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Isoperimetric-Type Inequalities for Subharmonic Functions on the Polydisk, Capacity, Transportation Approach, and Related Problems

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1. Introduction

The isopermetric problem can be stated as follows:
(A) Among all closed curves in the plane enclosing a fixed area, which curve (if any) minimizes the perimeter?
This question can be shown to be equivalent to the following problem:
(B) Among all closed curves in the plane of fixed perimeter, which curve (if any) maximizes the area of its enclosed region?
Although the circle appears to be an obvious solution to the problem, proving this fact is rather difficult. More precisely, if l is the length of a closed curve and A is the area of the planar region that it encloses, then the isoperimetric inequality states that

$$4\pi A \leqslant l^2, \tag{1}$$

and that the equality holds if and only if the curve is a circle. Dozens of proofs of the isoperimetric inequality have been found, see for example [3, 8, 15, 16, 20, 39, 48, 59, 60] and the literature cited there. In particular we highly recommend Expository Lectures by Andrejs Treibergs, [59, 60], to the interested reader as introduction in the subject, and more advanced Lectures by Druet [15] and Fusco [20]. For recent developments concerning geometric and functional inequalities, Optimal Transport and Applications, which incudesboth an overview of current knowledge and an update on the most recent advancements, see for example [18, 29, 45, 61, 62, 67] and the literature cited there.

In this paper we prove various versions of isoperimetric inequality of two types (i) for the polydisk related to recent work of M. Marković [31] and (ii) for the capacity. We also outline a few simple proofs of the isoperimetric inequality related to our previous papers and give short review of known results using some novelty.

The content of the paper is as follows. In Section 2 we consider the version of the Isoperimetric inequality for logarithmical plurisubharmonic function (as an example see Theorem 2.4) and for the polydisk and in Section 3 related results.

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In Section 4 we give a simple proof of isoperimetric inequality and in particular of the following result due to Carleman: Among all ring domains A with given area and with given area of the "holes" the domain bounded by two concentric circles gives the greatest value of modulus of A. Some generalizations and applications of these result are given(see for example Theorem 4.5). Possible connections between results concerning polydisk and capacity is indicated in Proposition 4.7. Our review includes Lax's Proof of the Isoperimetric Inequality with some novelty (see for example Proposition 4.9) and Papus-Guldinus theorem. Short discussion concerning Isoperimetric inequality in space is given in Section 5; in particular in subsections 5.2 and 5.3 using a transportation approach by Knothe and Brenier maps, we outline proofs of anisotropic isoperimetric inequality, and in subsection 5.5 we shortly consider isoperimetric inequality for Euclidean polyhedra cf. [5]. Finally in section 6 some results concerning Abel summability which we need are considered.

2. Isoperimetric inequality in polydisk

We will employ the following notation. Let $B^n(x, r) = \{z \in \mathbb{R}^n : |z - x| < r\}$, $S^{n-1}(x, r) = \partial B^n(x, r)$ (abbreviated $S(x, r)$) and let $\mathbb{B}^n$, $\mathbb{S} = \mathbb{S}^{n-1}$ stand for the unit ball and the unit sphere in $\mathbb{R}^n$, respectively. In particular, by $\mathbb{U}$ or $\mathbb{B}$ we denote the unit disk, by $\mathbb{T}$ the unit circle; we write $\mathbb{U}' = \mathbb{U} \setminus \{0\}$ for the punctured disk and $\mathbb{E} = \{z : |z| > 1\}$ for the exterior of circle. For $r > 0$, we denote by $\mathbb{U}_r$ and $\mathbb{T}_r$ the disk and circle of radius r with center at the origin respectively.

Frequently, Fourier and Laurent coefficient of function g we denote $\hat{g}(n)$ or $\hat{g}_n$. It is convenient to identify a function $g : \mathbb{T} \to \mathbb{C}$ with $t \mapsto g(e^{it})$, $t \in [0, 2\pi]$.

By notation $g \sim \sum_{n=-\infty}^{\infty} \hat{g}_n e^{-int}$ we denote that the series on the right is Fourier series of g.

Let $0 < p \leqslant \infty$. For a function $f : \mathbb{U} \to \mathbb{C}$ we define

$$\begin{aligned} M_p(f; r) &= \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{it})|^p dt \right)^{1/p}, && \text{if } 0 < p < \infty \\ M_\infty(f; r) &= \sup_{t \in [0, 2\pi]} |f(re^{it})|, && \text{if } p = \infty \end{aligned}$$

and $\|f\|_p = \lim\limits_{r \to 1} M_p(f; r)$.

For a function f holomorphic in the polydisc $\mathbb{U}^n$ we define

$$\begin{aligned} M_p(f; r) &= \left(\int_{\mathbb{T}^n} |f(r\omega|^p d\sigma(\omega) \right)^{1/p}, && \text{if } 0 < p < \infty \\ M_\infty(f; r) &= \sup_{\omega \in \mathbb{T}^n} |f(r\omega)|, && \text{if } p = \infty \end{aligned}$$

where $\sigma = \sigma_n$ is Haar measure on n-torus $\mathbb{T}^n$ and $\|f\|_p = \lim\limits_{r \to 1} M_p(f; r)$.

If $\|f\|_p$ is finite we say that f belongs to the Hardy class $H^p(\mathbb{U}^n)$ and write $f \in H^p(\mathbb{U}^n)$. It turns out that if $f \in H^p(\mathbb{U}^n)$, then there exists the finite limit

$$f_*(\omega) = f^*(\omega) = \lim_{r \to 1} f(r\omega) \quad \text{a.e, on } \mathbb{T}^n,$$

and the boundary function f^* belongs to $L^p(\mathbb{T}^n)$. Moreover $\|f^*\|_p = \|f\|_p$. Let

$$P_r(t) = \frac{1 - r^2}{1 - 2r\cos(t) + r^2}$$

denote the Poisson kernel.

If $\psi \in L^1[0, 2\pi]$ and

$$h(z) = \frac{1}{2\pi} \int_0^{2\pi} P_r(\theta - t)\, \psi(t)\, dt,$$

then the function $h = P[\psi]$ so defined is called Poisson integral of ψ.

