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Covering Groupoids of Categorical Rings

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Abstract. A *categorical group* is a kind of categorization of group and similarly a categorical ring is a categorization of ring. For a topological group X, the fundamental groupoid πX is a group object in the category of groupoids, which is also called in literature *group-groupoid* or 2-*group*. If X is a path connected topological group which has a simply connected cover, then the category of covering groups of X and the category of covering groupoids of πX are equivalent. Recently it was proved that if (X, x_0) is an H-group, then the fundamental groupoid πX is a categorical group and the category of the covering spaces of (X, x_0) is equivalent to the category of covering groupoids of the categorical group πX.

The purpose of this paper is to present similar results for rings and categorical rings.

Introduction

If X is a topological group, the fundamental groupoid πX is a *group-groupoid* which is a group object in the category of groupoids [7]. This notion is also known as an internal category in the category of groups [18]. This functor gives an equivalence of the category of the covering groups of a topological group X whose underlying space is locally nice and the category of the covering groupoids of πX [6, Proposition 2.3] (see also [15]). Recently the notion of monodromy groupoid for topological group-groupoids was developed in [16]; and normality and quotients of group-groupoids were developed in [17].

In [11] it was proved that if (X, x_0) is an H-group (group case of Definition 2.1), then the fundamental groupoid πX is a weak categorical group and also proved that the category of the covering spaces of an H-group (X, x_0) is equivalent to the category of covering groupoids of the weak categorical group πX. In this paper we will give similar results for rings and categorical rings.

1. Preliminaries on covering groupoids and covering spaces

A *groupoid* G on $\mathrm{Ob}(G)$ is a small category in which each morphism is an isomorphism. Thus G has a set of morphisms, a set $\mathrm{Ob}(G)$ of *objects* together with functions $s, t\colon G \to \mathrm{Ob}(G)$, $\epsilon\colon \mathrm{Ob}(G) \to G$ such that $s\epsilon = t\epsilon = 1_{\mathrm{Ob}(G)}$, the identity map. The functions s, t are called *initial* and *final* point maps respectively. If $a, b \in G$ and $t(a) = s(b)$, then the *product* or *composite* $b \circ a$ exists such that $s(b \circ a) = s(a)$ and $t(b \circ a) = t(b)$. Further, this composite is associative, for $x \in \mathrm{Ob}(G)$ the element $\epsilon(x)$ denoted by 1_x acts as the identity, and

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each element a has an inverse a^{-1} such that $s(a^{-1}) = t(a)$, $t(a^{-1}) = s(a)$, $a \circ a^{-1} = (\epsilon t)(a)$, $a^{-1} \circ a = (\epsilon s)(a)$. The map $G \to G$, $a \mapsto a^{-1}$, is called the *inversion*. So a group is a groupoid with only one object.

In a groupoid G for $x, y \in \mathrm{Ob}(G)$ we write $G(x, y)$ for the set of all morphisms with initial point x and final point y. We say G is *connected* if for all $x, y \in \mathrm{Ob}(G)$, $G(x, y)$ is not empty and *simply connected* if $G(x, y)$ has only one morphism. For $x \in \mathrm{Ob}(G)$ we denote the star $\{a \in G \mid s(a) = x\}$ of x by G_x. The set $G(x, x)$ of all morphisms from x to x is a group, called *object group* at x and denoted by $G(x)$.

Let G and H be groupoids. A *morphism* from H to G is a pair of maps $f : H \to G$ and $O_f : \mathrm{Ob}(H) \to \mathrm{Ob}(G)$ such that $s \circ f = O_f \circ s$, $t \circ f = O_f \circ t$ and $f(b \circ a) = f(b) \circ f(a)$ for all $(b, a) \in H_s \times_t H$. For such a morphism we simply write $f : H \to G$.

A morphism $p \colon \widetilde{G} \to G$ of groupoids is called a *covering morphism* and $\widetilde{G}$ *a covering groupoid* of G if for each $\tilde{x} \in \mathrm{Ob}(\widetilde{G})$ the restriction $p_x \colon (\widetilde{G})_{\tilde{x}} \to G_{p(\tilde{x})}$ of p is bijective. A covering morphism $p \colon \widetilde{G} \to G$ is called *connected* if both $\widetilde{G}$ and G are connected.

For example if $p \colon \widetilde{X} \to X$ is a covering map of topological spaces, then the induced morphism $\pi p \colon \pi\widetilde{X} \to \pi X$ of the fundamental groupoids is a covering morphism of groupoids.

A connected covering morphism $p \colon \widetilde{G} \to G$ is called *universal* if $\widetilde{G}$ covers every cover of G, i.e., if for every covering morphism $q \colon \widetilde{H} \to G$ there is a unique morphism of groupoids $\tilde{p} \colon \widetilde{G} \to \widetilde{H}$ such that $q\tilde{p} = p$ (and hence $\tilde{p}$ is also a covering morphism), this is equivalent to that for $\tilde{x}, \tilde{y} \in \mathrm{Ob}(\widetilde{G})$ the set $\widetilde{G}(\tilde{x}, \tilde{y})$ has not more than one element.

