G}^Z_\alpha(Y)$ such that $\tau_1 \subseteq \tau_2$. Then, $t(\tau_1) \subseteq t(\tau_2)$.
(2) Let t_1 and t_2 be two topologies on the set $\mathbf{B}^\alpha(Y,Z)$ such that $t_1 \subseteq t_2$. Then, $\tau(t_1) \subseteq \tau(t_2)$.

Proof. Follows easily from Definitions 3.1 and 3.2. □

Notations. (1) Let τ and t be two topologies on $\mathbf{G}^Z_\alpha(Y)$ and $\mathbf{B}^\alpha(Y,Z)$, respectively.
(i) By $s(\tau)$ we denote the family

$$\{(\mathbb{H}^\alpha, U) : \mathbb{H}^\alpha \in \tau,\ U \in \mathcal{O}(Z)\}.$$

(ii) By $r(t)$ we denote the family

$$\{(\mathcal{H}^\alpha, U) : \mathcal{H}^\alpha \in t,\ U \in \mathcal{O}(Z)\}.$$

(2) Suppose that $F : X \times Y \to Z$ is a coordinately Borel map of class α. By $\overline{F}$ we denote the map of $X \times \mathcal{O}(Z)$ into the set $\mathbf{G}^Z_\alpha(Y)$, for which $\overline{F}(x,U) = F_x^{-1}(U)$ for every $x \in X$ and $U \in \mathcal{O}(Z)$.
(3) Let G be a map of X into $\mathbf{B}^\alpha(Y,Z)$. By $\overline{G}$ we denote the map of $X \times \mathcal{O}(Z)$ into $\mathbf{G}^Z_\alpha(Y)$, for which $\overline{G}(x,U) = (G(x))^{-1}(U)$ for every $x \in X$ and $U \in \mathcal{O}(Z)$.

Definition 3.4. Let τ be a topology on $\mathbf{G}^Z_\alpha(Y)$. We say that a map M of $X \times \mathcal{O}(Z)$ into $\mathbf{G}^Z_\alpha(Y)$ is a *Borel map of class α, with respect to the first variable* if for every fixed element $U \in \mathcal{O}(Z)$, the map $M^U : X \to (\mathbf{G}^Z_\alpha(Y), \tau)$ is a Borel map of class α, where $M^U(x) = M(x,U)$ for every $x \in X$.

Definition 3.5. Let $\mathcal{A}$ be a class of G_δ-spaces.
(1) A topology τ on $\mathbf{G}^Z_\alpha(Y)$ is called *coordinately Borel $\mathcal{A}$-splitting topology of class α* if for every space $X \in \mathcal{A}$ the following implication holds: if the map $F : X \times Y \to Z$ is coordinately Borel of class α, then the map

$$\overline{F} : X \times \mathcal{O}(Z) \to (\mathbf{G}^Z_\alpha(Y), \tau)$$

is Borel of class α, with respect to the first variable.
(2) A topology τ on $\mathbf{G}^Z_\alpha(Y)$ is called *coordinately Borel $\mathcal{A}$-admissible topology of class α* if for every space $X \in \mathcal{A}$ and for every map

$$G : X \to \mathbf{B}^\alpha(Y, Z)$$

the following implication holds: if the map

$$\overline{G} : X \times \mathcal{O}(Z) \to (\mathbf{G}^Z_\alpha(Y), \tau)$$

is Borel of class α, with respect to the first variable, then the map

$$\widetilde{G} : X \times Y \to Z$$

is coordinately Borel of class α.

Remark. If $\mathcal{A}$ is the class of all G_δ-spaces, then we use the notions of coordinately Borel splitting of class α and coordinately Borel admissible of class α, instead of the notions of coordinately Borel $\mathcal{A}$-splitting of class α and coordinately Borel $\mathcal{A}$-admissible of class α, respectively.

Proposition 3.6. *Let $\mathcal{A}$ be a class of G_δ-spaces. The following statements are true:*
(1) Let τ be a topology on $\mathbf{G}^Z_\alpha(Y)$. If the topology $t(\tau)$ on $\mathbf{B}^\alpha(Y, Z)$ is coordinately Borel $\mathcal{A}$-splitting of class α, then the topology τ is coordinately Borel $\mathcal{A}$-splitting of class α.
(2) Let τ be a topology on $\mathbf{G}^Z_\alpha(Y)$. If $|s(\tau)| \leq \aleph_0$ and the topology τ is coordinately Borel $\mathcal{A}$-splitting of class α, then the topology $t(\tau)$ on $\mathbf{B}^\alpha(Y, Z)$ is coordinately Borel $\mathcal{A}$-splitting of class α.