A real valued function $u(x, y)$ defined on some open domain $\Omega \subset \mathbb{C}$ is said to be harmonic if it is locally real part of an analytic function (at any point of its domain there is a neighborhood U and analytic function f on U such that $u = Ref$ on U). If $f = u + iv$, then u and v satisfy the Cauchy-Riemann equations $u_x = v_y$ and $u_y = -v_x$ in Ω. Hence $\Delta u = \Delta v = 0$. A function $u(x, y)$ defined on some open domain $\Omega \subset \mathbb{C}$ is said to have as a conjugate a function $v(x, y)$ if and only if they are respectively real and imaginary part of a holomorphic function $f(z)$ of the complex variable $z := x + iy \in \Omega$. That is, v is conjugate to u if $f(z) := u(x, y) + iv(x, y)$ is holomorphic on Ω. As a first consequence of the definition, they are both harmonic real-valued functions on Ω. Moreover, the conjugate of u, if it exists, is unique up to an additive constant. Also, u is conjugate to v if and only if v is conjugate to $-u$. If u is harmonic on some open domain $\Omega \subset \mathbb{C}$ then function $2u_z = u_x - iu_y$ is analytic and it has locally a primitive function and if Ω is simply connected then it has a primitive function on Ω.

So any harmonic function always admits a conjugate function whenever its domain is simply connected, and in any case it admits a conjugate locally at any point of its domain. Equivalently, v is conjugate to u in Ω if and only if u and v satisfy the Cauchy-Riemann equations in Ω.

Let $sh^1(\mathbb{U})$ denote the class of subharmonic function u in $\mathbb{U}$ which is not identical to ∞ and such that u has a harmonic majorant in $h^1(\mathbb{U})$; it means that $I(|u|, r) = \int_0^{2\pi} |u(re^{it})| dt$ bounded for $r \in (r_0, 1)$.

It is a Littlewood theorem that radial limit u^* exists a.e. for $u \in sh^1(\mathbb{U})$.

A function p , defined in a domain D, will be said to be of class PL in D provided the following conditions are satisfied there. (i) p is continuous. (ii) $p \geqslant 0$. (iii) $\ln p$ is subharmonic in the part of D where $p(u, v) > 0$. If p is of class PL, then p is subharmonic. Indeed, at points where $p = 0$ the condition (ii) of Riesz obviously is satisfied; and elsewhere the fact that $\ln p$ is subharmonic implies that $p(u, v)$ is subharmonic. For $0 < p < \infty$, denote by h^p_{PL} (the notation $\mathcal{R}^p$ is also used,see [28]) the class of function f in PL in the unit disk (the sub class of PL) with bounded $M_p(f; r)$ in $r \in (0, 1)$. It is important to note the following "principle" has important applications.

Theorem 2.1 (cf. Theorem 1, Lozinski [28]). *For some fixed* $0 < p < \infty$*, suppose that* $f \in h^p_{PL}$*, that is*
(i1) *f is a log-subharmonic function in the unit disk,*
(i2) *with bounded $M_p(f; r)$ in $r \in (0, 1)$.*
Then there exists an analytic function $a \in H^p$ such that
$|a(z)| = f(z)$ a.e. for $z \in \mathbb{T}$ and
$f(z) \leqslant |a(z)|$ for $z \in \mathbb{U}$.

We outline a proof as follows. Suppose that f is a positive function on $\mathbb{T}$ and that $\ln f \in L^1(T)$ and set $h = P[\ln f]$. Since $\mathbb{U}$ is simply-connected domain h has a conjugate harmonic function $\tilde{h}$. Let $H = h + i\tilde{h}$, and $a = e^H$. It is clear that $|a| = e^h$. Since h is harmonic in $\mathbb{U}$ and $f \leqslant e^h$. By Jensen's ineqality $|a|^p \leqslant P[f^p]$ and therefore $a \in H^p$.

In some application the following elementary form is enough: If f is a log-subharmonic function in the unit disk, continuous in the closed unit disk, then there exists an analytic function $a \in H^\infty$ such that $|a(z)|$ is continuous in the closed unit disk,
$|a(z)| = f(z)$ for $z \in \mathbb{T}$ and
$f(z) \leqslant |a(z)|$ for $z \in \mathbb{U}$.

This principle lies behind the proofs of Fejér-Riesz-Lozinski inequality, cf. [28], and Proposition 2.2.

If curvature of a surface is nonpositive, then metric density ρ is log-subharmonic function and we can use principle of log-subharmonic function, which reduces the the proof of isoperimetric inequality to the analytic case.

We refer to the next proposition as Isoperimetric inequality for log-subharmonic functions.

Proposition 2.2 ([28, Theorem 4] , see also [42]). *For a log-subharmonic function $\varphi : \mathbb{B} \to \mathbb{R}$, $\varphi \in h^1(\mathbb{B})$ the following sharp inequality holds*

$$\int_{\mathbb{B}} |\varphi(z)|^2 dxdy \leqslant \frac{1}{4\pi}\left(\int_0^{2\pi} |\varphi(e^{it})|dt\right)^2. \tag{2}$$

The equality is attained if and only if $\varphi(z) = \frac{b}{|1-az|^2}$, $a \in \mathbb{B}$, $b \in \mathbb{R}$.

The solution to the isoperimetric problem is usually expressed in the form of an inequality that relates the length l of a closed curve and the area A of the planar region that it encloses. The isoperimetric inequality states that

$$4\pi A \leqslant l^2, \tag{3}$$

and that the equality holds if and only if the curve is a circle. Dozens of proofs of the isoperimetric inequality have been found.

In [10] Carleman gave a beautiful proof of the isoperimetric inequality, reducing it to an inequality for holomorphic functions on the unit disc. For more details and other proofs and generalizations of the isoperimetric inequality we refer to [3, 40, 42, 48, 51], and [18]. In this section we discus isoperimetric inequality for generalized polydisks.

In particular, if f is a fuction of one complex variable the statement (ii.0) of Proposition 2.3 is reduced to (A1): If $f \in H^p(\mathbb{B})$, $0 < p < \infty$, then

$$\int_{\mathbb{B}} |f(z)|^{2p} dxdy \leqslant \frac{1}{4\pi}\left(\int_0^{2\pi} |f(e^{it})|^p dt\right)^2. \tag{4}$$

The inequality (4) has been rediscovered several times, see [24] a nd [14], and dates at least back to Carleman [7].