For any groupoid morphism $p \colon \widetilde{G} \to G$ and an object $\tilde{x}$ of $\widetilde{G}$ we call the subgroup $p(\widetilde{G}(\tilde{x}))$ of $G(p\tilde{x})$ the *characteristic group* of p at $\tilde{x}$.

If $p \colon \widetilde{X} \to X$ is a covering map of topological spaces, then the induced morphism $\pi p \colon \pi\widetilde{X} \to \pi X$ of fundamental groupoids is a covering morphism of groupoids [5, 10.2.1].

Let $p \colon \widetilde{G} \to G$ be a covering morphism of groupoids and $q \colon H \to G$ a morphism of groupoids. If there exists a unique morphism $\tilde{q} \colon H \to \tilde{G}$ such that $q = p\tilde{q}$ we just say q *lifts* to $\tilde{q}$ by p. We recall the following theorem from Brown [5, 10.3.3] which gives an important criteria to have the lifting maps on covering groupoids. For a useful application of this theorem see for example the proof of Theorem 3.14.

Theorem 1.1. *Let $p \colon \widetilde{G} \to G$ be a covering morphism of groupoids, $x \in \mathrm{Ob}(G)$ and $\tilde{x} \in \mathrm{Ob}(\widetilde{G})$ such that $p(\tilde{x}) = x$. Let $q \colon K \to G$ be a morphism of groupoids such that K is connected and $z \in \mathrm{Ob}(K)$ such that $q(z) = x$. Then the morphism $q \colon K \to G$ uniquely lifts to a morphism $\tilde{q} \colon K \to \widetilde{G}$ such that $\tilde{q}(z) = \tilde{x}$ if and only if $q[K(z)] \subseteq p[\widetilde{G}(\tilde{x})]$, where $K(z)$ and $\widetilde{G}(\tilde{x})$ are the object groups.*

From Theorem 1.1 the following corollary follows.

Corollary 1.2. *Let $p \colon (\widetilde{G}, \tilde{x}) \to (G, x)$ and $q \colon (\widetilde{H}, \tilde{z}) \to (G, x)$ be connected covering morphisms with characteristic groups C and D respectively. If $C \subseteq D$, then there is a unique covering morphism $r \colon (\widetilde{G}, \tilde{x}) \longrightarrow (\widetilde{H}, \tilde{z})$ such that $p = qr$. If $C = D$, then r is an isomorphism.*

We assume the usual theory of covering maps. All spaces X are assumed to be locally path connected and semi locally simply connected, so that each path component of X admits a simply connected cover.

Recall that a covering map $p \colon \widetilde{X} \to X$ of connected spaces is called *universal* if it covers every cover of X in the sense that if $q \colon \widetilde{Y} \to X$ is another cover of X then there exists a map $r \colon \widetilde{X} \to \widetilde{Y}$ such that $p = qr$ (hence r becomes a cover). A covering map $p \colon \widetilde{X} \to X$ is called *simply connected* if $\widetilde{X}$ is simply connected. Note that a simply connected cover is a universal cover.

We call a subset U of X *liftable* if it is open, path connected and U lifts to each cover of X, that is, if $p \colon \widetilde{X} \to X$ is a covering map, $\imath \colon U \to X$ is the inclusion map, and $\tilde{x} \in \widetilde{X}$ satisfies $p(\tilde{x}) = x \in U$, then there exists a map (necessarily unique) $\hat{\imath} \colon U \to \widetilde{X}$ such that $p\hat{\imath} = \imath$ and $\hat{\imath}(x) = \tilde{x}$.

It is an easy application that U is liftable if and only if it is open, path connected and for all $x \in U$, the fundamental group $\pi_1(U, x)$ is mapped to the singleton by the morphism $\imath_\star \colon \pi(U, x) \to \pi(X, x)$ induced by the inclusion map $\imath \colon U \to X$. Remark that if X is a semi locally simply connected topological space, then each point $x \in X$ has a liftable neighbourhood.

We recall the following topology called *lifted topology* from [5, 10.5]

Let X be a semi-locally simply connected space and $q\colon \widetilde{G} \to \pi X$ a covering morphism of groupoids. Let $\widetilde{X} = \mathrm{Ob}(\widetilde{G})$ and $p = \mathrm{Ob}(q)\colon \widetilde{X} \to X$. If $\mathcal{U}$ is the class of all liftable subsets of X and U is an element of $\mathcal{U}$, then by Theorem 1.1 the induced morphism $\imath_\star\colon \pi(U,x) \to \pi(X,x)$ lifts to the morphism of groupoids $\tilde{\imath}\colon \pi(U,x) \to (\widetilde{G},\tilde{x})$. The set $\tilde{\imath}(U) = \widetilde{U}$ is called the *lifting* of U. The set of all such liftings of U for all U in $\mathcal{U}$ is written $\widetilde{\mathcal{U}}$. Since X is semi-locally simply connected the set $\widetilde{\mathcal{U}}$ covers $\widetilde{X}$ and is a base for the open sets of a topology on $\widetilde{X}$. This topology on $\widetilde{X}$ is called *lifted topology* of X.