Proof. (1) Suppose that the topology $t(\tau)$ on $\mathbf{B}^\alpha(Y, Z)$ is a coordinately Borel $\mathcal{A}$-splitting topology of class α, $X \in \mathcal{A}$, and $F : X \times Y \to Z$ is a coordinately Borel map of class α. It suffices to prove that the map

$$\overline{F} : X \times \mathcal{O}(Z) \to (\mathbf{G}^Z_\alpha(Y), \tau)$$

is a Borel map of class α, with respect to the first variable. Let $U \in \mathcal{O}(Z)$ and $\mathbb{H}^\alpha \in \tau$. We need to prove that $\overline{F}_U^{-1}(\mathbb{H}^\alpha) \in \mathbf{\Sigma}^0_\alpha(X)$. We have

$$\begin{aligned} \overline{F}_U^{-1}(\mathbb{H}^\alpha) &= \{x \in X : \overline{F}_U(x) = F_x^{-1}(U) = \widehat{F}(x)^{-1}(U) \in \mathbb{H}^\alpha\} \\ &= \widehat{F}^{-1}((\mathbb{H}^\alpha, U)). \end{aligned}$$

Since $F : X \times Y \to Z$ is a coordinately Borel map of class α, the map

$$\widehat{F} : X \to \mathbf{B}^\alpha_{t(\tau)}(Y, Z)$$

is a Borel map of class α. Thus, $\overline{F}_U^{-1}(\mathbb{H}^\alpha) \in \mathbf{\Sigma}^0_\alpha(X)$.

(2) Suppose that the topology τ on $\mathbf{G}^Z_\alpha(Y)$ is a coordinately Borel $\mathcal{A}$-splitting topology of class α, $X \in \mathcal{A}$, and $F : X \times Y \to Z$ is a coordinately Borel map of class α. It suffices to prove that the map $\widehat{F} : X \to \mathbf{B}^\alpha_{t(\tau)}(Y, Z)$ is a Borel map of class α. Let

$$\bigcup_{i \in I} (\bigcap_{j \in J_i} (\mathbb{H}^\alpha_j, U_j)) \in t(\tau),$$

where $|I| \leq \aleph_0$, $|J_i| < \aleph_0$, $\mathbb{H}_j^\alpha \in \tau$, and $U_j \in \mathcal{O}(Z)$ for every $j \in J_i$ and $i \in I$. Then, we have

$$\widehat{F}^{-1}(\bigcup_{i\in I}(\bigcap_{j\in J_i}(\mathbb{H}_j^\alpha, U_j))) = \bigcup_{i\in I}(\bigcap_{j\in J_i}\widehat{F}^{-1}(\mathbb{H}_j^\alpha, U_j)) = \bigcup_{i\in I}(\bigcap_{j\in J_i}\overline{F}_{U_j}^{-1}(\mathbb{H}_j^\alpha)).$$

Since the map $\overline{F}_{U_j}$ is a Borel map of class α and $|I| \leq \aleph_0$, we have that

$$\widehat{F}^{-1}(\bigcup_{i\in I}(\bigcap_{j\in J_i}(\mathbb{H}_j^\alpha, U_j))) \in \mathbf{\Sigma}_\alpha^0(X).$$

Thus, the map $\widehat{F}$ is a Borel map of class α. $\square$

Corollary 3.7. *Let τ be a topology on $\mathbf{G}_\alpha^Z(Y)$. Then, the following statements are true:*
(1) If the topology $t(\tau)$ on $\mathbf{B}^\alpha(Y,Z)$ is a coordinately Borel splitting topology of class α, then the topology τ is a coordinately Borel splitting topology of class α.
(2) If $|s(\tau)| \leq \aleph_0$ and the topology τ is a coordinately Borel splitting topology of class α, then the topology $t(\tau)$ on $\mathbf{B}^\alpha(Y,Z)$ is a coordinately Borel splitting topology of class α.

Proposition 3.8. *Let $\mathcal{A}$ be a class of G_δ-spaces. The following statements are true:*
(1) Let t be a topology on $\mathbf{B}^\alpha(Y,Z)$. If the topology $\tau(t)$ on $\mathbf{G}_\alpha^Z(Y)$ is a coordinately Borel $\mathcal{A}$-splitting topology of class α, then the topology t on $\mathbf{B}^\alpha(Y,Z)$ is a coordinately Borel $\mathcal{A}$-splitting topology of class α.
(2) Let t be a topology on $\mathbf{B}^\alpha(Y,Z)$. If $|r(t)| \leq \aleph_0$ and the topology t is a coordinately Borel $\mathcal{A}$-splitting topology of class α, then the topology $\tau(t)$ is a coordinately Borel $\mathcal{A}$-splitting topology of class α.