We refer to this result as H^p-version of isoperimetric inequality (for one complex variable). It seems that approach to the proof of (A1) via the Cauchy-Shwartz inequality is due to Carleman[10]; we rediscavered this approach, see for example [40]. By using a similar approach as Carleman, Strebel in his book ([56], Theorem 19.9) proved (4). In the case $p = 2$, (4) reduces to

$$\int_U g(z)^4 dxdy \leqslant \frac{1}{4\pi}\left[\int_0^{2\pi} |g(e^{it})|^2 dt\right]^2, \tag{5}$$

which is just the Carleman inequality.

Let γ be a rectifiable Jordan closed curve of length L and $D = Int(\gamma)$ and ϕ a conformal of B onto D. An application of the Carleman inequality to $g = (\phi')^{1/2}$ shows that $4\pi A \leqslant L^2$.

Vukotić [63] also rediscovered some results obtained in [36].

An n-dimensional multi-index is an n-tuple $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ of non-negative integers (i.e. an element of the n-dimensional set of natural numbers, denoted $\mathbb{N}_0^n$ or $\mathbb{Z}_+^n$). For $z = \underline{z} = (z_1, \cdots z_n) \in \mathbb{C}^n$, we define $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$. In the theory of functions of several complex variables, a branch of mathematics, a polydisc is a Cartesian product of discs.

More specifically, if we denote by $D(z, r)$ the open disc of center z and radius r in the complex plane, then an open polydisc is a set of the form $D(z_1, r_1) \times \cdots \times D(z_n, r_n)$. In particular, $\mathbb{U}^n = \mathbb{B} \times \cdots \times \mathbb{B}$ and $\mathbb{T}^n = \mathbb{T} \times \cdots \times \mathbb{T}$.

Haar measure on n-torus is $d\sigma_n(e^{it_1}, \cdots, e^{it_n}) = \frac{1}{(2\pi)^n} dt_1 \cdots dt_n$; dm_n is the normilized measure on $\mathbb{U}^n$.

Let F be holomorphic in polydisk $\mathbb{U}^n$. By $\varphi_n(p; r)$, we denote the function

$$\varphi_n(p; r) = \varphi_n(p; r, F) = \left[\int_{\mathbb{T}^n} |F(rz)|^p d\sigma\right]^2 - \int_{\mathbb{U}^n} |F(rz)|^{2p} dm_n,$$

where m_n and σ_n are Lebesgue measure on $\mathbb{U}^n$ and $\mathbb{T}^n$. We define $\delta_n(p, r) = r^{2n}\varphi_n(p; r)$ and if $n = 1$, we write $\varphi(p; r)$ and $\delta(p, r)$ instead of $\varphi_1(p; r)$ and $\delta_1(p, r)$, respectively.

Proposition 2.3. (i.0) *Let F be holomorphic in polydisk $\mathbb{U}^n$. Then $\varphi_n(2;r)$ is nondecreasing in $r \in (0,1)$.*
(ii.0) *Let $F \in H^p(\mathbb{U}^n)$, $0 < p < \infty$. Then*

$$\int_{\mathbb{U}^n} |F(z)|^{2p} dm_n \leqslant \left[\int_{\mathbb{T}^n} |F(z)|^p d\sigma\right]^2 .$$

(iii.0) *If $n = 1$, $\varphi(p;r)$ is nondecreasing in $r \in (0,1)$, cf. [52].*

Proof. (i.0) If $F(z) = \sum_{\alpha\in\mathbb{Z}_+^n} c(\alpha)z^\alpha$, then

$$L(r) := \int_{\mathbb{T}^n} |F(rz)|^2 d\sigma = \sum_{\alpha\in\mathbb{Z}_+^n} A(\alpha) r^{2|\alpha|} ,$$

where $A(\alpha) = \sum_{\beta+\gamma=\alpha} |c(\beta)c(\gamma)|^2$. Note that $F^2(z) = \sum_{\alpha\in Z_+^n} d(\alpha)z^\alpha$, where $d(\alpha) = \sum_{\beta+\gamma=\alpha} c(\beta)c(\gamma)$.
Then

$$A(r) = \int_{\mathbb{U}^n} |F(rz)|^4 dm_n = \sum_{\alpha\in\mathbb{Z}_+^n} B(\alpha) r^{2|\alpha|},$$

where $B(\alpha) = |d(\alpha)|^2(1+\alpha_1)^{-1}(1+\alpha_2)^{-1}\dots(1+\alpha_n)^{-1}$. Hence

$$\varphi_n(2;r) = \sum_{\alpha\in\mathbb{Z}_+^n} (A(\alpha) - B(\alpha)) r^{2|\alpha|} .$$

By the Cauchy-Shwartz inequality, $A(\alpha) \geqslant B(\alpha)$ and (a) follows.
(ii.0) Since $|F|$ is logarithmical subharmonic, by Proposition 2.2, we get (ii.0).
For $p = 2$ it follows from previous Proposition.
(ii.0) Now we suppose that F is holomorphic in $\mathbb{B}$. By the change of variables $w = \rho z$,

$$\varphi(p;r) = \varphi(p;r,F) = \left[\int_{\mathbb{T}^n} |F(rz)|^p d\sigma\right]^2 - \int_{\mathbb{B}} |F(w)|^{2p} dm_n = \left[\frac{1}{2\pi}\int_{\mathbb{T}_r} |F(w)|^p d|w|\right]^2 - \frac{1}{\pi}\int_{\mathbb{B}_r} |F(w)|^{2p} dudv.$$

Let $0 \leqslant r \leqslant R < 1$. Using Blaschke product, we can find an analytic function g such that
A2 $|F(z)|^p \leqslant |g(z)|^2$ on $\mathbb{B}_R$ and $|F(z)|^p = |g(z)|^2$ on $\mathbb{T}_R$.
By Proposition 2.3, $\delta(2,r) = \rho^2\varphi_n(2;\rho)$, $0 \leqslant \rho < 1$, is a nondecreasing function. Hence

$$\left[\frac{1}{2\pi}\int_{\mathbb{T}_R} |g(w)|^2 d|w|\right]^2 \geqslant \left[\frac{1}{2\pi}\int_{\mathbb{T}_r} |g(w)|^2 d|w|\right]^2 + \frac{1}{\pi}\int_{\mathbb{U}_r} |F(w)|^4 dm_n.$$