For the lifted topology on $\widetilde{X}$, the following results, which we need in some proofs are given.

Theorem 1.3. [5, 10.5.3] *Let $f\colon Z \to X$ be a continuous map. If the groupoid morphism $\pi f\colon \pi Z \to \pi X$ lifts to a morphism $\tilde{f}\colon \pi Z \to \widetilde{G}$, then $f' = \mathrm{Ob}(\tilde{f})\colon Z \to \widetilde{X}$ is continuous and is a lifting of $f\colon Z \to X$*

Theorem 1.4. [5, 10.5.5] *The lifted topology is the only topology on $\widetilde{X}$ such that*

(a) $p\colon \widetilde{X} \to X$ is a covering map;

(b) there is an isomorphism $r\colon \widetilde{G} \to \pi\widetilde{X}$ such that $\pi p \circ r = q$

The following result, which is very useful for some proofs, say Theorem 3.11 and Corollary 3.15, is known as Covering Homotopy Theorem [19, Theorem 10.6]. In Theorem 3.12 a parallel result is given for the homotopies of functors.

Theorem 1.5. *Let $p\colon \widetilde{X} \to X$ be a covering map and Z a connected space. Consider the commutative diagram of continuous maps*

$$\begin{tikzcd} Z \arrow[r, "\tilde{f}"] \arrow[d, "j"'] & \widetilde{X} \arrow[d, "p"] \\ Z \times I \arrow[r, "F"'] \arrow[ur, dashed, "\widetilde{F}"] & X \end{tikzcd}$$

where $j\colon Z \to Z\times I$, $j(z) = (z,0)$ for all $z \in Z$. Then there is a unique continuous map $\widetilde{F}\colon Z\times I \to \widetilde{X}$ such that $p\widetilde{F} = F$ and $\widetilde{F}j = \tilde{f}$. □

As a corollary of this theorem if the maps $f,g\colon Z \to X$ are homotopic, then their respective liftings $\tilde{f}$ and $\tilde{g}$ are homotopic. If f and g are homotopic, there is a continuous map $F\colon Z\times I \to X$ such that $F(z,0) = f(z)$ and $F(z,1) = g(z)$. So there is a continuous map $\widetilde{F}\colon Z\times I \to \widetilde{X}$ as in Theorem 1.5. Here $p\widetilde{F}(z,0) = F(z,0) = f(z)$ and $p\widetilde{F}(z,1) = F(z,1) = g(z)$. By the uniqueness of the liftings we have that $\widetilde{F}(z,0) = \tilde{f}(z)$ and $\widetilde{F}(z,1) = \tilde{g}(z)$. Therefore $\tilde{f}$ and $\tilde{g}$ are homotopic.

2. Coverings of H-rings

An *H-group* can be thought as a group where the group axioms are satisfied up to homotopies rather than equalities [19, p.324]. The following definition is a ring version of the definition of an H-group.

Definition 2.1. An *H-ring* is a pointed topological space (X,x_0) with continuous maps

$$m\colon X\times X,(x_0,x_0) \to (X,x_0),(x,x') \mapsto x+x'$$

$$n\colon X\times X,(x_0,x_0) \to (X,x_0),(x,x') \mapsto xx'$$

$$u\colon (X,x_0) \to (X,x_0), x \mapsto -x$$

such that the following homotopies hold:

(i) homotopy associativity of $m : m(1 \times m) \simeq m(m \times 1)$;

(ii) homotopy unit: $m\,\iota_1 \simeq 1 \simeq m\,\iota_2$, where $\iota_1, \iota_2 \colon X \to X \times X$ are injections defined by $\iota_1(x) = (x, x_0)$ and $\iota_2(x) = (x_0, x)$;

(iii) homotopy inverse: $m(1, u) \simeq c \simeq m(u, 1)$; where $c \colon X \to X$ is the constant map at x_0;

(iv) homotopy commutativity of m: $m \simeq m'$, where m' is defined by $m'(x, x') = m(x', x)$ for $x, x' \in X$;

(v) homotopy associativity of $n : n(1 \times n) \simeq n(n \times 1)$;

(vi) left distributivity of n on m: $n(1 \times m) \simeq m(n_1, n_2)$ where n_1 and n_2 are defined by $n_1(x, y, z) = xy$ and $n_2(x, y, z) = xz$;

(vii) right distributivity of n on m: $n(m \times 1) \simeq m(n_2, n_3)$; where n_3 is defined by $n_3(x, y, z) = yz$.
□