Proof. The proof is similar to the proof of Proposition 3.6. $\square$

Corollary 3.9. *Let t be a topology on $\mathbf{B}^\alpha(Y,Z)$. Then, the following statements are true:*
(1) If the topology $\tau(t)$ on $\mathbf{G}_\alpha^Z(Y)$ is a coordinately Borel splitting topology of class α, then the topology t on $\mathbf{B}^\alpha(Y,Z)$ is a coordinately Borel splitting topology of class α.
(2) If $|r(t)| \leq \aleph_0$ and the topology t is a coordinately Borel splitting topology of class α, then the topology $\tau(t)$ is a coordinately Borel splitting topology of class α.

Proposition 3.10. *Let $\mathcal{A}$ be a class of G_δ-spaces. The following statements are true:*
(1) Let τ be a topology on $\mathbf{G}_\alpha^Z(Y)$. If τ is a coordinately Borel $\mathcal{A}$-admissible topology of class α, then the topology $t(\tau)$ on $\mathbf{B}^\alpha(Y,Z)$ is a coordinately Borel $\mathcal{A}$-admissible topology of class α.
(2) Let τ be a topology on $\mathbf{G}_\alpha^Z(Y)$. If $|s(\tau)| \leq \aleph_0$ and $t(\tau)$ is a coordinately Borel $\mathcal{A}$-admissible topology of class α, then τ is a coordinately Borel $\mathcal{A}$-admissible topology of class α.

Proof. (1) Suppose that the topology τ on $\mathbf{G}_\alpha^Z(Y)$ is a coordinately Borel $\mathcal{A}$-admissible topology of class α, $X \in \mathcal{A}$, and $G : X \to \mathbf{B}_{t(\tau)}^\alpha(Y,Z)$ is a Borel map of class α. It suffices to prove that the map $\widetilde{G} : X \times Y \to Z$ is a coordinately Borel map of class α. We need to prove that the map

$$\overline{G} : X \times \mathcal{O}(Z) \to (\mathbf{G}_\alpha^Z(Y), \tau)$$

is a Borel map of class α, with respect to the first variable. Indeed, let $U \in \mathcal{O}(Z)$ and $\mathbb{H}^\alpha \in \tau$. Then, we have

$$\begin{aligned} \overline{G}_U^{-1}(\mathbb{H}^\alpha) &= \{x \in X : \overline{G}_U(x) = G(x)^{-1}(U) \in \mathbb{H}^\alpha\} \\ &= G^{-1}((\mathbb{H}^\alpha, U)). \end{aligned}$$

Since G is a Borel map of class α, $G^{-1}((\mathbb{H}^\alpha, U)) \in \mathbf{\Sigma}_\alpha^0(X)$ and, therefore, $\overline{G}_U^{-1}(\mathbb{H}^\alpha) \in \mathbf{\Sigma}_\alpha^0(X)$. Thus, the map $\overline{G}$ is a Borel map of class α, with respect to the first variable.

(2) Suppose that the topology $t(\tau)$ on $\mathbf{B}^\alpha(Y,Z)$ is a coordinately Borel $\mathcal{A}$-admissible topology of class α, $X \in \mathcal{A}$, $G : X \to \mathbf{B}^\alpha(Y,Z)$, and

$$\overline{G} : X \times \mathcal{O}(Z) \to (\mathbf{G}_\alpha^Z(Y), \tau)$$

is a Borel map of class α, with respect to the first variable. We need to prove that the map $\widetilde{G} : X \times Y \to Z$ is a coordinately Borel map of class α. It suffices to prove that the map

$$G : X \to \mathbf{B}^{\alpha}_{t(\tau)}(Y, Z)$$

is a Borel map of class α. Indeed, let

$$\bigcup_{i\in I}(\bigcap_{j\in J_i}(\mathbb{H}^{\alpha}_j, U_j)) \in t(\tau),$$

where $|I| \leq \aleph_0$, $|J_i| < \aleph_0$, $\mathbb{H}^{\alpha}_j \in \tau$, and $U_j \in \mathcal{O}(Z)$ for every $j \in J_i$ and $i \in I$. Then, we have

$$G^{-1}(\bigcup_{i\in I}(\bigcap_{j\in J_i}(\mathbb{H}^{\alpha}_j, U_j))) = \bigcup_{i\in I}(\bigcap_{j\in J_i} G^{-1}(\mathbb{H}^{\alpha}_j, U_j)) = \bigcup_{i\in I}(\bigcap_{j\in J_i} \overline{G}_{U_j}^{-1}(\mathbb{H}^{\alpha})).$$