Combining the above we get (c). □

In several variables it is not possible to use a Blaschke product, but we can use subharmonic functions.
We say that, a nonnegative function u upper semi continuous defined in domain D is logarithmical subharmonic if $u = 0$ or $\log u$ is subharmonic in D.
A function $f : G \to \mathbb{R} \cup \{-\infty\}$, with domain $G \subset \mathbb{C}^n$ is called plurisubharmonic if it is upper semi-continuous, and for every complex line $\{a + bz \mid z \in \mathbb{C}\} \subset \mathbb{C}^n$with $a, b \in \mathbb{C}^n$ the function $z \mapsto f(a + bz)$ is a subharmonic function on the set $\{z \in \mathbb{C} \mid a + bz \in G\}$.
Let D_1 and D_2 be a simply connected domains in $\mathbb{C}$ with a rectifiable boundaries and $D = D_1 \times D_2$ be a generalized polydisk with distinguished boundary $\partial_0 D = \partial D_1 \times \partial D_2$.

Theorem 2.4. *Let u be a logarithmical plurisubharmonic function in $D = D_1 \times D_2$ and continuous on $\overline{D}$. Then*

$$\int_D u^2 dV \leqslant \frac{1}{(4\pi)^2} L^2 \,, \tag{6}$$

where $L = \int_{\partial_0 D} u\, ds = \int_{\partial_0 D} u(z_1, z_2)|dz_1||dz_2|$.

Define χ by $\chi(z_2) = \int_{\partial D_1} u(z_1, z_2)\, ds$, where $ds = |dz_1|$; it is logarithmical subharmonic function in $z_2 \in D_2$, (see for example [53], rusian edition p.64). For a fixed $z_2 \in D_2$, since $u(z_1, z_2)$ logarithmical subharmonic function in $z_1 \in D_1$, by Proposition 2.2,

$\int_{D_1} u^2(z_1, z_2) dx_1 dy_1 \leqslant \frac{1}{4\pi} \left(\int_{\partial D_1} u(z_1, z_2) ds\right)^2 = \frac{1}{4\pi}\chi^2(z_2)$. By definition of χ and Fubini's theorem,

$$\int_{\partial D_2} \chi(z_2)|dz_2| = \int_{\partial D_2} \left(\int_{\partial D_1} u(z_1, z_2)|dz_1|\right)|dz_2| \tag{7}$$

$$= \int_{\partial_0 D} u(z_1, z_2)|dz_1||dz_2| = L \,. \tag{8}$$

Since χ is logarithmical subharmonic function in $z_2 \in D_2$, again by Proposition 2.2,

$$\int_{D_2} \chi^2(z_2)\, dx_2 dy_2 \leqslant \frac{1}{4\pi} \left(\int_{\partial D_2} \chi(z_2)|dz_2|\right)^2 = \frac{1}{4\pi} L^2. \tag{9}$$

Hence, $\int_D u^2 dV \leqslant \frac{1}{(4\pi)^2} \left(\int_{\partial D} u ds\right)^2$.

In particular, as a corollary of Theorem 2.4, we find:

Proposition 2.5. *If $f \in H^p(D)$ and $g \in H^q(D)$, $0 < p, q < \infty$, then*

$$\int_D |f|^p |g|^q dV \leqslant \frac{1}{(4\pi)^2} \int_{\partial_0 D} |f|^p ds \int_{\partial_0 D} |g|^q ds \,. \tag{10}$$

By Cauchy-Shwartz inequality,

$$\int_D |f|^p |g|^q dV \leqslant \left(\int_D |f|^{2p} dV\right)^{1/2} \left(\int_D |g|^{2q} dV\right)^{1/2} \tag{11}$$

and therefore

$$\int_D |f|^p |g|^q dV \leqslant \frac{1}{(4\pi)^2} \int_{\partial_0 D} |f|^p ds \int_{\partial_0 D} |g|^q ds \,. \tag{12}$$

See also [24, 31]:

3. Further and related results

In [31] M. Markovic proved an isoperimetric inequality for holomorphic functions in the unit polydisc $\mathbb{U}^n$. As a corollary he derives an inclusion relation between weighted Bergman and Hardy spaces of holomorphic functions in the polydisc which generalizes the classical Hardy- Littlewood relation $H^p \subset A^{2p}$. Also, he extends some results due to Burbea. In order to describe these results we need first some definitions.

There are two standard generalizations of Hardy spaces on a hyperbolic simple connected plain domain D. One is immediate, by using harmonic majorants, denoted by $H^p(G)$. Let $0 < p < \infty$ a holomorphic function f on a domain G is in $H^p(G)$ if the subharmonic function $|f|^p$ has a harmonic majorant v on D. It can be proved that there is a unique harmonic majorant u_f, which will be called the least harmonic majorant of f. Fixed $z_0 \in G$ and set $|f|_p = (u_f(z_0))^{1/p}$.

The second is due to Smirnov, usually denoted by $E^p(D)$. The definitions can be found in the tenth chapter of the book of Duren [14]. These generalizations coincide if and only if the conformal mapping of D onto the unit disc is a bi-Lipschitz mapping (by [[14], Theorem 10.2]); for example this occurs if the boundary is C^1 with Dini-continuous normal (Warschawski's theorem, see [64]). The previous can be adapted for generalized polydiscs (see the paper of Kalaj [24]). Let $\underline{D}$ be the generalized polydomain given as the Cartesian product of n open Jordan domains $D_1, ..., D_n$: $\underline{D} = \prod_{i=1}^n D_i$ and ϕ_k, $k = 1, \cdots n$, the conformal mapping of D_k onto the unit disc, and $\underline{\phi} = (\phi_1, \cdots, \phi_n)$. In particular, $H^p(D^n) = E^p(D^n)$, if the distinguished boundary $\partial_0(D)$ is sufficiently smooth, which means ϕ_k, $k = 1, \cdots n$ are sufficiently smooth.