We remark that the axioms of the associativity of m, the unit and the inverse can be respectively stated by the commutativity up to homotopy of the following diagrams:

$$\begin{tikzcd} X \times X \times X \arrow[r, "m\times 1"] \arrow[d, "1\times m"'] & X \times X \arrow[d, "m"] \\ X \times X \arrow[r, "m"'] & X \end{tikzcd}$$

$$\begin{tikzcd} X \times X \arrow[dr, "m"'] & X \arrow[l, "\iota_1"'] \arrow[r, "\iota_2"] \arrow[d, "1_X"] & X \times X \arrow[dl, "m"] \\ & X & \end{tikzcd}$$

and

$$\begin{tikzcd} X \times X \arrow[dr, "m"'] & X \arrow[l, "(u,1)"'] \arrow[r, "(1,u)"] \arrow[d, "c"] & X \times X \arrow[dl, "m"] \\ & X & \end{tikzcd} .$$

The axioms of the associativity of n, left and right distributiveness can be restated by the commutativity up to homotopy of the following diagrams:

$$\begin{tikzcd} X \times X \times X \arrow[r, "n\times 1"] \arrow[d, "1\times n"'] & X \times X \arrow[d, "n"] \\ X \times X \arrow[r, "n"'] & X \end{tikzcd}$$

$$\begin{tikzcd} X \times X \times X \arrow[r, "1\times m"] \arrow[d, "(n_1,n_2)"'] & X \times X \arrow[d, "n"] \\ X \times X \arrow[r, "m"'] & X \end{tikzcd}$$

$$\begin{array}{ccc} X\times X\times X & \xrightarrow{m\times 1} & X\times X \\ {\scriptstyle (n_2,n_3)}\downarrow & & \downarrow{\scriptstyle n} \\ X\times X & \xrightarrow[m]{} & X \end{array}.$$

Example 2.2. A topological ring X is an H-ring since the group addition

$$m\colon X\times X,(0,0)\to (X,0)$$

the multiplication

$$n\colon X\times X,(0,0)\to (X,0)$$

and the inverse map $u\colon (X,0)\to (X,0)$ are continuous; and the axioms of Definition 2.1 are satisfied for equalities rather than homotopies. □

A *morphism* between H-rings (X,x_0) and (Y,y_0) is a continuous based map $f\colon (X,x_0)\to (Y,y_0)$ with the homotopies:

$$f\circ m_X\simeq m_Y\circ (f\times f)$$

$$f\circ n_X\simeq n_Y\circ (f\times f).$$

By the fact that the vertical union of commutative diagrams up to homotopy is also commutative up to homotopy, H-rings and morphisms between them constitute a category denoted by HRing.

3. Coverings of categorical rings

For the homotopies of functors we first need the following fact whose proof is straightforward.

Proposition 3.1. Let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ be categories and $F\colon \mathcal{C}\times\mathcal{D}\to\mathcal{E}$ a functor. Then for $x\in \mathrm{Ob}(\mathcal{C})$ and $y\in\mathrm{Ob}(\mathcal{D})$ we have the induced functors

$$F(x,-)\colon \mathcal{D}\to\mathcal{E}$$

$$F(-,y)\colon \mathcal{C}\to\mathcal{E}.$$

□

We write $\mathcal{J}$ for the simply connected groupoid whose objects are 0 and 1; and whose non-identity morphisms are $\iota\colon 0\to 1$ and $\bar{\iota}\colon 1\to 0$. As similar to the homotopy of continuous functions, the homotopy of functors is defined as follows.

Definition 3.2. [5, p.228] Let $f,g\colon \mathcal{C}\to\mathcal{D}$ be functors. These functors are called *homotopic* and written $f\simeq g$ if there is a functor $F\colon \mathcal{C}\times\mathcal{J}\to\mathcal{D}$ such that $F(-,0)=f$ and $F(-,1)=g$. □

Proposition 3.3. [5, 6.5.10] If the maps $f,g\colon X\to Y$ are homotopic, then the induced morphisms $\pi f,\pi g\colon \pi X\to \pi Y$ of the fundamental groupoids are homotopic.

Definition 3.4. Let $f,g\colon \mathcal{C}\to\mathcal{D}$ be two functors. We call f and g are *naturally isomorphic* if there exists a natural isomorphism $\sigma\colon f\to g$. □

Theorem 3.5. [11, Theorem 3.5] *The functors $f,g\colon \mathcal{C}\to\mathcal{D}$ are homotopic in the sense of Definition 3.2 if and only if they are naturally isomorphic.*

A *group-groupoid* is a group object in the category of groupoids [7]; equivalently, it is an internal category and hence an internal groupoid in the category of groups ([18], [1]) . An alternative name, quite generally used, is "2-group", see for example [4]. Recently for example in [1], [12] and [13] some results on group-groupoids have been generalized to the internal groupoids for certain algebraic categories. A *ring-groupoid* which is an internal groupoid in the category of ring, in other words a ring object in the category of groupoids is defined in [14] as follows:

Definition 3.6. A *ring-groupoid* G is a groupoid endowed with a ring structure such that the following additive, product and inverse maps are the morphisms of groupoids:

- $m\colon G \times G \to G$, $(a, b) \mapsto a + b$;
- $n\colon G \times G \to G$, $(a, b) \mapsto ab$;
- $u\colon G \to G$, $a \mapsto -a$;
- $0\colon \{\star\} \to G$, where $\{\star\}$ is singleton.

