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The Unions of Dense Metrizable Subspaces with Certain Local Properties

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Abstract. Many important examples of topological spaces can be represented as a union of a finite or countable collection of metrizable subspaces. However, it is far from clear which spaces in general can be obtained in this way. Especially interesting is the case when the subspaces are dense in the union.

We present below several results in this direction. In particular, we show that if a Tychonoff space X is the union of a countable family of dense metrizable locally compact subspaces, then X itself is metrizable and locally compact. We also prove a similar result for metrizable locally separable spaces. Notice in this connection that the union of two dense metrizable subspaces needn't be metrizable. Indeed, this is witnessed by a well-known space constructed by R.W. Heath.

1. Introduction

Given a class $\mathcal{P}$ of topological spaces with some nice properties, it is natural and instructive to investigate which spaces can be represented as the union of a countable collection of members of $\mathcal{P}$. In this article, we consider some special versions of this question. In each of these versions, $\mathcal{P}$ is a certain subclass of the class of metrizable spaces. It is well-known that the union of two metrizable subspaces needn't be metrizable - just take the Alexandroff compactification (by one point) of an uncountable discrete space. A systematic study of topological spaces which can be represented as the union of a finite collection of metrizable subspaces has been undertaken in [2]. Special attention was given there to the case when the subspaces are dense in the union.

This paper is closely related to [2]. In particular, we study in it the unions of countable collections of dense metrizable locally separable subspaces. The main conclusion here is that every space of this kind is again metrizable and locally separable. We also observe that (see Theorem 3.1) that if $\gamma = \{X_n : n \in \omega\}$ is a countable family of dense metrizable locally compact subspaces of a space Z, then their union $X = \cup\{X_n : n \in \omega\}$ is also a locally compact and metrizable subspace of Z. The last section contains some new results on the finite unions of not necessarily dense metrizable subspaces.

By "a space" we understand a Tychonoff topological space. Notation and terminology are as in [5].

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2. General observations and preliminary resultson unions of dense subspaces

Given an arbitrary topological property $\mathcal{P}$, we should not expect that the union of two dense subspaces with the property $\mathcal{P}$ would have $\mathcal{P}$. This can be seen from the following example which shows that there exists a non-normal space which is the union of two dense normal open subspaces.

Example 2.1. *Let T be the Deleted Tychonoff Plank. Then $T = [0, \omega_1] \times [0, \omega] \setminus \{(\omega_1, \omega)\}$ is Tychonoff but is not normal. Let A be the set of all points of T with first coordinate ω_1 and B be the set of all points with second coordinate ω. Then $U = T \setminus A = [0, \omega_1) \times [0, \omega]$ and $V = T \setminus B = [0, \omega_1] \times [0, \omega)$ are open, dense normal subspaces of T. Clearly, $T = U \cup V$.*

Some basic observations on properties of spaces represantable as the union of two dense metrizable subspaces can be found in [2]. In particular, we use below the following two well known facts:

Proposition 2.2. *If a space X is the union of a countable family of dense metrizable subspaces, then X has a σ-disjoint base (and therefore, X is a space with a point-countable base).*

Example 2.3. *There exists a non-metrizable space X which is the union of two dense metrizable subspaces and has a uniform base. This is the famous Heath space described in [8] (see also [5][5.4.B]). In this paper, we denote this space by H_1. The space H_1 is the union of two discrete (hence, metrizable, locally compact and locally separable) subspaces, one of which is open and another is closed. But it can be also represented as the union of two dense metrizable (non-discrete) subspaces. The space H_1 is also Čech-complete and has a uniform base. However, the space H_1 is not normal [8]. Therefore, it is not paracompact and not metrizable.*

It seems rather natural to conjecture at this point that the non-metrizability of the union of two dense metrizable subspaces can occur *only* when paracompactness is lost under the union. However, this is not so: it has been shown in [2] that there exists a paracompact non-metrizable space X which is the union of two dense metrizable subspaces. In this connection, we are going to establish below that a certain paracompactness type property is preserved by finite dense unions, at least in the class of first-countable spaces. This result will play an essential role in the proofs of some addition theorems.

