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On Lipschitz Mappings of the Unit Circle onto a Convex Curve and their Extension

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In celebration of Matti Vuorinen's 65-th birthday

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Abstract. Among the other results, in this paper we prove the following result: Given a $L-$biLipschitz mapping of the unit circle onto a convex Jordan curve, there exists a $\frac{3}{2}L^3$-biLipschitz extension of the plane onto itself.

1. Introduction

Let n be a positive integer and $\mathbf{R}^n$ be the Euclidean space with the Euclidean norm $|\cdot|$. Given a set $C \subseteq \mathbf{R}^n$ and a function $u : C \to \mathbf{R}^n$, and $L > 0$, we say that u is L Lipschitz (L biLipschitz) if, for any $x \neq y \in C$, one has

$$|u(y) - u(x)| \leq L|y - x|\,; \tag{1}$$

and

$$|x - y|/L \leq |u(y) - u(x)| \leq L|y - x|\,. \tag{2}$$

The least such a constant L is denoted by $\mathbf{Lip}(f)$ and $\mathbf{biLip}(f)$ respectively.

Motivated by the Kirszbrauns theorem the following problem naturally arises.

Problem 1.1. *a) Given a bi-Lipschitz embedding $f : \mathbf{R} \to \mathbf{C}$ of the real line $\mathbf{R}$ into the complex plane $\mathbf{C}$ with the bi-Lipschitz constant L find the bi-Lipschitz extension with the minimal Lipschitz constant L'.*

b) Given a bi-Lipschitz embedding $f : \mathbf{T} \to \mathbf{C}$ of the unit circle $\mathbf{T}$ into the complex plane $\mathbf{C}$ with the bi-Lipschitz constant L find the bi-Lipschitz extension with the minimal Lipschitz constant L'.

Problem 1.1 a) has in one direction the optimal solution. Namely Kovalev in [5], by using Beurling-Ahlfors extension achieved $L' = CL$, with a a universal constant C. In the same paper he showed that L' is in general bigger than L.

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It seems that Problem 1.1 b) is much more delicate. Some pioneering work on this topic has been done by Tukia and Väisälä ([7–9]). Among the other results Tukia in [7] proved the following theorem. Let $f : \mathbf{T} \to \mathbf{C}$ be an L bi-Lipschitz map. Then there exists an extension $F : \mathbf{C} \to \mathbf{C}$ which is also bi-Lipschitz, with constant $\widetilde{L}$ depending only on L. In a recent paper Daneri and Pratelli [1] achieved $L' = CL^4$ with a universal constant C.

We will consider this problem for two special cases:

- γ is a convex curve. We will show for this special case that $L' \leq \frac{3}{\sqrt{2}}L^3$ (Theorem 2.3).
- γ is a starlike w.r.t. origin, and the parametrization is polar. We will show that if $f(t) = r(t)e^{it}$ is $L-$bi-Lipschitz, then $L' \leq \frac{3}{2}L^3$ (Theorem 3.1). In particular if γ is convex, then $L' \leq \frac{3}{2}L$.

Since the polar parametrization is a special parametrization and it not yields the optimal distortion of mappings of the unit circle onto a curve γ, then the following problem naturally arises.

Problem 1.2. *a) For a given rectifiable Jordan curve γ find a homeomorphism $f : \gamma \to \mathbf{T}$ with the smallest Lipschitz constant L.*

b) For a given rectifiable Jordan curve γ find a homeomorphism $f : \mathbf{T} \to \gamma$ with the smallest Lipschitz constant L.

We will solve Problem 1.2 a) and Problem 1.2 b) for certain classes of curves (see the fourth section). In the same section, we will prove that there exists a $1-$Lipschitz mapping between two convex curves and surfaces, provided the second is inside the first one (Theorem 4.13).

2. bi-Lipschitz extension of convex embedding

First we prove a lemma in more general setting.

Lemma 2.1. *Let γ be a Jordan curve that lies between parallel lines p and q in the complex plane $\mathbb{C}$ such that there is a line s which is orthogonal to p containing two common points with γ. If f is a Lipschitz homeomorphism between γ and the unit circle $\mathbb{T}$, then $\mathbf{Lip}(f) \geq 2/\mathrm{dist}(p, q)$.*

Proof. By applying an isometry of the Euclidean plane $\mathbb{C}$, we can assume that $p = \{z : \operatorname{Im} z = \mathrm{dist}(p, q)/2\}$ and $q = \{z : \operatorname{Im} z = -\mathrm{dist}(p, q)/2\}$ and that s is the imaginary axis. Let $s \cap \gamma = \{P, Q\}$. Let $A = f(P)$ and $B = f(Q)$. If A and B are opposite points, then $\mathbf{Lip}(f) \geq 2/\mathrm{dist}(P, Q) \geq 2/\mathrm{dist}(p, q)$. In the opposite case the points A and B lie in a common semicircle.