$\square$

Here note that in this definition, the axioms involving the ring can be stated in terms of functors as follows:

(i) associativity of m : $m(1 \times m) = m(m \times 1)$;

(ii) unit: $m\,\imath_1 = 1_G = m\,\imath_2$, where $\imath_1, \imath_2\colon G \to G \times G$ are injections defined by $\imath_1(a) = (a, 0)$ and $\imath_2(a) = (0, a)$;

(iii) inverse: $m(1, u) = c = m(u, 1)$; where $c\colon G \to G$ is the constant map at 0;

(iv) commutativity of m: $m = m'$, where m' is defined by $m'(a, b) = m(b, a)$ for $a, b \in G$;

(v) associativity of n : $n(1 \times n) = n(n \times 1)$;

(vi) left distributivity of n on m: $n(1 \times m) = m(n_1, n_2)$; where n_1 and n_2 are defined by $n_1(a, b, c) = ab$ and $n_2(a, b, c) = ac$;

(vii) right distributivity of n on m: $n(m \times 1) = m(n_2, n_3)$; where n_3 is defined by $n_3(a, b, c) = bc$.

In the definition of ring-groupoid if we require these functors to be homotopic rather than equal, we obtain the definition of *weak categorical ring*. In the literature the group case of this notion is called *categorical group* (see [8] and [10])

Definition 3.7. Let $\mathcal{G}$ be a groupoid. Let

$$\oplus\colon \mathcal{G} \times \mathcal{G} \to \mathcal{G}, (a, b) \mapsto a + b$$

$$\odot\colon \mathcal{G} \times \mathcal{G} \to \mathcal{G}, (a, b) \mapsto ab$$

and

$$\ominus\colon \mathcal{G} \to \mathcal{G}, a \mapsto -a$$

be functors called respectively *additive*, *product* and *inverse*. Let $0 \in \mathrm{Ob}(\mathcal{G})$ be an object. If the following conditions are satisfied, then we call $(\mathcal{G}, \oplus, \odot, \ominus, 0)$ a *weak categorical ring* and write just $\mathcal{G}$:

1. the functors $\oplus(1 \times \oplus), \oplus(\oplus \times 1)\colon \mathcal{G} \times \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ are homotopic;
2. the functors $0 \oplus 1, 1 \oplus 0\colon \mathcal{G} \to \mathcal{G}$ defined by $(0 \oplus 1)(a) = 0 + a$ and $(1 \oplus 0)(a) = a + 0$ for $a \in \mathcal{G}$ are homotopic to the identity functor $\mathcal{G} \to \mathcal{G}$;

3. the functors $\oplus(1,\ominus), \oplus(\ominus,1)\colon \mathcal{G} \to \mathcal{G}$ defined by $\oplus(1,\ominus)(a) = a + (-a)$ and $\oplus(\ominus,1)(a) = -a + a$ are homotopic to the constant functor $0\colon \mathcal{G} \to \mathcal{G}$;

4. the functors $\oplus$ and $\oplus'$ are homotopic, where $a \oplus' b = b \oplus a$;

5. the functors $\odot(1\times\odot), \odot(\odot\times 1)\colon \mathcal{G}\times\mathcal{G}\times\mathcal{G} \to \mathcal{G}$ are homotopic;

6. the functors $\odot(1\times\oplus)$ and $\oplus(\odot_1,\odot_2)$ defined by $\odot(1\times\oplus)(a,b,c) = a(b+c)$ and $\oplus(\odot_1,\odot_2)(a,b,c) = ab+ac$ are homotopic, where $\odot_1$ and $\odot_2$ are defined by $\odot_1(a,b,c) = ab$ and $\odot_2(a,b,c) = ac$;

7. the functors $\odot(\oplus\times 1)$ and $\oplus(\odot_2,\odot_3)$ defined by $\odot(\oplus\times 1)(a,b,c) = (a+b)c$ and $\oplus(\odot_2,\odot_3)(a,b,c) = (ac)+(bc)$, where $\odot_3$ is defined by $\odot_3(a,b,c) = bc$ and $\odot_2(a,b,c) = ac$.
□

Here we note that in this definition if these functors are equal rather than homotopic, then the weak categorical ring is called a *strict categorical ring* which is a ring-groupoid.