Proposition 2.4. *Suppose that $\gamma = \{U_\alpha : \alpha \in \Lambda\}$ is a family of open subsets of a first-countable space X, and A is a dense subset of X such that γ is point-countable at every point of A (that is, $|\{U \in \gamma : x \in U\}| \leq \omega$, for each $x \in A$). Then γ is point-countable at each $x \in X$.*

Proof. Take any $x \in (X \setminus A)$. Since X is first-countable, there exists a countable base $\mathcal{B}(x)$ at the point x. Clearly, for each $U_\alpha \in \gamma$ which contains x, there exists $B_n \in \mathcal{B}(x)$ such that $B_n \subset U_\alpha$. Since A is dense in X, and γ is point-countable at every point of A, each $B_n \in \mathcal{B}(x)$ is contained by not more than countably many $U_\alpha \in \gamma$. Therefore, the family $\{U_i : x \in U_i \in \gamma\}$ is countable. □

A space X is said to be *metalindelöf* if every open cover of X can be refined by a point-countable open cover.

Theorem 2.5. *Suppose that a first-countable space X is the union of two (of a countable family) of metalindelöf dense subspaces. Then X is metalindelöf.*

Proof. Let $X = A \cup B$, where A and B are dense metalindelöf subspaces of X. Take any open cover $\mathcal{U}$ of X. Clearly, $\mathcal{U}_A = \{U \cap A : U \in \mathcal{U}\}$ covers A. Hence, there exists a point-countable refinement $\mathcal{V}$ of $\mathcal{U}_A$ by open subsets of A which covers A. For each $V \in \mathcal{V}$ we choose an open subset O_V of X such that $V = O_V \cap A$ and O_V is contained in a member of $\mathcal{U}$. The family $\gamma_1 = \{O_V : V \in \mathcal{V}\}$ of open subsets of X is point-countable at every point of A. Therefore, by Proposition 2.4, the family γ_1 is point-countable at every point of X. In a similar way, we define a family γ_2 of open subsets of X which is point-countable at every point of X, refines $\mathcal{U}$ and covers B. Then $\gamma_1 \cup \gamma_2$ is a point-countable open refinement of $\mathcal{U}$. □

The next result immediately follows from the last statement.

Corollary 2.6. *If X is the union of a countable family of dense metrizable subspaces, then X is metalindelöf.*

Let us show that metrizability is preserved under countable dense unions of locally compact subspaces.

Observe that if A and B are dense locally compact subspace of a space X, then $Z = A \cup B$ is a locally compact subspace of X. This is so, since every dense locally compact subset of a space is open and every compact subspace of a space is closed. We will also need below the following two simple factts:

Theorem 2.7. *The union of any family of locally compact dense subspaces of a space is locally compact.*

Lemma 2.8. *Every separable subspace of a space with a point-countable base has a countable base.*

The next lemma is a version of Alexandroff's result in [1][Lemma.4.1]. Since it plays an essential role below, we present its proof for the sake of completeness.

Lemma 2.9. *If a space X is covered by a point-countable family of open separable subspaces of X, then $X = \bigoplus_{s\in S} X_s$, where all X_s are separable.*

Proof. Let X be covered by a point-countable family $\mathcal{V}$ of open separable subspaces of X. Clearly, each member of $\mathcal{V}$ meets only countably many members of $\mathcal{V}$. Define an equivalence relation "$\equiv$ " on $\mathcal{V}$ as follows: For any V_α, $V_\beta \in \mathcal{V}$, $V_\alpha \equiv V_\beta$ if there exists a finite sequence $V_0, V_1, ..., V_k$ of members of $\mathcal{V}$ such that $V_0 = V_\alpha$ and $V_k = V_\beta$, and $V_i \cap V_{i+1} \neq \emptyset$ for $i = 0, 1, 2, ..., k-1$.
For any $V \in \mathcal{V}$, $[V]$ denotes the equivalence class of V.
Claim. $[V]$ is countable.
Define $\mathcal{A}_0 = \{V\}$ and $\mathcal{A}_k = \{V_\beta \in \mathcal{V} :$ there exists $V_\alpha \in \mathcal{A}_{k-1}$ such that $V_\alpha \cap V_\beta \neq \emptyset\}$. By using mathematical induction, we can verify that $\mathcal{A}_k$ is countable for each integer k. Since $[V] = \bigcup_k \mathcal{A}_k$, it follows that $[V]$ is countable.
Let S be an index set enumerating distinct equivalence classes of the relation "$\equiv$ ". Take the union $X_s = \bigcup \mathcal{V}_s$ for each distinct equivalence class $\mathcal{V}_s$. Since $\mathcal{V}_s$ is countable and each member of $\mathcal{V}_s$ is separable, X_s is separable for each $s \in S$. Therefore X can be covered by the pairwise disjoint open separable subspaces $\{X_s\}_{s\in S}$. It follows that $X = \bigoplus_{s\in S} X_s$ where all X_s are separable. □

The next statement is a part of the folklore. It easily follows from some classical results of P.S. Alexandroff and A.S. Mischenko. We present its proof for the sake of completeness.