Next, we consider A' and B', the points on the unit circle which are opposite A and B, respectively. Specifically, parameterize the portion of the circle from A to B' as $C(t)$ and from A' to B as $C'(t)$ such that $C(t)$ and $C'(t)$ are opposite points on the circle for every $t \in [0, 1]$. Note that by the homeomorphism, $f^{-1}(A')$ and $f^{-1}(B')$ lie on the same side of the y-axis. Then by the intermediate value theorem, there is a t such that $f^{-1}(C(t))$ and $f^{-1}(C'(t))$ have the same x-coordinate. But that implies $|f^{-1}(C(t)) - f^{-1}(C'(t))| \leq \mathrm{dist}(p, q)$. Then we have

$$\begin{aligned} 2 &= |f(f^{-1}(C(t))) - f(f^{-1}(C'(t)))| \\ &\leq \mathbf{Lip}(f)|f^{-1}(C(t)) - f^{-1}(C'(t))| \\ &\leq \mathbf{Lip}(f)\mathrm{dist}(p, q). \end{aligned}$$

This implies the desired result. □

Now we recall some facts [2]. Let γ be a smooth starlike Jordan curve w.r.t. the origin in $\mathbf{C}$ such that every tangent line of γ is disjoint from the origin. We will recall some properties of γ. Let $s \mapsto r(s)e^{is}$ be the polar parametrization of γ. The tangent t_s of γ at $\zeta = r(s)e^{is}$ is defined by

$$y = r(s)e^{is} + (r'(s) + ir(s))e^{is}x, \quad x \in \mathbf{R}.$$

Following the notations in , the acute angle α_s between ζ and the positive oriented tangent at ζ is given by

$$\cot\alpha_s = \frac{r'(s)}{r(s)}. \tag{3}$$

Let $G : \mathbf{T} \to \gamma$ be a continuous locally injective function from the unit circle $\mathbf{T}$ onto the star-like Jordan curve γ smooth almost everywhere. Then

$$g(t) = \rho(t)e^{i\psi(t)} = G(e^{it}),\ \ t \in [0, 2\pi)$$

is a parametrization of γ. If G is an orientation preserving mapping then ψ obviously is monotone increasing. Suppose that g is differentiable. Since $r(\psi(t)) = \rho(t)$, we deduce that $\rho'(t) = r'(\psi(t)) \cdot \psi'(t)$. Hence

$$r'(\psi(t)) = \frac{\rho'(t)}{\psi'(t)}. \tag{4}$$

By (4) and (3) we obtain

$$\rho'(t) = \rho(t)\psi' \cot\alpha_{\psi(t)}. \tag{5}$$

If $f(e^{it}) = \rho(t)e^{i\psi(t)}$, $\varphi(t) = \rho(t)e^{i\psi(t)}$ we set here and in the sequel

$$F(z) = |z| f(z/|z|). \tag{6}$$

Then the following relations has been established ([2]):

$$\begin{aligned} h(t) &:= \|DF(re^{it})\| \\ &= \frac{1}{2}\left(|\varphi'(t) + i\varphi(t)| + |\varphi'(t) - i\varphi(t)|\right) \\ &= \frac{\rho(t)}{2}\left\{\sqrt{\frac{(\psi'(t))^2}{\sin^2\alpha_\psi} + 2\psi'(t) + 1} + \sqrt{\frac{(\psi'(t))^2}{\sin^2\alpha_\psi} - 2\psi'(t) + 1}\right\} \end{aligned} \tag{7}$$

and

$$\begin{aligned} g(t) &:= \|D(F^{-1})(F(re^{it})\| \\ &= \frac{1}{2}\left((|\varphi'(t) - i\varphi(t)| - |\varphi'(t) + i\varphi(t)|)\right)^{-1} \\ &= \frac{\sqrt{\frac{(\psi'(t))^2}{\sin^2\alpha_\psi} + 2\psi'(t) + 1} + \sqrt{\frac{(\psi'(t))^2}{\sin^2\alpha_\psi} - 2\psi'(t) + 1}}{2\rho(t)\psi'(t)}. \end{aligned} \tag{8}$$

From (7) and (8) we obtain

$$\mathbf{Lip}(F) = \|h\| := \operatorname{ess\,sup}_t h(t) \le \operatorname{ess\,sup}_t \frac{\rho(t)}{\sqrt{2}}\sqrt{\frac{(\psi'(t))^2}{\sin^2\alpha_\psi} + 1} \tag{9}$$

and

$$\mathbf{Lip}(F^{-1}) = \|g\| := \operatorname{ess\,sup}_t g(t) \le \operatorname{ess\,sup}_t \frac{\sqrt{\frac{(\psi'(t))^2}{\sin^2\alpha_\psi} + 1}}{\sqrt{2}\rho(t)\psi'(t)}. \tag{10}$$

Thus

$$\mathbf{biLip}(F) = \max\{\|g\|, \|h\|\}.$$

In particular, if $\psi(s) \equiv s$, then

$$h(t) = \frac{\rho(t)}{2}(|\csc\alpha_t| + \sqrt{\csc^2\alpha_t + 3}) \tag{11}$$

and

$$g(t) = \frac{1}{2\rho(t)}(|\csc\alpha_t| + \sqrt{\csc^2\alpha_t + 3}). \tag{12}$$

Lemma 2.2. *Let $f(e^{it}) = \rho(t)e^{i\phi(t)}$ be a $L-$biLipschitz homeomorphism of the unit circle $\mathbf{T}$ onto a Jordan curve γ starlike w.r.t. origin and smooth almost everywhere. Then for almost every t we have*

$$\frac{1}{L} \leq \frac{\rho(t)|\psi'(t)|}{\sin\alpha_\psi} \leq L \tag{13}$$

and if in addition γ is convex then

$$\frac{1}{\rho(t)\sin\alpha_\psi} \leq \frac{1}{\mathrm{dist}(\gamma, 0)}. \tag{14}$$

In the opposite direction, if

$$\frac{\rho(t)|\psi'(t)|}{\sin\alpha_\psi} \leq L \tag{15}$$

then f is $\pi L/2$-Lipschitz.