Since the map $\overline{G}_{U_j}$ is a Borel map of class α and $|I| \leq \aleph_0$, we have that

$$G^{-1}(\bigcup_{i\in I}(\bigcap_{j\in J_i}(\mathbb{H}^{\alpha}_j, U_j))) \in \mathbf{\Sigma}^0_{\alpha}(X).$$

Thus, the map G is a Borel map of class α. □

Corollary 3.11. *Let τ be a topology on $\mathbf{G}^Z_{\alpha}(Y)$. The following statements are true:*
(1) If τ is a coordinately Borel admissible topology of class α, then the topology $t(\tau)$ on $\mathbf{B}^{\alpha}(Y, Z)$ is a coordinately Borel admissible topology of class α.
(2) If $|s(\tau)| \leq \aleph_0$ and $t(\tau)$ is a coordinately Borel admissible topology of class α, then τ is coordinately Borel admissible topology of class α.

Proposition 3.12. *Let $\mathcal{A}$ be a class of G_{δ}-spaces. The following statements are true:*
(1) Let t be a topology on $\mathbf{B}^{\alpha}(Y, Z)$. If t is a coordinately Borel $\mathcal{A}$-admissible topology of class α, then the topology $\tau(t)$ on $\mathbf{G}^Z_{\alpha}(Y)$ is a coordinately Borel $\mathcal{A}$-admissible topology of class α.
(2) Let t be a topology on $\mathbf{B}^{\alpha}(Y, Z)$. If $|r(t)| \leq \aleph_0$ and the topology $\tau(t)$ is a coordinately Borel $\mathcal{A}$-admissible topology of class α, then the topology t is a coordinately Borel $\mathcal{A}$-admissible topology of class α.

Proof. The proof is similar to the proof of Proposition 3.10. □

Corollary 3.13. *Let t be a topology on $\mathbf{B}^{\alpha}(Y, Z)$. The following statements are true:*
(1) If t is a coordinately Borel admissible topology of class α, then the topology $\tau(t)$ on $\mathbf{G}^Z_{\alpha}(Y)$ is a coordinately Borel admissible topology of class α.
(2) If $|r(t)| \leq \aleph_0$ and the topology $\tau(t)$ is a coordinately Borel admissible topology of class α, then the topology t is a coordinately Borel admissible topology of class α.

Definition 3.14. A topology on $\mathbf{B}^{\alpha}(Y, Z)$ (respectively, on $\mathbf{G}^Z_{\alpha}(Y)$) is said to be a *family-Borel topology of class* α if it is dual to a topology on $\mathbf{G}^Z_{\alpha}(Y)$ (respectively, to a topology on $\mathbf{B}^{\alpha}(Y, Z)$).

Proposition 3.15. *Let τ be a topology on $\mathbf{G}^Z_{\alpha}(Y)$. If the set γ is a subbasis for τ, then the set*

$$s(\gamma) = \{(\mathbb{H}^{\alpha}, U) : \mathbb{H}^{\alpha} \in \gamma,\ U \in \mathcal{O}(Z)\}$$

is a subbasis for $t(\tau)$.

Proof. Let $\mathbb{H}^{\alpha} \in \tau$ and $U \in \mathcal{O}(Z)$. Assume that $f \in (\mathbb{H}^{\alpha}, U)$. Then, there exist finite elements $\mathbb{H}^{\alpha}_0, \ldots, \mathbb{H}^{\alpha}_n$ of γ such that

$$f^{-1}(U) \in \mathbb{H}^{\alpha}_0 \cap \ldots \cap \mathbb{H}^{\alpha}_n \subseteq \mathbb{H}^{\alpha}.$$

Therefore,

$$f \in (\mathbb{H}^{\alpha}_0 \cap \ldots \cap \mathbb{H}^{\alpha}_n, U) \subseteq (\mathbb{H}^{\alpha}, U).$$

Thus, any element of the subbasis $s(\tau)$ of $t(\tau)$ is a union of finite intersections of elements of the set $s(\gamma)$, which means that this set is also a subbasis for $t(\tau)$. □

Proposition 3.16. *Let t be a topology on $\mathbf{B}^\alpha(Y,Z)$. If the set s is a subbasis for t, then the set*

$$r(s) = \{(\mathcal{H}^\alpha, U) : \mathcal{H}^\alpha \in s,\ U \in \mathcal{O}(Z)\}$$

is a subbasis for $\tau(t)$.