Note that the additive $\oplus\colon \mathcal{G}\times\mathcal{G}\to\mathcal{G}$ is a functor if and only if

$$(b\circ a) + (d\circ c) = (b+d)\circ(a+c) \tag{1}$$

and similarly the product $\odot\colon \mathcal{G}\times\mathcal{G}\to\mathcal{G}$ is a functor if and only if

$$(b\circ a)(d\circ c) = (bd)\circ(ac) \tag{2}$$

for $a,b,c,d\in\mathcal{G}$ whenever the composites $b\circ a$ and $d\circ c$ are defined. Since $\ominus\colon \mathcal{G}\to\mathcal{G}, a\mapsto -a$ is a functor we have $-(b\circ a) = (-b)\circ(-a)$ when the groupoid composite $b\circ a$ is defined and $-(1_x) = 1_{(-x)}$ for $x\in \mathrm{Ob}(\mathcal{G})$.

Theorem 3.8. *If (X,x_0) is an H-ring, then the fundamental groupoid πX is a weak categorical ring.*

Proof: Since (X,x_0) is an H-ring there are continuous maps

$$m\colon X\times X,(x_0,x_0)\to(X,x_0)$$

$$n\colon X\times X,(x_0,x_0)\to(X,x_0)$$

$$u\colon (X,x_0)\to(X,x_0)$$

with the homotopies (i)-(vii) in Definition 2.1. Then by these maps we have the following induced morphisms of fundamental groupoids:

$$\oplus = \pi m\colon \pi X\times\pi X,(x_0,x_0)\to(\pi X,x_0)$$

$$\odot = \pi n\colon \pi X\times\pi X,(x_0,x_0)\to(\pi X,x_0)$$

$$\ominus = \pi u\colon \pi(X,x_0)\to\pi(X,x_0)$$

By Proposition 3.3 we have the following homotopies of the functors induced by the homotopies (i)-(vii) in Definition 2.1:

(i) $\oplus(1\times\oplus)\simeq\oplus(\oplus\times 1)$;

(ii) $0\oplus 1\simeq 1\oplus 0\simeq 1$;

(iii) $\oplus(1,\ominus)\simeq c\simeq\oplus(\ominus,1)$;

(iv) $\oplus\simeq\oplus'$;

(v) $\odot(1\times\odot)\simeq\odot(\odot\times 1)$;

(vi) $\odot(1\times\oplus)\simeq\oplus(\odot_1,\odot_2)$;

(vii) $\odot(\oplus \times 1) \simeq \oplus(\odot_2, \odot_3)$

Therefore πX becomes a weak categorical ring as required.

□

Definition 3.9. Let $\mathcal{G}$ and $\mathcal{H}$ be two weak categorical rings. A morphism of weak categorical rings is a morphism $f\colon \mathcal{H} \to \mathcal{G}$ of groupoids such that:

(i) the functors $f\oplus, \oplus(f \times f)\colon \mathcal{H} \times \mathcal{H} \to \mathcal{G}$ are homotopic;

(ii) the functors $f\odot, \odot(f \times f)\colon \mathcal{H} \times \mathcal{H} \to \mathcal{G}$ are homotopic;

(iii) $f(0_{\mathcal{H}})$ is isomorphic to $0_{\mathcal{G}}$, where $0_{\mathcal{G}}$ and $0_{\mathcal{H}}$ are respectively the base points of $\mathcal{G}$ and $\mathcal{H}$.
□

Therefore we have a category of weak categorical rings and morphisms between them. Denote this category $\mathsf{CatRing}$.

Proposition 3.10. If $p\colon (\widetilde{X}, \tilde{x}_0) \to (X, x_0)$ is a morphism of H-rings, then the induced map $\pi p\colon \pi\widetilde{X} \to \pi X$ is a morphism of weak categorical rings.

Proof: As we define above if $p\colon (\widetilde{X}, \tilde{x}_0) \to (X, x_0)$ is a morphism of H-rings, we have the homotopies of maps $f \circ m_X \simeq m_Y \circ (f \times f)$ and $f \circ n_X \simeq n_Y \circ (f \times f)$. Then by Proposition 3.3 we have the homotopies of the functors $\pi f \circ \pi m_X \simeq \pi m_Y \circ (\pi f \times \pi f)$ and $\pi f \circ \pi n_X \simeq n_Y \circ (\pi f \times \pi f)$. The other details follow.

□

Let (X, x_0) be an H-ring. Then we have a category $\mathsf{Cov}_{\mathsf{Hring}}/(\mathsf{X}, \mathsf{x}_0)$ of covering morphisms of H-rings $f\colon (\widetilde{X}, \tilde{x}_0) \to (X, x_0)$ based on (X, x_0). Similarly we have another category $\mathsf{Cov}_{\mathsf{CatRing}}/\pi\mathsf{X}$ of morphisms $p\colon \widetilde{G} \to \pi X$ of weak categorical rings, which are covering morphisms on the underlying groupoids.

Theorem 3.11. Let (X, x_0) be an H-ring such that the underlying space has a simply connected cover. Then the categories $\mathsf{Cov}_{\mathsf{Hring}}/(\mathsf{X}, \mathsf{x}_0)$ and $\mathsf{Cov}_{\mathsf{CatRing}}/\pi\mathsf{X}$ are equivalent.