Proof. The relation (13) follows from the simple relation

$$\lim_{s\to t} \frac{|f(e^{it}) - f(e^{is})|}{|e^{it} - e^{is}|} = \left|\frac{\partial\rho(t)e^{i\psi(t)}}{\partial t}\right| = \frac{\rho(t)|\psi'(t)|}{\sin\alpha_\psi}.$$

Let T_t be the tangent of the curve γ at $f(t)$. Then we have $\mathbf{dist}(T_t, 0) \geq \mathbf{dist}(\gamma, 0) = \min_t \rho(t)$. Further by Sine Rule we have

$$\frac{\mathbf{dist}(T_t, 0)}{\rho(t)} = \sin\alpha_\psi.$$

This implies (14). Relation (15) follows from the simple relations

$$\frac{|f(e^{ix}) - f(e^{iy})|}{|e^{ix} - e^{iy}|} \leq \frac{\pi}{2}\frac{|f(e^{ix}) - f(e^{iy})|}{|x-y|} \leq \frac{\pi}{2}\sup_t |\partial_t f(e^{it})|.$$

□

Theorem 2.3. *If f is a $L-$biLipschitz convex embedding of the unit circle into $\mathbf{C}$, then it provides a $\frac{3}{\sqrt{2}}L^3$ bi-Lipschitz extension onto the whole complex plane*

Proof. Let $K = f(\mathbf{T})$ and let $D = \mathbf{Diam}(K) = |a - b|$, where $a, b \in K$. Let $z \in [a, b]$ and γ_z be the part of the line orthogonal to $[a, b]$ inside the curve K and let $d_0 = \max_{z\in\gamma} |\gamma_z| = |\gamma_p|$ for some $p \in [a, b]$. Let c be the center of $[a, b]$ and assume w.l.g. that c is between a and p. Since the triangle determined by a and the segment γ_p is inside K, because K is convex, and since $|a - c| \geq |p - c|$, it follows that $d = |\gamma_c| \geq d_0/2 = |\gamma_p|/2$. By Lemma 2.1 we have $\mathbf{Lip}(f^{-1}) \geq 2/d_0$, i.e. $d_0 \geq 2/L$. Thus

$$d \geq \frac{1}{L}. \tag{16}$$

Further, $a = f(e^{i\alpha})$ and $b = f(e^{i\beta})$ and therefore $|a - b| \le L|e^{i\alpha} - e^{i\beta}|$ and this implies that

$$D \le 2L. \tag{17}$$

Let ζ be the center of γ_c and, by using a translation, which are bi-Lipschitz invariant transformation we can assume that $\zeta = 0$. Let F be the radial extension of f. Then the minimal angle that makes the tangent of the curve γ at a given point z with the point z is bigger than $\arctan(\frac{d}{D}) \ge \arctan(1/(2L^2))$. Thus

$$\sin\alpha \ge \frac{1}{\sqrt{1 + 4L^4}}. \tag{18}$$

Further we have

$$\frac{1}{\rho(t)\sin\alpha_\psi} \le \frac{2}{d}. \tag{19}$$

$$\frac{1}{L} \le \frac{\rho(t)|\psi'(t)|}{\sin\alpha_\psi} \le L. \tag{20}$$

Thus

$$\begin{aligned}\frac{1}{\sqrt{2}}\sup_t \sqrt{\frac{1}{\rho^2(t)\sin^2\alpha_\psi} + \frac{1}{\rho^2(t)(\psi'(t))^2}} &\le \frac{1}{\sqrt{2}}\sqrt{\frac{4}{d^2} + \frac{L^2}{\sin^2\alpha}}\\ &\le \frac{1}{\sqrt{2}}\sqrt{5L^2 + 4L^6} \le \frac{3\sqrt{2}}{2}L^3.\end{aligned}$$

On the other hand

$$\sup_t \frac{\rho(t)}{\sqrt{2}}\sqrt{\frac{(\psi'(t))^2}{\sin^2\alpha_\psi} + 1} \le \frac{\sqrt{L^2 + 4L^2}}{\sqrt{2}} = \frac{\sqrt{5}}{\sqrt{2}}L.$$

This finishes the proof.

□

3. bi-Lipschitz extension of polar parametrization

Theorem 3.1. *Let $f(e^{it}) = \rho(t)e^{it}$ be a $L-$bi-Lipschitz mapping of the unit circle onto a starlike Jordan curve Γ. Then it provides a $\frac{3}{2}L^3-$ bi-Lipschitz extension of the complex plane onto itself. The power 3 is the best possible if we restrict ourselves to radial extensions. If in addition γ is convex then it provides a $\frac{3}{2}L$ bi-Lipschitz extension.*

The result follows from the following two lemmas

Lemma 3.2. *If $f(e^{it}) = \rho(t)e^{it}$ is a $L-$bi-Lipschitz homeomorphism of the unit circle onto the starlike Jordan curve γ, then* $\mathbf{Lip}(F) \le \frac{3}{2}L$.