Proposition 2.10. *Every locally compact space with a point-countable base is metrizable.*

Proof. Let C be a point-countable base for a locally compact space X. Since every point of X has a compact neighbourhood with a point-countable base, a well-known result of Miščenko (see [9]) implies that X is covered by a subcollection $\mathcal{B}$ of C whose members are separable. It follows by Lemma 2.9 that $X = \bigoplus_{s\in S} X_s$, where all X_s are open separable subspaces of X. Also, by Lemma 2.8, all X_s are metrizable. Therefore, X is metrizable. □

3. The main results

Now we present our first result on dense unions of metrizable spaces.

Theorem 3.1. *If $\gamma = \{X_n : n \in \omega\}$ is a countable family of dense metrizable locally compact subspaces of a space Z, then the subspace $X = \cup\{X_n : n \in \omega\}$ of Z is also locally compact and metrizable.*

Proof. Clearly, each X_n is an open subspace of X. Since each X_n has a point-countable base, it follows that X also has a point-countable base. Therefore, by Theorem 2.7 and Proposition 2.10, the space X is metrizable and locally compact. □

Notice that there are no restrictions on the cardinality of the family of subspaces in Theorem 2.7. Let us show in this connection that the restriction in Theorem 3.1 that the family γ is countable cannot be dropped: there exists a space which is neither metrizable, nor paracompact, but can be represented as the union of some uncountable family of dense metrizable locally compact subspaces.

Example 3.2. *Let $L = [0, \omega_1) \times [0, 1)$ be the Long Line. Thus, $L = \{(\alpha, i) : \alpha < \omega_1$ and $i \in [0, 1)\}$ is given the topology generated by the lexicographic order. For $\beta \in \omega_1$, we denote by M_β the set of all $(\alpha, 0) \in L$, where α is a countable limit ordinal such that $\beta \leq \alpha < \omega_1$. Put $D_\beta = L \setminus M_\beta$, for $\beta \in \omega_1$. Then each D_β is open and dense in L, since M_β is closed and nowhere dense in L. It is also clear that each D_β is metrizable and locally compact. Obviously, $L = \cup\{D_\beta : \beta \in \omega_1\}$, that is, L is the union of an uncountable chain of dense open metrizable locally compact subspaces. However, the space L is not metrizable and is not paracompact, even though it is locally metrizable and locally compact.*

3.1. Dense unions of metrizable locally separable subspaces

Let us now consider the more delicate case of the unions of countable families of dense metrizable locally separable subspaces and establish that the spaces of this kind have the same structure.

The next statement, belonging to P.S. Alexandroff, easily follows from Lemma 2.9.

Lemma 3.3. *[1] A space X is metrizable and locally separable if and only if $X = \bigoplus_{s\in S} X_s$, where all spaces X_s are separable and metrizable.*

We also need the next well-known fact:

Lemma 3.4. *If U is an open subset of a space X, and A is a dense subset of X, then $A \cap U$ is dense in U.*

Theorem 3.5. *Suppose that a space X is the union of countably many subspaces A_n, where each A_n is a dense in X metrizable locally separable subspace of X. Then X is metrizable and locally separable.*

Proof. Since A_n is metrizable and locally separable, we can fix a point-countable base $\mathcal{P}_n$ in the space A_n such that each member of $\mathcal{P}_n$ is separable. For every $V \in \mathcal{P}_n$, we can choose an open subset $U(V)$ of X such that $U(V) \cap A_n = V$. Put $\mathcal{B}_n = \{U(V) : V \in \mathcal{P}_n\}$. Since A_n is dense in X, it follows from Lemma 3.4 that every member of $\mathcal{B}_n$ is separable, and that $\mathcal{B}_n$ is a base for X at every point of A_n. Clearly, $\mathcal{B}_n$ is also point-countable at every point of A_n. Observe that X is, obviously, first-countable. Now it follows from Proposition 2.4 that the family $\mathcal{B}_n$ is point-countable at every point of X. Clearly, the family $\mathcal{W} = \bigcup_{n\in\omega} \mathcal{B}_n$ is a point-countable base for X such that every member of $\mathcal{W}$ is separable. Then, by Lemma 2.8 and Lemma 2.9, X is metrizable and locally separable. □

The next theorem resembles the last result, but does not follow from it. The assumptions and the conclusion in it are weaker than in the preceding theorem.