Let G be a simply-connected domain with rectifiable Jordan boundary γ. The set $E^p(G)$ consists of all functions holomorphic in G, such that for every function in it there is a sequence of closed rectifiable Jordan curves $\Gamma_n \subset G$ such that Γ_n tends to γ and $\underline{M}_p(f) = \sup \int_{\Gamma_n} |f|^p |dz| < \infty$; $|f|_p = \underline{M}_p(f)^{1/p}$. This definition was proposed by M.V. Keldysh and M.A. Lavrent'ev , and is equivalent to V.I. Smirnov's definition in which curves γ_r are used instead of $\Gamma_n = \Gamma_n(f)$. These curves γ_r are the images of the circles T_r under some univalent conformal mapping φ from the disc onto the domain G, and the supremum is taken over all $0 < r < 1$. The classes $E^p(G)$ are the most known generalization of Hardy spaces and are related in the following way: $f \in E^p(G)$ if and only if $f \circ \varphi(\varphi')^{1/p} \in H^p(\mathbb{U})$.

This definition can be extended for generalized polydiscs G^n; let Γ_r, $0 < r < 1$, be the Cartesian product of n Jordan curves γ_r and

$$\underline{M}_p(f) = \sup_{0<r<1} \int_{\Gamma_r} |f|^p |d\underline{z}| < \infty\,, \tag{13}$$

where $|d\underline{z}| = |dz_1| \cdots |dz_n|$. Set $|f|_p = \underline{M}_p(f)^{1/p}$.

A bounded simply-connected domain G with a rectifiable Jordan boundary in the complex plane $\mathbb{C}$ having the following property: there is a univalent conformal mapping φ from the disc onto the domain G such that the harmonic function $\ln|\varphi'|$ can be written as the Poisson integral of its non-tangential boundary values $\ln|\varphi'(e^{it}|$. These domains were introduced by V.I. Smirnov in 1928 in the course of investigating the completeness of a system of polynomials in the Smirnov class $E^2(G)$. The problem of the existence of non-Smirnov domains with rectifiable Jordan boundaries was solved by M.V. Keldysh and M.A. Lavrentiev , who gave a sophisticated and intricate construction of such domains and of the corresponding mapping functions φ, with the additional property $|\varphi'(e^{it})| = 1$ that for almost-all e^{it}. In several complex variables, the Cauchy integral formula can be generalized to polydiscs (see Hörmander book 1966 [23], Theorem 2.2.1). Let D be the polydisc given as the Cartesian product of n open discs $D_1, ..., D_n$: $D = \prod_{i=1}^n D_i$.

Suppose that f is a holomorphic function in D continuous on the closure of D. Then

$$f(\zeta) = \frac{1}{(2\pi i)^n} \int \cdots \iint_{\partial D_1 \times \cdots \times \partial D_n} \frac{f(z_1, \ldots, z_n)}{(z_1 - \zeta_1) \ldots (z_n - \zeta_n)} dz_1 \ldots dz_n, \tag{14}$$

where $\zeta = (\zeta_1, ..., \zeta_n) \in D$.

If we set $K_n(z, w) = \prod_{k=1}^n \frac{1}{z_k \overline{w_k}}$ and $dW = \frac{1}{(2\pi i)^n} dw_1 \ldots dw_n$, we can rewrite the Cauchy integral formula in the form: $f(z) = \int_D K_n(z, w) f(w) dW$.

$E^2(D^n)$ is a Hilbert space with the reproducing kernel given by

$$K_{D^n}(z, w) = K_n(z, w) (\prod_{k=1}^n \phi'_k(z_k) \phi'_k(w_k))^{1/2}\,. \tag{15}$$

Proposition 3.1 ([31]). *Let f be an upper semi-continuous function on a product $D \times G$ of domains $D \subset \mathbb{R}^n$ and $G \subset \mathbb{R}^p$. Let μ be a positive measure on G and $E \subset G$ such that $\mu(E) < \infty$. Then $f(x) = \int_E f(x,y)d\mu(y), x \in D$ is (logarithmically) subharmonic if $f(\cdot, y)$ is (logarithmically) subharmonic for all (almost all with respect to the measure μ) $y \in G$.*

The following lemma has a role in finding extremal function in corresponding inequalities.

Lemma 3.2 (Lemma 2.5,[31]). *If $f \in H^p(\mathbb{U}^2)$, then $g(z) = \int_{-\pi}^{\pi} |f(z, e^{it})|^p dt$ is logarithmically subharmonic and belongs to the space h^1_{PL}.*

Let V_n be the volume measure in the space C^n and λ be the Poincare metric on the generalized polydisc G^n. By vector-valued function we mean $\mathbb{C}^l$-valued for some integer l. We allow vector-valued holomorphic functions $f = (f_1, f_2, ..., f_l)$ to belong to the spaces $E^p(G^n)$ if they satisfy the growth condition (13) with $\|\cdot\|$ instead of $|\cdot|$. Note that if $f = (f_1, f_2, ..., f_l)$ vector-valued holomorphic function then $\lambda(z) = (\sum_{j=1}^{l} |f_j(z)|^2)^{1/2}$ is (logarithmically) subharmonic.

Theorem 3.3 ([31]). *Suppose that G is simply-connected and that the distinguished boundary $\partial_0(G^n)$ of the generalized polydisc G^n, is sufficiently smooth. If $f_j \in H^{p_j}(G^n)$, $0 < p_j < \infty$, $j = 1, \cdots, m$, be holomorphic vector-valued functions on a generalized polydisc G^n, then*

$$\int_{G^n} \prod_{j=1}^{m} |f_j|^{p_j} \lambda^{2-m} dV_n \leqslant \prod_{k=1}^{m} \underline{M}_{p_j}(f_j)\,.$$

For complex-valued functions f, the equality in the above inequality occurs if and only if either some of the f_j, $j = 1, \cdots, m$ are identically equal to zero or if for some point $w \in G^n$, and constants $a_j \neq 0$ or $b_j \neq 0$, $f_j = 0$, the functions have the following form $f_j^{p_j} = a_j^{p_j} K_n(\cdot, w)^2 = b_j^{p_j} \prod_{k=1}^{n} \psi'_k, j = 1, \cdots, m$, where K_n is the reproducing kernel for the domain G^n and $\psi_k, k = 1, \cdots, n$, are conformal mappings of G_k onto $\mathbb{U}$.