Proof. From (11) and (13) we have

$$\mathbf{Lip}(F) = \|h\| = \text{ess sup}_t \frac{\rho(t)}{2}(|\csc\alpha_t| + \sqrt{\csc^2\alpha_t + 3}) \le \frac{3}{2}\frac{\rho(t)}{\sin\alpha_t} \le \frac{3}{2}L.$$

□

Lemma 3.3. *If $f(e^{it}) = \rho(t)e^{it}$ is a $L-$bi-Lipschitz homeomorphism of the unit circle onto the starlike Jordan curve γ, then* $\mathbf{Lip}(F^{-1}) \le \frac{3}{2}L^3$. *If the curve γ is convex then* $\mathbf{Lip}(F^{-1}) \le \frac{3}{2}L$.

Proof. We use the following proposition ([3]).

Proposition 3.4. *If $f(e^{it}) = r(t)e^{it}$ is a homeomorphism of the unit circle onto the starlike Jordan curve γ, and $|z| \le r_0$ is an inscribed circle inside γ touching γ in a point, then* $\mathbf{Lip}(f) = 1/r_0$.

We now use (12) and (14) in order to obtain

$$\begin{aligned}\mathbf{Lip}(F^{-1}) = \|g\| &= \operatorname{ess\,sup}_t \frac{1 + \sqrt{1 + 3\sin^2\alpha_t}}{2r(t)\sin\alpha_t} \\ &\le \frac{3}{2r} \le \frac{3}{2}L,\end{aligned}$$

provided that γ is convex. If γ is not convex, then from (13) and Proposition 3.4 we have

$$\begin{aligned}\mathbf{Lip}(F^{-1}) = \|g\| &= \operatorname{ess\,sup}_t \frac{1 + \sqrt{1 + 3\sin^2\alpha_t}}{2r(t)\sin\alpha_t} \\ &\le L \cdot \operatorname{ess\,sup}_t \frac{1 + \sqrt{1 + 3\sin^2\alpha_t}}{2r(t)^2} \\ &\le \frac{3}{2}L^3.\end{aligned}$$

In order to prove that the power 3 is optimal we take the following example. Let $A = 1$, $B = i/L$, $C = -1$ and $D = -i/L$ and construct a curvilinear (concave) rectangle γ with vertices A, B, C and D symmetric with respect to both axis and such that the angle at points A and B is approximately $2\arcsin(1/L)$ and in the other two vertices is approximately $2\arcsin(1/L^2)$. See Fig 1. Let $f(t) = \rho(t)e^{it}$ be polar parametrization of γ. Then by Proposition 3.4, $\mathbf{Lip}(f^{-1}) \asymp L$. *Here and in the sequel $A \asymp B$ means that there exists an absolute constant C such that* $1/C \le A/B \le C$. By Lemma 3.2 we have $\mathbf{Lip}(f) \le 3/2L$. But for $F = |z|f(z/|z|)$ we have $\mathbf{Lip}(F^{-1}) \asymp L^3$. Thus the power 3 is optimal in this context.

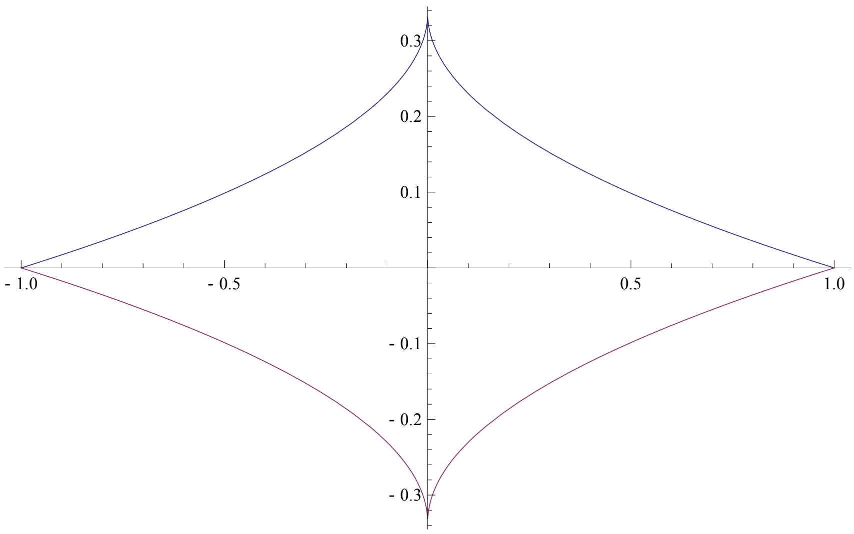

Figure 1: The curve γ for $L = 3$.

□

The following example implies that the set of polar parametrization of a curve w.r.t. some interior point are not optimal bi-Lipschitz mappings.