Proof. Let $\mathcal{H}^\alpha \in t$ and $U \in \mathcal{O}(Z)$. Assume that $A \in (\mathcal{H}^\alpha, U)$. Then, there exists an element $f \in \mathcal{H}$ such that $A = f^{-1}(U)$. There exist finite elements $\mathcal{H}_0^\alpha, \ldots, \mathcal{H}_n^\alpha$ of s such that

$$f \in \mathcal{H}_0^\alpha \cap \ldots \cap \mathcal{H}_n^\alpha \subseteq \mathcal{H}.$$

Therefore,

$$A \in (\mathcal{H}_0^\alpha \cap \ldots \cap \mathcal{H}_n^\alpha, U) \subseteq (\mathcal{H}^\alpha, U).$$

Thus, any element of the subbasis $r(t)$ of $\tau(t)$ is a union of finite intersections of elements of the set $r(s)$, which means that this set is also a subbasis for $\tau(t)$. □

Example 3.17. *Below we give some family-Borel topologies of class α on $\mathbf{B}^\alpha(Y,Z)$.*

(1) For every $y \in Y$ we set

$$\mathbf{G}_\alpha^Z(y) = \{A \in \mathbf{G}_\alpha^Z(Y) : y \in A\}.$$

Let τ be the topology on $\mathbf{G}_\alpha^Z(Y)$ for which the family of all sets $\mathbf{G}_\alpha^Z(y)$, where $y \in Y$ is as subassis. By Proposition 3.15 the set

$$\{(\mathbf{G}_\alpha^Z(y), U) : y \in Y,\ U \in \mathcal{O}(Z)\}$$

is a subbasis for $t(\tau)$. It is easy to see that

$$(\mathbf{G}_\alpha^Z(y), U) = (\{y\}, U),$$

for every $y \in Y$ and $U \in \mathcal{O}(Z)$. Therefore, $t(\tau) = t_p^\alpha$, which means that the point-Borel topology of class α is a family-Borel topology of class α.

(2) Let $\mathcal{M} \subseteq \mathcal{P}(Y)$. The $t_{\mathcal{M}}^\alpha$ topology on $\mathbf{B}^\alpha(Y,Z)$ is the one having all sets

$$(M, U) = \{f \in \mathbf{B}^\alpha(Y,Z) : f(M) \subseteq U\}$$

as subbasis, where $M \in \mathcal{M}$ and $U \in \mathcal{O}(Z)$.

For every $M \in \mathcal{M}$ we set

$$\mathbf{G}_\alpha^Z(M) = \{A \in \mathbf{G}_\alpha^Z(Y) : M \subseteq A\}.$$

Let τ be the topology on $\mathbf{G}_\alpha^Z(Y)$ for which the family of all sets $\mathbf{G}_\alpha^Z(M)$, where $M \in \mathcal{M}$ is as subassis. By Proposition 3.15 the set

$$\{(\mathbf{G}_\alpha^Z(M), U) : M \in \mathcal{M},\ U \in \mathcal{O}(Z)\}$$

is a subbasis for $t(\tau)$. It is easy to see that

$$(\mathbf{G}_\alpha^Z(M), U) = (M, U)$$

for every $M \in \mathcal{M}$ and $U \in \mathcal{O}(Z)$. Therefore, $t(\tau) = t_{\mathcal{M}}^\alpha$, which means that the topology $t_{\mathcal{M}}^\alpha$ is a family-Borel topology of class α.

Proposition 3.18. *Let $\mathbb{H}^\alpha$ be a subset of $\mathbf{G}_\alpha^Z(Y)$ and $U \in \mathcal{O}(Z)$. Then,*

$$\Big((\mathbb{H}^\alpha, U), U\Big) \subseteq \mathbb{H}^\alpha.$$

Proof. We have

$$\Big((\mathbb{H}^\alpha, U), U\Big) = \{f^{-1}(U) : f \in (\mathbb{H}^\alpha, U)\} = \{f^{-1}(U) : f^{-1}(U) \in \mathbb{H}^\alpha\} \subseteq \mathbb{H}^\alpha.$$