In particular, for $n = 1$ and $m = 2$ and in the case of complex-valued functions, the above inequality reduces to the result of Mateljević and Pavlović [40]. Burbea [9] extended Carlemans inequality by proving the following. Let $g \in H^p$ in the unit disc $\mathbb{U}$, and let m be an integer $m \geqslant 2$, then

$$\frac{m-1}{\pi} \int_U g(z)^{mp} (1 - |z|)^{m-2} dxdy \leqslant \left[\frac{1}{2\pi} \int_0^{2\pi} |g(e^{it})|^p dt \right]^m . \tag{16}$$

In the case $m = 2$, $p = 2$, the inequality reduces to Carleman's inequality mentioned in the Section 2.

In [55], the integration formula is proved:

(B1) $\int_S g(z_1) d\sigma(z) = \frac{m-1}{\pi} \int_U g(z)^{2m} (1 - |z|)^{m-2} dxdy$,

which may be called integration by slices, and the following result:

(B2) If $z' = (z_1, \cdots, z_{n-1})$, then

$f^0(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{it} z) dt$, $f^1(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(z', e^{it} z_n) dt$,

and

$\int_S f(z) d\sigma(z) = \int_S f^0(z) d\sigma(z) = \int_{\mathbb{B}_{n-1}} f^1(z) dv(z')$.

Note if z' is picked, then f^1 is independent of z_n; for example if f holomorphic then $f^1(z) = f(z', 0)$.

Let $\mathbb{U}^n$ and $\mathbb{B}_n$ be the unit polydisc and the unit all in $\mathbb{C}^n$, respectively. In [50] it is proved that if f is in the Hardy space $H^2(\mathbb{U}^n)$, then f is in $H^{2n}(\mathbb{B}_n)$ and that the norm of the inclusion is equal to one. More precisely, if $f \in H^2(\mathbb{U}^n)$

$$\int_{\partial \mathbb{B}_n} |f(w)|^{2n} dm_n \leqslant \left[\int_{\mathbb{T}^n} |f(z)|^2 d\sigma \right]^n .$$

If f depends on one variable only, then the result also reduces to the Carleman inequality.

By (B1) this inequality coincides with Burbea inequality (16) for $m = n; p = 2$ and $f(z) = f(z_1)$, that is if f actually depends only on one complex variable.

4. Isoperimetric inequality, capacity and Wirtinger inequality

The isoperimetric problem is to determine a plane figure of the largest possible area whose boundary has a specified length. If $L = L(\gamma)$ is the circumference of a closed Jordan rectifiable curve γ in the plane and the area of a plane region it encloses $A = A(\gamma)$, then $A \leqslant L^2/4\pi$.

A version which includes self-intersecting curve is outlined in [35, 36].

Proposition 4.1 ([35]). *Let K be positively oriented unit circle and let a curve γ be defined by $w = \phi(e^{i\theta}), 0 \leqslant \theta \leqslant 2\pi$, is of bounded variation and $\phi(e^{i\theta}) \sim \sum_{n=-\infty}^{\infty} \hat{\phi}_n\, e^{-in\theta}$. Then the sign area bounded by this curve is $A(\phi) = \frac{i}{2}\int_K \phi\, d\overline{\phi}$.*

If we denote by $L = |\gamma|$ length of the curve γ, then by isoperimetric inequality, $A(\phi) \leqslant L^2/4\pi$.

By the next Proposition 4.2, $A(\phi) = \pi \sum\limits_{-\infty}^{\infty} n|\hat{\phi}_n|^2 < \infty$.

Let a curve γ be defined on $[0, 2\pi]$ and $\gamma \sim \sum_{n=-\infty}^{\infty} \hat{\gamma}_n\, e^{-in\theta}$. If γ is of bounded variation, closed and continuous curve, then the sign area determined by γ is given by $A(\gamma) = \frac{i}{2}\int_0^{2\pi} \gamma\, d\overline{\gamma} = \pi \sum\limits_{-\infty}^{\infty} n|\hat{\gamma}_n|^2 < \infty$.

Proposition 4.2. *Then*

$$A(\gamma) = \pi \sum_{-\infty}^{\infty} n|\hat{\gamma}_n|^2 < \infty\,. \tag{17}$$

Proof. Set $g = P[\gamma]$ and $a_n = n|\hat{\gamma}_n|$. For $0 \leqslant r < 1$, we define $B(\gamma_r) := \frac{i}{2}\int_0^{2\pi} \gamma_r\, d\overline{\gamma} = \pi \sum\limits_{-\infty}^{\infty} n r^{|n|}|\hat{\gamma}_n|^2$ and $B(\gamma_r) \to A(\gamma)$ when $r \to 1_-$. Hence the series (17) is Abel summable. By Theorem 6.3, $n|\hat{\gamma}_n| \to 0$ and therefore $n^2|\hat{\gamma}_n|^2 \to 0$. Since $na_n = n(n|\hat{\gamma}_n|^2) \to 0$, by Theorem 6.2 (Tauber's convergence theorem) it is convergent in the ordinary sense as well. □

If we set

$A_+(\gamma) = \pi \sum\limits_{1}^{\infty} n|\hat{\gamma}_n|^2$, $A_-(\gamma) = \pi \sum\limits_{-\infty}^{1} n|\hat{\gamma}_n|^2$, it is clear that $A(\gamma) = A_+(\gamma) + A_-(\gamma)$ and $A(\gamma) \leqslant A_+(\gamma)$.

If γ is closed Jordan positively oriented curve, the oriented area is the same as the usual area of $Int(\gamma)$.

Let γ be closed Jordan curve and $\tilde{G} = Ext(\gamma)$. By *Riemann's theorem* there is a conformal mapping

$$f(z) = \lambda\, z + a_0 + \frac{a_1}{z} + \cdots + \frac{a_k}{z^k} + \cdots \tag{4}$$

of $\mathbb{E}$ onto $\tilde{G}$. For $\rho \geqslant 1$, set $\gamma_\rho(t) = f(\rho e^{it})$, $s(\rho) = A(\gamma_\rho)$, $\tau(\rho) = \rho^{-2}s(\rho)$ and $|\gamma|_1^* = \inf_c \int_0^{2\pi} |\gamma(t) - c| dt$.

Theorem 4.3. *Under the above hypothesis*
(i.1) *τ is not decreasing in* $[1, \infty]$
(ii.1) $\tau(\rho) \to \pi|\lambda|^2$ *if* $\rho \to +\infty$.
(iii.1) $A(\gamma) \leqslant \pi|\lambda|^2$
(iv.1) $|\lambda| \leqslant |\gamma|_1^*$
(v.1) $2\pi|\lambda| \leqslant L$, *where L is length of γ.*

For (iii.1) see also [36, 40].