Example 3.5. *Every polar parametrization of the unit circle of onto the triangle $\Delta = \Delta(A, B, C)$ with the vertices $A = -1/L$, $B = 1/L$, $C = iL$, w.r.t any inner point, has at least a quadratic growth of bi-Lipschitz constant as a function of L, but the following mapping and its radial extension has linear growth of distortion w.r.t. L. Define $\varphi : \mathbf{T} \to \Delta$ by*

$$\varphi(e^{it}) = \begin{cases} \frac{1}{L} + \frac{2(-\frac{1}{L}+iL)t}{\pi} & 0 \le t \le \pi/2 \\ iL + \frac{2(-\frac{1}{L}-iL)(-\frac{\pi}{2}+t)}{\pi} & \pi/2 \le t \le \pi \\ \frac{-3\pi+2t}{L\pi} & \pi \le t \le 2\pi \end{cases}.$$

Then it can be proved that $\mathbf{Lip}(\varphi) = \frac{2\sqrt{L^2+L^{-2}}}{\pi}$ *and* $\mathbf{Lip}(\varphi^{-1}) = \frac{\pi L}{2}$.

Let $f(e^{it}) = \varphi(e^{it}) - i/L$ and define $F(z) = |z| f(z/|z|)$. Then

$$\mathbf{Lip}(F) \asymp L.$$

and

$$\mathbf{Lip}(F^{-1}) \asymp L.$$

4. Minimal distortion of mappings between Jordan curve and the unit circle

To motivate the problem which we consider in this section, assume that γ is a rectifiable Jordan curve of length $|\gamma|$. Let $g : [0, |\gamma|] \to \gamma$ be arc-length parametrization and define the mapping $f(e^{it}) = g(t\frac{|\gamma|}{2\pi})$. Then

$$\frac{|f(e^{it}) - f(e^{is})|}{|e^{it} - e^{is}|} = \frac{|f(e^{it}) - f(e^{is})|}{|t - s|}\frac{|t - s|}{|e^{it} - e^{is}|} \le \frac{|\gamma|}{2\pi}\frac{\pi}{2} = \frac{|\gamma|}{4}.$$

Thus

$$\mathbf{Lip}(f) \le \frac{|\gamma|}{4}. \tag{21}$$

The questions arises, is the inequality (21) sharp, and is the arc-length parametrization optimal for the Lipschitz constant for the mapping f. The answer to the first question is affirmative, since we can consider a rectangle which is approximately equal to the interval $[-\pi/2, \pi/2]$. Then the arc-length parametrization of the rectangle provides the Lipschitz constant approximately equal to $\pi/2$. However the constant $|\gamma|/4$ is far from optimal if γ is the unit circle. In this case of course the Lipschitz constant is 1. In order to find the optimal Lipschitz constant of mappings between certain open domains in Euclidean spaces or manifolds, we arrive to infinity-harmonic equations. Since we do not involve into this equation, we skip the details. We recall the first specific examples of infinity harmonic mapping

Example 4.1 (Infinity - harmonic curves). *([6]) Any regular curve $X : (a, b) \to (M^m, g)$ is an infinity-harmonic map provided it is parameterized by arc length. Then* $\mathbf{Lip}(f) = 1$.

Example 4.1 states that the best Lipschitz constant throughout all parameterizations of a curve is attained by an arc-length parametrization. Now we consider the class of mappings from the unit circle and a curve γ in complex plane. The question arises what is the least distortion throughout the class. Since chordal metric in the unit circle is not a Riemannian metric, we cannot obtain any conclusion from Example 4.1.

In the rest of the section we will see that the arc-length parameterization in very rare cases yields the optimal Lipschitz constant.

Example 4.2. *a) Assume that $\gamma = \partial([-1, 1]^2)$. Let g be the arc-length parametrization: $g : [0, 8] \to \gamma$ and define $f(e^{it}) = g(\frac{8}{2\pi}t)$. Then*

$$\mathbf{Lip}(f) = \max_{|t| \le \pi/2} \frac{2\sqrt{2}|x|}{\pi|\sin x|} = \sqrt{2}$$

and $\mathbf{Lip}(f^{-1}) = \pi/(2\sqrt{2})$. *Thus the parametrization* f *yields the smaller bi-Lipschitz constant* $L = \sqrt{2}$ *than the bi-Lipschitz constant of polar parametrization:*

$$F(e^{is}) = \min\left\{\frac{1}{|\sin s|}, \frac{1}{|\cos s|}\right\} e^{is}$$

which is equal to 2 ([2]). Since $d((-1,-1),(1,1)) \leq 2L$, *we have* $L \geq \sqrt{2}$. *Thus* $\sqrt{2}$ *is the minimal bi-Lipschitz constant.*

b) If $Q = [-\pi/6, \pi/6] \times [-2\pi/6, 2\pi/6]$ *and* γ *is its boundary, then the arc-length parametrization* $f : S^2 \to \gamma$, $f(1) = \pi/3$, *does not provide the optimal Lipschitz constant. Namely*

$$\mathbf{Lip}(f) = \frac{|f(\pi/2) - f(\pi/6)|}{|e^{\pi/2} - e^{i\pi/6}|} = \frac{2\pi}{3\sqrt{3}} > \frac{\mathbf{diam}(\gamma)}{2} = \frac{\sqrt{5}\pi}{6}.$$

The second example of infinity harmonic mapping is

Example 4.3 (Projection to the unit sphere). *Assume that* $f : \Sigma \to S^{n-1}$ *is a projection of an open subset* $U \subset \mathbf{R}^n$ *to the unit sphere:* $f(x) = \frac{x}{|x|}$. *Then (see [3])* $\mathbf{Lip}(f) = 1/\mathbf{dist}(\Sigma, 0)$.