Proof: Let $p\colon (\widetilde{X}, \tilde{x}_0) \to (X, x_0)$ be a morphism of H-rings which is a covering map on the underlying spaces. Then by Proposition 3.10, the induced morphism $\pi p\colon \pi\widetilde{X} \to \pi X$ is a morphism of weak categorical rings which is a covering morphism of underlying groupoids. So in this way we have a functor

$$\pi\colon \mathsf{Cov}_{\mathsf{Hring}}/(\mathsf{X}, \mathsf{x}_0) \to \mathsf{Cov}_{\mathsf{CatRing}}/\pi\mathsf{X}.$$

Conversely we define another functor

$$\eta\colon \mathsf{Cov}_{\mathsf{CatRing}}/\pi\mathsf{X} \to \mathsf{Cov}_{\mathsf{Hring}}/(\mathsf{X}, \mathsf{x}_0)$$

as follows:

Let $q\colon \widetilde{G} \to \pi X$ be a morphism of weak categorical rings which is a covering morphism on the underlying groupoids. Then by Theorem 1.4 there is a *lifted topology* on $\widetilde{X} = \mathrm{Ob}(\widetilde{G})$ and an isomorphism $r\colon \widetilde{G} \to \pi\widetilde{X}$ such that $p = O_q\colon \widetilde{X} \to X$ is a covering map and $q = \pi p \circ r$. Hence weak categorical ring structure on $\widetilde{G}$ transports via r to $\pi\widetilde{X}$. So we have the morphisms of groupoids

$$\widetilde{\oplus}\colon \pi\widetilde{X} \times \pi\widetilde{X} \longrightarrow \pi\widetilde{X}$$

$$\widetilde{\odot}\colon \pi\widetilde{X} \times \pi\widetilde{X} \longrightarrow \pi\widetilde{X}$$

$$\widetilde{\ominus}\colon \pi\widetilde{X} \longrightarrow \pi\widetilde{X}$$

such that $\pi p \circ \widetilde{\oplus} = \oplus \circ (\pi p \times \pi p)$, $\pi p \circ \widetilde{\odot} = \odot \circ (\pi p \times \pi p)$ and $\pi p \circ \widetilde{\ominus} = \ominus \circ \pi p$. These morphisms induce respectively the following maps which are continuous by Theorem 1.3:

$$\tilde{m} \colon \widetilde{X} \times \widetilde{X} \longrightarrow \widetilde{X}$$

$$\tilde{n} \colon \widetilde{X} \times \widetilde{X} \longrightarrow \widetilde{X}$$

$$\tilde{u} \colon \widetilde{X} \longrightarrow \widetilde{X}$$

Since (X, x_0) is an H-ring with the maps

$$m \colon X \times X, (x_0, x_0) \to (X, x_0)$$

$$n \colon X \times X, (x_0, x_0) \to (X, x_0)$$

$$u \colon (X, x_0) \to (X, x_0)$$

we have the homotopies (i)-(vii) in Definition 2.1. Then by Theorem 1.5 we have the same homotopies for $\tilde{m}$, $\tilde{n}$ and $\tilde{u}$. Therefore $(\widetilde{X}, \tilde{x}_0)$ becomes an H-ring and $p \colon (\widetilde{X}, \tilde{x}_0) \to (X, x_0)$ is a covering morphism of H-rings. The other details of the proof follow from the equivalence of the underlying spaces and groupoids. □

As similar to Theorem 1.5, due to the following theorem the liftings of homotopic functors are also homotopic.

Theorem 3.12. [11, Theorem 3.12] *Let $p \colon (\widetilde{G}, \widetilde{x}) \to (G, x)$ be a covering morphism of groupoids. Suppose that K is a simply connected groupoid, i.e, for each $x, y \in \mathrm{Ob}(K)$, $K(x, y)$ has only one morphism. Let $f, g \colon (K, z) \to (G, x)$ be the morphisms of groupoids such that f and g are homotopic. Let $\tilde{f}$ and $\tilde{g}$ be the liftings of f and g respectively. Then $\tilde{f}$ and $\tilde{g}$ are also homotopic.*

Definition 3.13. Let $\mathcal{G}$ be a weak categorical ring, $0 \in \mathrm{Ob}(\mathcal{G})$ the base point and let $\widetilde{G}$ be just a groupoid. Suppose $p \colon \widetilde{G} \to \mathcal{G}$ is a covering morphism of groupoids and $\tilde{0} \in \mathrm{Ob}(\widetilde{G})$ such that $p(\tilde{0}) = 0$. We say that the weak categorical ring structure of $\mathcal{G}$ *lifts* to $\widetilde{G}$ if there exists a weak categorical ring structure on $\widetilde{G}$ with the base point $\tilde{0} \in \mathrm{Ob}(\widetilde{G})$ such that $p \colon \widetilde{G} \to \mathcal{G}$ is a morphism of weak categorical rings □