Note that from (iii.1) and (v.1), we immediately find a version of isopermetric inequality: $A(\gamma) \leqslant \pi|\lambda|^2 \leqslant \frac{L^2}{4\pi}$. Note that in [59, 60] Treibergs also gives several arguments which depend on more elementary geometric and analytic inequalities.

Proof. Since

$$\tau(\rho) = \rho^{-2}s(\rho) = \pi \sum_{k=-\infty}^{1} k|a_k|^2\rho^{2k-2} = \pi|\lambda|^2 + \pi \sum_{k=-\infty}^{1} k|a_k|^2\rho^{2k-2}$$

we get (i.1) and (ii.1).
(iii.1) follows from $A_+(\gamma) = \pi|\lambda|^2$.
Using $\lambda = \int_0^{2\pi} \gamma(t)e^{-it}dt$ and
$i\lambda = \int_0^{2\pi} \gamma'(t)e^{-it}dt$ we find (iv.1) and (v.1) respectively. □

As a corollary of (iii.1) and (v.1), we get the isoperimetric inequality for simple curve: $4\pi A \leqslant L^2$.
The next example shows that the estimate (iv.1) can be better than (v.1). Let $r > 1$, $z_k = e^{ik2\pi/n}$, $w_k = rz_k$ and P_n polygon $z_1w_1z_2w_2\cdots z_{n-1}w_{n-1}z_nw_nz_1$, then $l(P_n) \to \infty$ and it is clear that $|P_n|_1^* \leqslant 2\pi r$.

4.1. *area-modulus inequality*

For $0 < r < R$, let $A(r, R) = \{r < |z| < R\}$ be the annulus with inner radius r and other radius R.
A domain A is ring if A^c has exactly two components. By topology, ∂A has also two components C_1 and C_2. Denote by $\Gamma = \Gamma_A$ the collection of curves $\gamma \subset A$ connecting C_1 and C_2.
There is $A(r_1, r_2)$ and conformal maping ϕ of $A(r_1, r_2)$ onto A. Modulus of A is defined as

$$M(A) = \frac{\log(r_2/r_1)}{2\pi}.$$

Theorem 4.4. *Let $F \subset D$ and $A = D \setminus F$ topological annulus. Then*

$$e^{4\pi M(A)} area(F) \leqslant area(D). \tag{18}$$

If equality holds in (4.4), then A is a circular regular ring.

If we set $S_0 = areaF = \pi r_0^2$ and $S_1 = areaD = \pi r_1^2$, and $A_0 = A(r_0, r_1)$ then:

$$4\pi M(A) \leqslant \ln \frac{S_1}{S_0} = 4\pi\, M(A_0)\,.$$

Hence we rewrite Theorem 4.4 respectively in the form:
(I.1) $M(A) \leqslant M(A(r_1, r_2))$. We can also restate Theorem 4.4:
(I.1′) Consider the family of all doubly-connected plane domains bounded by an outer curve C_1 and an inner curve C_0. For each domain D, let A_i be the area bounded by C_i, $i = 0, 1$. Then among all domains conformally equivalent to a given one, the minimum of A_1/A_0 is attained by a circular annulus.
We give here a proof due to Szegö (see [48] [1]) based on the isoperimetric inequality.
Let $r_0 < |z| < r_1$ be a given annulus, and let D be its image under a conformai map $f(z)$. Let $L(r)$ be the length of the image of $|z| = r$, and $A(r)$ the area enclosed. Then $4\pi A(r) \leqslant L^2(r) \leqslant 2\pi A'(r)/A(r)$ and $2/r \leqslant 2A'(r)/A(r)$, $r_0 < r < r_1$. Integrating from r_0 to r_1, yields $2\ln\frac{r_1}{r_0} \leqslant \ln\frac{A_1}{A_0}$, or $\frac{r_1^2}{r_0^2} \leqslant \frac{A_1}{A_0}$, which proves the theorem.

Proof. There exists an annulus $A_r = \{r < |z| < 1\}$ and a conformal mapping $\phi : A_r \to A$, $\phi(z) = \sum a_kz^k$.
Let $\Gamma_\rho = \phi \circ K_\rho$ and $G_\rho = Int(\Gamma_\rho)$. Then

$$s(\rho) = area(G_\rho) = \sum k|a_k|^2\rho^{2k}.$$

Let

$$\tau(\rho) = \rho^{-2}s(\rho) = \sum k|a_k|^2\rho^{2k-2}$$

Since $k(2k-2) \geqslant 0$, τ' is non negative and τ is increasing function and consequently $\tau(1) \geqslant \tau(r)$ and therefore

$$\frac{s(1)}{s(r)} \geqslant \frac{1}{r^2} = e^{2\ln\frac{1}{r}}. \tag{19}$$

Hence, since $M(A) = M(A_r) = \ln\frac{1}{r}/2\pi$, it follows $s(r)e^{4\pi mod(A)} \leqslant s(1)$. Since $s(1) = area(D)$ and $s(r) = area(F)$, this yields (4.4).

If equality holds in (4.4), then equality holds in (19). Hence $a_k = 0, k \neq 1$, and therefore $\phi(z) = a_0 + a_1 z$. □

If function ϕ is analytic on an annulus $A_r = \{r < |z| < 1\}$, then $\tau(\rho) = \rho^{-2}s(\rho)$ is not decreasing.

As a corollary $A(r_1, R_1)$ and $A(r_2, R_2)$ are conformally equivalent if and only if $R_1/r_1 = R_2/r_2$.

Let f be a holomorphic function on $A(r, R)$, $r < \rho < R$, $\Gamma_\rho = f \circ K_\rho$, where K_ρ positively oriented circle of radius ρ withe center at the origin.