As a consequence of Lemma 2.1 and Example 4.3 we obtain the following theorem.

Theorem 4.4. *Assume that* γ *is a curve bounding a starlike domain* Ω *w.r.t a point* a *such that* $D(a, r) \subset \Omega \subset \mathbf{Strip}(p, q)$, *where* $\mathbf{Strip}(p, q)$ *is a strip domain bounded by two parallel lines* p, q *whose distance is* $2r$. *Then the mapping with the least constant of distortion between* γ *and the unit circle is the mapping* $f(z) = \frac{z-a}{|z-a|}$, *whose Lipschitz constant is* $1/r$.

Remark 4.5. *We can cover by the Theorem 4.4 the following special cases: the regular polygon with even number of edges, the ellipse, the rectangle etc. The regular triangle does not satisfy the condition of Theorem 4.4, however it can be proved that the mapping* $f(x) = (z - a)/|z - a|$, *where* a *is the center of triangle, yields the minimal constant of distortion equal to* $1/r$ *where* r *is the radius of inscribed circle to the triangle. If the triangle is not regular, then the mappings* $f(z) = (z - a)/|z - a|$ *are not optimal in this context. It seems that in this case the optimal Lipschitz constant is* $\sqrt{3}/x$, *where* x *is the length of the side of minimal regular triangle inscribed in a given triangle. It will be of interest to find the optimal mappings provided that* γ *is a convex Jordan curve.*

In the following proposition are obtained some approximately sharp two-sided estimates of a bi-Lipschitz constant provided that the curve is very close to the unit circle. However as we will see in the rest of the paper, the estimates are far from optimal.

Proposition 4.6. *Let* $f : \mathbf{T} \to \gamma$ *be an arc-length parametrization of a* C^2 *smooth symmetric convex curve of the length* 2π. *Then*

$$\kappa_{\min} \leq \mathbf{biLip}(f) \leq \kappa_{\max},$$

where $\kappa_{\min}$ *and* $\kappa_{\max}$ *are minimal and maximal curvature of* γ, *respectively.*

Proof. By the condition of lemma, there exists a smooth mapping $\beta : [0, 2\pi] \to [0, 2\pi]$ such that $f(e^{is}) = \int_0^s e^{i\beta(t)} dt$. Moreover $\beta'(t) = \kappa_t$, where κ_t is the curvature of the curve γ at the point $f(e^{it})$. Thus

$$\frac{|f(s) - f(t)|^2}{|e^{is} - e^{it}|^2} = \frac{\left|\int_s^t e^{i\beta(\tau)} d\tau\right|^2}{4\sin^2 \frac{t-s}{2}}.$$

By differentiating the previous equality with respect to t, we see that the stationary points t of the quantity $\frac{A}{B} = \frac{|f(s)-f(t)|^2}{|e^{is}-e^{it}|^2}$ satisfy the differential equation

$$\frac{A}{B} = \frac{A'}{B'}.$$

Thus

$$\frac{|f(s)-f(t)|^2}{|e^{is}-e^{it}|^2}=\frac{\int_t^s \cos(\beta(\tau)-\beta(s))d\tau}{\sin(s-t)}.$$

By differentiating now the quantity $\frac{A'}{B'}$, we see that the stationary points t of it satisfy the equation

$$\frac{A'}{B'}=\frac{A''}{B''}.$$

Thus minimum and maximum of the function

$$\frac{|f(s)-f(t)|}{|e^{is}-e^{it}|}$$

is bigger (smaller) than the minimum and the maximum of the function

$$F(s,t)=\frac{\cos(\beta(t)-\beta(s))}{\cos(t-s)}.$$

Let $x=t-s$ and $\phi(x)=\beta(x+s)-\beta(s)$. Let

$$h(x)=\frac{\cos\phi(x)}{\cos x}.$$

Then h is differentiable in $[0,\pi]$. The stationary points of h satisfy the equation

$$\phi'\frac{\sin\phi(x)}{\cos x}-\frac{\sin x}{\cos x}h=0.$$

Therefore

$$h^2(x)=(\phi'(x))^2\sin^2\phi(x)+\cos^2\phi(x).$$

Since

$$2\pi=\phi(2\pi)-\phi(0)=\int_0^{2\pi}\phi'(x)dx,$$

we have that $\min_x(\phi'(x))\le 1\le \max_x(\phi'(x))$. It follows that

$$\min_x(\phi'(x))^2\le h^2(x)\le\max_x(\phi'(x))^2.$$

The conclusion is

$$\min\beta'(x)\le\frac{|f(s)-f(t)|}{|e^{is}-e^{it}|}\le\max\beta'(x).$$

□

In order to obtain the main result of this section, we prove first the following technical lemma.