□

Proposition 3.19. *Let $\mathcal{H}^\alpha$ be a subset of $\mathbf{B}^\alpha(Y, Z)$ and $U \in \mathcal{O}(Z)$. Then,*

$$\big((\mathcal{H}^\alpha, U), U\big) \supseteq \mathcal{H}^\alpha.$$

Proof. Let $f \in \mathcal{H}^\alpha$. Then, $f^{-1}(U) \in (\mathcal{H}^\alpha, U)$ and, therefore, $f \in \big((\mathcal{H}^\alpha, U), U\big)$. □

Proposition 3.20. *Let $\mathbb{H}^\alpha$ be a subset of $\mathbf{G}^Z_\alpha(Y)$, $\mathcal{H}^\alpha$ be a subset of $\mathbf{B}^\alpha(Y, Z)$, and $U \in \mathcal{O}(Z)$. Then,*

$$\Big(\big((\mathbb{H}^\alpha, U), U\big), U\Big) = (\mathbb{H}^\alpha, U)$$

and

$$\Big(\big((\mathcal{H}^\alpha, U), U\big), U\Big) = (\mathcal{H}^\alpha, U).$$

Proof. By Proposition 3.18 we have $\big((\mathbb{H}^\alpha, U), U\big) \subseteq \mathbb{H}^\alpha$ and, therefore,

$$\Big(\big((\mathbb{H}^\alpha, U), U\big), U\Big) \subseteq (\mathbb{H}^\alpha, U).$$

Also, by Proposition 3.19 for $\mathcal{H}^\alpha = (\mathbb{H}^\alpha, U)$ we have $\Big(\big((\mathbb{H}^\alpha, U), U\big), U\Big) \supseteq (\mathbb{H}^\alpha, U)$. Hence,

$$\Big(\big((\mathbb{H}^\alpha, U), U\big), U\Big) = (\mathbb{H}^\alpha, U).$$

Now, by Proposition 3.19 we have $\big((\mathcal{H}^\alpha, U), U\big) \supseteq \mathcal{H}^\alpha$ and, therefore,

$$\Big(\big((\mathcal{H}^\alpha, U), U\big), U\Big) \supseteq (\mathcal{H}^\alpha, U).$$

Also, by Proposition 3.18 for $\mathbb{H}^\alpha = (\mathcal{H}^\alpha, U)$ we have

$$\Big(\big((\mathcal{H}^\alpha, U), U\big), U\Big) \subseteq (\mathcal{H}^\alpha, U).$$

Thus, $\Big(\big((\mathcal{H}^\alpha, U), U\big), U\Big) = (\mathcal{H}^\alpha, U)$. □

Definition 3.21. Let τ be a topology on $\mathbf{G}^Z_\alpha(Y)$ and t be a topology on $\mathbf{B}^\alpha(Y, Z)$. The pair (τ, t) is called a *pair of mutually dual topologies of class α* if $\tau = \tau(t)$ and $t = t(\tau)$.

Proposition 3.22. *Let τ_0 be a topology on $\mathbf{G}^Z_\alpha(Y)$ and t_0 a topology on $\mathbf{B}^\alpha(Y, Z)$. Then,*

$$t(\tau_0) = t\Big(\tau\big(t(\tau_0)\big)\Big)$$

and

$$\tau(t_0) = \tau\Big(t\big(\tau(t_0)\big)\Big).$$

Therefore, $\Big(\tau\big(t(\tau_0)\big), t(\tau_0)\Big)$ and $\Big(\tau(t_0), t\big(\tau(t_0)\big)\Big)$ are pairs of mutually dual topologies of class α.

Proof. The $t(\tau_0)$ topology on $\mathbf{B}^\alpha(Y, Z)$ has as subbasis all the sets of the form $(\mathbb{H}^\alpha, U)$, where $\mathbb{H}^\alpha \in \tau_0$ and $U \in \mathcal{O}(Z)$. Moreover, we observe that the topology $t\Big(\tau\big(t(\tau_0)\big)\Big)$ on $\mathbf{B}^\alpha(Y, Z)$ has as subbasis all the sets of the form $\Big(\big((\mathbb{H}^\alpha, U), U\big), U\Big)$, where $\mathbb{H}^\alpha \in \tau_0$ and $U \in \mathcal{O}(Z)$. By Proposition 3.20 we have $t(\tau_0) = t\Big(\tau\big(t(\tau_0)\big)\Big)$. Similar we can see that $\tau(t_0) = \tau\Big(t\big(\tau(t_0)\big)\Big)$. □

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