Theorem 3.6. *Suppose that X is the union of countably many subspaces A_n, where each A_n is a dense in X, metalindelöf, first-countable, locally separable subspace of X. Then X is also metalindelöf, first-countable, and locally separable.*

Proof. First, observe that X is locally separable, by Lemma 3.4. Since each A_n is a dense first-countable subspace of X and X is regular, it follows that X is first-countable. Let $\mathcal{U}$ be any open cover of X. Then $\mathcal{U}_{A_n} = \{U \cap A_n : U \in \mathcal{U}\}$ covers A_n. Therefore, $\mathcal{U}_{A_n}$ has a point-countable refinement $\mathcal{V}_n$. For each $V \in \mathcal{V}_n$ we choose an open subset O_V of X such that $V = O_V \cap A_n$ and O_V is contained in a member of $\mathcal{U}$. Then the family $\gamma_n = \{O_V : V \in \mathcal{V}_n\}$ is point-countable at every point of A_n. By Proposition 2.4, each γ_n is point-countable at every point of X. Put $\gamma = \bigcup_n \gamma_n$. Then, clearly, γ is a point-countable open refinement of $\mathcal{U}$. □

Example 3.7. *Theorems 3.5 and 3.6 cannot be extended to the unions of uncountable families of dense metrizable separable subspaces. We establish this fact here, assuming the Continuum Hypothesis CH. Indeed, using CH, S. Franklin and M. Rajagopalan have constructed a separable space P_1 which is the union of an uncountable chain of separable metrizable subspaces, is locally compact, and contains a topological copy of the space ω_1 as a closed subspace [6], [5][3.12.17(c)]. Clearly, the space P_1 is not metalindelöf, and hence, it is not metrizable.*

On the other hand, we should mention another simple fact: local separability, as well as separability, is obviously preserved by arbitrary dense unions.

3.2. Some applications

We denote by $\mathcal{M}_{fu}$ the class of spaces which can be represented as the union of a finite family of metrizable subspaces. The class $\mathcal{M}_{fu}$ is much wider than the class of subspaces of unions of finite families of dense metrizable subspaces.

Let us show that it is possible to apply the results obtained in this section to members of $\mathcal{M}_{fu}$ and to learn certain new facts concerning them. The key role in this study belongs to the next lemma from [2]:

Lemma 3.8. *[2] Suppose that a space X is the union of a finite family μ of subspaces. Then there exists a finite disjoint family η of open subspaces of X such that $\cup\eta$ is dense in X, and for every $V \in \eta$ there exists a subfamily ν of μ such that $V \cap M$ is dense in V for every $M \in \nu$, and $V \subset \cup\nu$.*

The above lemma can be applied to many situations [2]. We just mention two of such applications below.

Theorem 3.9. *Suppose that a space X is the union of a finite collection μ of metrizable locally compact subspaces. Then X has a dense open metrizable locally compact subspace.*

Proof. This theorem obviously follows from Lemma 3.8 and Theorem 3.1. □

Similarly, with the help of Theorem 3.5, the following theorem is proved:

Theorem 3.10. *Suppose that a space X is the union of a finite collection μ of metrizable locally separable subspaces. Then X, and every subspace Y of X, has a dense open metrizable locally separable subspace.*

Our next result concerns the following well-known property of metric spaces: every pseudocompact subspace of a metrizable space X is closed in X and compact. The same property holds in all Eberlein compacta and in some function spaces with the topology of pointwise convergence. See about this [10]. A. Grothendieck has shown that certain classical function spaces have the following weaker property: if for every infinite subset of a subspace A of a space Z there exists in Z a point of accumulation (in this case we say that A is *countably compact in Z*), then the closure of A in Z is compact [7]. We contribute to this topic with the following theorem:

Theorem 3.11. *If a space X is the union of a countable family of dense metrizable subspaces, and a subset A of X is countably compact in X, then the closure of A in X is metrizable and compact.*

Proof. Indeed, X has a point-countable base. Therefore, the closure F of A in X is also a space with a point-countable base. Since A is dense in F and countably compact in F, it follows from a result in [3] that F is compact and metrizable. □

The word "dense" in the above statement cannot be dropped. Mrowka's space shows that Theorem 3.11 cannot be extended to the union of two non-dense metrizable (even discrete) subspaces. Indeed, this space M is not countably compact, but is first-countable and contains a countable dense subset A which is countably compact in M.

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