Lemma 4.7. *Let a,b and c be sides of a triangle. Then the function*

$$f(x)=\frac{b^2(a-x)^2+\left(b^2+c^2-a^2\right)(a-x)x+c^2x^2}{g^2(x)},x\in[0,a]$$

attains its maximum on $\{0,a\}$ where $g(x)=px+q$ or $g(x)=a\sin(ux+v)$, provided that $q>0$, $pa+q>0$ and $u,v>0$, $au+v<\pi$.

Proof. We will prove the lemma for the case $g(x) = a\sin(ux+v)$ by reducing it to the case $g(x) = px+q$ and by proving the last case. First of all for $x \in [0,a]$

$$a\sin(ux+v) \geq px+q = a\sin v - ax\left(\sin v - \sin(au+v)\right).$$

So in $[0,a]$

$$f(x) \leq h(x) := \frac{b^2(a-x)^2 + \left(-a^2+b^2+c^2\right)(a-x)x + c^2x^2}{(px+q)^2}.$$

Further

$$h'(x) = \frac{a\left(-(b-c)(b+c)(q-px) + a^2(-q+px) + a\left(-2b^2p+2qx\right)\right)}{(q+px)^3},$$

So $h'(x_0) = 0$ if and only if

$$x_0 = \frac{2ab^2p + \left(a^2+b^2-c^2\right)q}{(a^2+b^2-c^2)\,p + 2aq}.$$

If $x_0 \notin [0,a]$ we obtain that the maximum of h is in 0 or a and so the same hold for f.

So assume that $x_0 \in [0,a]$. Then

$$h''(x) = \frac{2a\left(a^2p(2q-px) + (b-c)(b+c)p(2q-px) + a\left(3b^2p^2 + q(q-2px)\right)\right)}{(q+px)^4}.$$

After some straightforward computations we obtain that

$$h''(x_0) = \frac{a\left(\left(a^2+b^2-c^2\right)p + 2aq\right)^4}{8\left(ab^2p^2 + (a^2+b^2-c^2)\,pq + aq^2\right)^3}.$$

We will prove that $h''(x_0) > 0$. It is enough to prove that the expression $I = ab^2p^2 + \left(a^2+b^2-c^2\right)pq + aq^2$ is positive. The quadratic form that appear in expression I is positive if and only if the discriminant is positive. This means that

$$a^2b^2 - (a^2+b^2-c^2)^2/4 \geq 0.$$

This inequality follows from the triangle inequalities $|a-b| \leq c \leq a+b$. So x_0 is a local minimum of h implying that

$$\max_{[0,a]} f(x) = \max_{[0,a]} h(x) = \max\{h(0), h(a)\} = \max\{f(0), f(a)\}.$$

□

Theorem 4.8. *Let P_n be a polygon and assume that f is a piecewise linear mapping of the unit circle onto P_n. Let $B = f(\mathbf{B})$ and $C = f(\mathbf{C})$ be two consecutive vertices of P_n. Assume that $X = f(\mathbf{X}) \in [B,C]$ and $A = f(\mathbf{A}) \in P_n \setminus [B,C]$. Then*

$$\frac{|AX|}{|\mathbf{AX}|} \leq \max\left\{\frac{|AB|}{|\mathbf{AB}|}, \frac{|AC|}{|\mathbf{AC}|}\right\}.$$

Proof. Let $\mathbf{A} = e^{i\alpha}$, $\mathbf{B} = e^{i\beta}$, $\mathbf{C} = e^{i\gamma}$ and assume that $\gamma \geq \beta$. Then a piecewise linear mappings satisfies the condition $X = f(\mathbf{X}) = f(e^{ix}) = Px + Q$, $\beta \leq x \leq \gamma$ and $f(e^{i\beta}) = B$, $f(e^{i\gamma}) = C$. Let $a = |BC|$, $b = |CA|$ and $c = |AB|$. Now define

$$g(x) = f(e^{i((\gamma-\beta)x/a+\beta)}) = \left(1 - \frac{x}{a}\right)B + \frac{x}{a}C$$

and apply the previous lemma to the function

$$F(x) = \frac{|g(x) - A|^2}{|e^{i((\gamma-\beta)x/a+\beta)} - e^{i\alpha}|^2}, \quad x \in [0,a].$$

□

Theorem 4.9. *Assume that γ is a convex curve inside a circle S of radius R. Then there is a $R-$Lipschitz homeomorphism of the unit circle onto γ.*

Proof. Assume that $g : [0, |\gamma|] \to \gamma$ is an arc-length parametrization and let $A_k = g((k/n)|\gamma|)$, $k = 1, \dots, n$, be points of the curve γ. Put $A_0 = A_n$ and assume that α_k, $k = 1, \dots, n-1$ are concave angles determined by points $A_{k-1}A_kA_{k+1}$. Assume that p_k are bisectrices of angles α_k. Assume that B_k belong to the arc $AB \cap p_k$. Define the mapping from S to the γ by $f_n(B_i) = A_i$, $i = 1, \dots, n$ and linearly from $[\widehat{B_k, B_{k+1}}]$ to $[A_k, A_{k+1}]$: $f(Re^{(\alpha t+\beta)i}) = at + b$.