Theorem 3.14. *Let $\widetilde{G}$ be a simply connected groupoid and $\mathcal{G}$ a weak categorical ring. Suppose that $p \colon \widetilde{G} \to \mathcal{G}$ is a covering morphism on the underlying groupoids. Let $0 \in \mathrm{Ob}(\mathcal{G})$ be the base point of $\mathcal{G}$ and $\tilde{0} \in \mathrm{Ob}\,\widetilde{G}$ such that $p(\tilde{0}) = 0$. Then the weak categorical ring structure of $\mathcal{G}$ lifts to $\widetilde{G}$.*

Proof: Since $\mathcal{G}$ is weak categorical ring, we have the following functors

- $\oplus \colon \mathcal{G} \times \mathcal{G} \to \mathcal{G}, (a, b) \mapsto a \oplus b$;
- $\odot \colon \mathcal{G} \times \mathcal{G} \to \mathcal{G}, (a, b) \mapsto a \odot b$;
- $\ominus \colon \mathcal{G} \to \mathcal{G}, a \mapsto a^{-1}$;
- $0 \colon \{\star\} \to \mathcal{G}$

with the homotopies of the functors (i)-(vii) in Definition 3.7. Since $\widetilde{G}$ is a simply connected groupoid by Theorem 1.1, the functors $\oplus$, $\odot$ and $\ominus$ lift respectively to the morphisms of groupoids:

$$\widetilde{\oplus} \colon \widetilde{G} \times \widetilde{G}, (\tilde{0}, \tilde{0}) \to (\widetilde{G}, \tilde{0})$$

$$\widetilde{\odot} \colon \widetilde{G} \times \widetilde{G}, (\tilde{0}, \tilde{0}) \to (\widetilde{G}, \tilde{0})$$

and

$$\widetilde{\ominus} \colon (\widetilde{G}, \tilde{0}) \to (\widetilde{G}, \tilde{0})$$

By Theorem 3.12 the axioms (i)-(vii) of Definition 3.7 are satisfied for the $\tilde{\oplus}$, $\tilde{\odot}$ and $\tilde{\ominus}$. Therefore $\widetilde{G}$ is a week categorical ring as required. □

The group structure of a connected topological group lifts to a simply connected covering space (e.g. [9]). In non-connected case this problem was studied in [20] (see also [2] for a lifting of R-module structure to the covering spaces and see [3] for setting R-module object in the category of groupoids). We now give a similar result for H-ring as a result of Theorem 3.14.

Corollary 3.15. Let (X, x_0) be an H-ring and $p\colon (\widetilde{X}, \tilde{x}_0) \to (X, x_0)$ a covering map. If $\widetilde{X}$ is a simply connected topological space, then H-ring structure of (X, x_0) lifts to $(\widetilde{X}, \tilde{x}_0)$, i.e, $(\widetilde{X}, \tilde{x}_0)$ is an H-ring and $p\colon (\widetilde{X}, \tilde{x}_0) \to (X, x_0)$ is a morphism of H-ring.

Proof: Since $p\colon (\widetilde{X}, \tilde{x}_0) \to (X, x_0)$ is a covering map, the induced morphism $\pi p\colon \pi\widetilde{X} \to \pi X$ is a covering morphism of groupoids. Since (X, x_0) is an H-ring by Theorem 3.8 πX is a weak categorical ring. Since $\widetilde{X}$ is simply connected the fundamental groupoid $\pi\widetilde{X}$ is a simply connected groupoid. So by Theorem 3.14 the weak categorical ring structure of πX lifts to $\pi\widetilde{X}$. So we have the induced morphisms of fundamental groupoids

$$\widetilde{\oplus}\colon \pi\widetilde{X} \times \pi\widetilde{X} \longrightarrow \pi\widetilde{X}$$

$$\widetilde{\odot}\colon \pi\widetilde{X} \times \pi\widetilde{X} \longrightarrow \pi\widetilde{X}$$

and

$$\widetilde{\ominus}\colon \pi\widetilde{X} \longrightarrow \pi\widetilde{X}$$

such that $\pi p \circ \tilde{\oplus} = \oplus \circ (\pi p \times \pi p)$ and $u \circ \pi p = \pi p \circ \tilde{u}$ and therefore we have the maps, which are continuous by Theorem 1.3.

$$\tilde{m}\colon \widetilde{X} \times \widetilde{X} \longrightarrow \widetilde{X}$$

$$\tilde{n}\colon \widetilde{X} \times \widetilde{X} \longrightarrow \widetilde{X}$$

$$\tilde{u}\colon \widetilde{X} \longrightarrow \widetilde{X}.$$

Since the axioms (i)-(vii) of Definition 2.1 are satisfied for m, n and u by Theorem 1.5 same axioms are satisfied for $\tilde{m}$, $\tilde{n}$ and $\tilde{u}$. Hence $(\widetilde{X}, \tilde{x}_0)$ becomes an H-ring and $p\colon (\widetilde{X}, \tilde{x}_0) \to (X, x_0)$ a morphism of H-rings, which is a covering map on the underlying spaces. □

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