Then $|B_kB_l| \geq |A_kA_l|$, $1 \leq l \leq n$ and by Theorem 4.8 we obtain that

$$\mathbf{Lip}(f_n) \leq \sup_{i \neq j} \frac{|f_n(A_i) - f_n(A_j)|}{|A_i - A_j|} \leq 1.$$

Now if $\epsilon > 0$, we can choose n and points $A_1, \dots A_n$ from γ satisfying $\max_i |A_i - A_{i+1}| < \epsilon$. Then by letting $\epsilon \to 0$, by using Arzela-Ascoli theorem the mappings f_n tends uniformly to a mapping f, of γ onto S with $\mathbf{Lip}(f) \leq 1$. Now a linear mapping of the unit circle onto the circle of radius R is $R-$Lipschitz. The composition of the corresponding mappings yields the desired Lipschitz mapping.

□

Corollary 4.10. *a) There exists a Lipschitz homeomorphism f of the unit circle onto the triangle Δ such that*

$$\mathbf{Lip}(f) = \begin{cases} \mathbf{Diam}(\Delta)/2, & \text{if } \Delta \text{ is an obsture triangle;} \\ R, & \text{if } \Delta \text{ is an acute-angled triangle,} \end{cases}$$

where R is the radius of outscribed triangle.

b) There exists a Lipschitz homeomorphism f of the unit circle onto the parallelogram P such that $\mathbf{Lip}(f) = \mathbf{Diam}(P)/2$.

c) Assume that P_n is a convex polygon inside a circle of radius R. Then there is a $R-$Lipschitz homeomorphism of the unit circle onto P_n.

Corollary 4.11 ("Archimedes' axiom"). *The length of a convex curve inside of the unit circle is smaller than 2π.*

Proof. Let $f : T \to \gamma$ be a homeomorphism with $\mathbf{Lip}(f) \leq 1$. Define $F(t) = f(e^{it})$. Then $|F'(t)| \leq 1$, and so

$$|\gamma| = \int_0^{2\pi} |F'(t)| dt \leq 2\pi.$$

□

Example 4.12. *Let Q_n be a regular polygon whose length is 2π, where n is an even integer. Assume that $e^{is} \to f(e^{is})$ is arc-length parametrization. Then*

$$\mathbf{Lip}(f) = \frac{\mathbf{Diam}(Q_n)}{2} = \frac{\pi}{n \sin \frac{\pi}{n}}.$$

We now make the following several dimensional generalization.

Theorem 4.13. *Assume that Σ is a convex surface inside another convex surface S. Then there is a $1-$Lipschitz homeomorphism of the surface S onto Σ. In particular*

$$\mathbf{Area}(\Sigma) \leq \mathbf{Area}(S). \tag{22}$$

Proof. In order to simplify the exposition, assume that Σ is two-dimensional. Further, assume that h is a homeomorphism of the unit sphere S^2 onto Σ and let $P(\varphi, \theta) : [0, 2\pi] \times [0, \pi] \to S^2$ be spherical coordinates. Assume that $P_n = \{(2k\pi/n, l\pi/n), 0 \leq k, l \leq n\}$ is a net in $[0, 2\pi] \times [0, \pi]$. Let $G = h \circ P$. Let $A_{k,l} = G(2k\pi/n, l\pi/n)$. Then $A_{k+1,l}$, $A_{k-1,l}$, $A_{k,l+1}$, $A_{k,l-1}$ and $A_{k,l}$ form a pyramid with the top at point $A_{k,l}$ and base laying in the plane π. Now the line throughout $A_{k,l}$ orthogonal to π intersects the surface S in a point $B_{k,l}$ which belongs to the same side of π as $A_{k,l}$. Define the discrete mappings $f_n(B_{k,l}) = A_{k,l}$ and extend it linearly between corresponding polyhedra. Since $\|A_{k,l} - A_{k',l'}\| \le \|B_{k,l} - B_{k',l'}\|$, by using Lemma 4.7, we obtain that $\mathbf{Lip}(f_n) \le 1$. Then by letting $n \to \infty$ we obtain that f_n tends uniformly to a mapping f which is $1-$Lipschitz continuous on a dense subset of S. By continuity we can extend f to be $1-$Lipschitz continuous in S. Moreover, since both surfaces are convex, f is a homeomorphism.

In order to deduce (22), observe that

$$\mathbf{Area}(\Sigma) = \mathbf{Area}(f(S)) = \int_{[0,2\pi]\times[0,\pi]} D_{f\circ Q},$$

where Q is a parametrization of S defined by $Q(\varphi,\theta) = \rho(\varphi,\theta)P(\varphi,\theta)$. If we fix a point $p = Q(\varphi,\theta)$ in S and consider $f'(p)$, as a mapping between tangent spaces TS_p and $T\Sigma_{f(p)}$, we obtain that $D_{f\circ Q}(\varphi,\theta) = |\det f'(p)|D_Q(\varphi,\theta)$. But $\det f'(p) = \lambda_1 \cdot \lambda_2$, where λ_1^2 and λ_2^2 are eigenvalues of the matrix $f'(p)\cdot(f'(p))^t$. Since $\|f'(p)\| = \max\{\lambda_1,\lambda_2\} \le 1$, we obtain that

$$D_{f\circ Q}(\varphi,\theta) \le D_Q(\varphi,\theta)$$

and therefore

$$\mathbf{Area}(\Sigma) \le \int_{[0,2\pi]\times[0,\pi]} D_Q = \mathbf{Area}(S).$$

□

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