Denote by $S(\rho) = S_f(\rho)$ the oriented area surrounded by Γ_ρ and set $\tau(\rho) = \tau_f(\rho) = \rho^{-2}S(\rho)$. The method of the proof of Theorem 4.4 can be used to prove more general result:

Theorem 4.5. *Let f be a holomorphic function on $A = A(r, R)$, $r < \rho < R$, $\Gamma_\rho = f \circ K_\rho$ and $S(\rho)$ the oriented area surrounded by Γ_ρ.*

Then $\tau(\rho) = \rho^{-2}S(\rho)$ is increasing, that is for $r < r_1 \leqslant R_1 < R$,

$$\frac{R_1^2}{r_1^2} \leqslant \frac{S(R_1)}{S(r_1)}.$$

Proof. Let $f(z) = \sum_{-\infty}^{+\infty} a_k z^k$, $z \in A$. It is known

$$S(\rho) = \int_{\Gamma_\rho} u\, dv = \frac{i}{2}\int_{\Gamma_\rho} w\, d\overline{w} = \frac{i}{2}\int_0^{2\pi} \Gamma_\rho\, d\,\overline{\Gamma_\rho}\,.$$

Hence

$$S(\rho) = \pi \sum_{-\infty}^{+\infty} k|a_k|^2 \rho^{2k}, \quad r < \rho < R\,.$$

Since $k(2k-2) \geqslant 0$, τ' is non negative and τ is not a decreasing function and consequently $\tau(R_1) \geqslant \tau(r_1)$. □

Theorem 4.6. (i.2) *τ_f is increasing on $[r, R]$, that is for $r < r_1 \leqslant R_1 < R$,*

$$\frac{R_1^2}{r_1^2} \leqslant \frac{S(R_1)}{S(r_1)}.$$

(ii.2) *Let A_1 be two ring domain in the plane and $\psi : A \to A_1$ be a covering with degree p.*

(iii.2) *Then $\tau_p(\rho) = \rho^{-2p}S(\rho)$ is increasing, that is for $r < r_1 \leqslant R_1 < R$,*

$$\frac{R_1^{2p}}{r_1^{2p}} \leqslant \frac{S(R_1)}{S(r_1)}.$$

(iv.2) *Let A and A_1 be two ring domains in the plane and let $\psi : A \to A_1$ be K-qr. Then $|deg\psi|M(A) \leqslant KM(A_1)$.*

The proof will appear in a forthcoming paper. If f is univalent, Theorem 4.6 (i.2) is reduced to Theorem A.

Beardon and Minda proved (Theorem 13.6, [4]):

If ψ is a holomorphic function. Then $|deg\psi|M(A) \leqslant M(A_1)$.

In addition, if ψ is a covering with degree p, then $M(A_1) = pM(A)$.

Note: Let γ be a closed rectifiable curve of length $L = l(\gamma)$. Then

$L \geqslant 2\pi cap\,\gamma$.

Let $G = [0, 1] \times \mathbb{B}$, and for each $\tau \in [0, 1]$ $f_\tau : \mathbb{B} \to C$ continuous mapping, and g homeomorphism of $[0, 1]$. Set $F = F_\tau(z) = (f_\tau(z), g(\tau))$ and $G^* = F(G)$. Define $cap_{av}(G^*) = (\int_0^1 cap^2(\tau)d\tau)^{1/2}$, where $cap(\tau) = cap(f_\tau(\mathbb{B})$.

Proposition 4.7. *Suppose that F is continuous and injective on G and that $G^* = F(G)$. Then $m_3(G^*) \leqslant \pi cap_{av}^2(G^*)$.*

Proof. The proof is based on the following facts:
(a) By Fubini's theorem $m_3(G^*) = \int_0^1 m_2(f_\tau(\mathbb{B}))d\tau$, and
(b) By Theorem 4.3(iii.1), $m_2(f_\tau(\mathbb{B})) \leqslant \pi cap^2(\tau)$. □

Using Goodman -Papus theorem we can prove the following.

Proposition 4.8. *Suppose that the volume V of a solid of revolution generated by rotating a plane figure F about an external axis and the distance d traveled by its geometric centroid. Then*

$$V \leqslant \pi d \mathrm{cap}^2(F) .$$

4.2. Wirtinger and Isoperametric inequality

The union of all closed spheres of radius r whose centeres lie in D is called the exterior parallel set D_r and denote by $A(r)$ the area of D_r. Let D be a convex domain in $\mathbb{R}^2$, r_0 the radius of the smallest circumbscribed circle in D and L be the length of ∂D. Then $A(r_0) = A + Lr_0 + \pi r_0^2 \leqslant 2Lr_0$; for a simple proof see [15]. Hence $A \leqslant Lr_0 - \pi r_0^2 \leqslant \frac{L^2}{4\pi}$.

An annulus $A = A(r, R)$ is a ring bounded by two concentric circles K_R and K_r with radii r, R ($r < R$). The annulus is said to enclose C if K_R encloses and meets C, while C encloses and meets K_R; to bi-enclose C if C passes at least four time between K_r and K_R. For a simple closed rectifiable curve γ we define the isoperimetric deficiency $\Delta(\gamma) = L^2 - 4\pi A$ and $\Delta_1(\gamma) = L^2/2\pi - 2A$, where $A = A(\gamma)$ and $L = L(\gamma)$ are respectively the sign area and the length of γ.

T. Bonnesen showed that for a convex closed curve C there is a unique annulus which bi-enclose C and established the inequality $2(R - r)^2 \leqslant \Delta_1(C)$. Fuglede extended Bonnesen result to any closed rectifiable curve γ in the plane.

If $z = \gamma = x + iy$, and $t = 2\pi s/L$, where s denotes natural parameter, then $\Delta_1(\gamma) = \int_0^{2\pi}[(x')^2 + y'^2]dt - 2\int_0^{2\pi} xy'dt$. Hence, since $[(x')^2 + y'^2] - 2xy' = (x - y')^2 + (x')^2 - (x)^2$, we find $\Delta_1(\gamma) = \int_0^{2\pi}(x - y')^2 dt + \int_0^{2\pi}[(x')^2 - (x)^2]dt$.

In the complex notation, $L^2 = \int_0^{2\pi} |z'(t)|^2 dt$, $2iA = (z', z)$ and $|iz' + z|^2 = |z'(t)|^2 + |z(t)|^2 + 2i(z', z)$. Hence $\Delta(\gamma) = L^2 - 4\pi A = 2\pi(\int_0^{2\pi}(|z'(t)|^2 + i(z', z))dt) = \pi(\int_0^{2\pi}(|z'(t)|^2 - |z(t)|^2 + |iz' + z|^2)dt$.

For a complex valued function f defined on $[a, b]$ we define $\underline{W}(f) := \underline{W}(f; [a, b]) = \int_0^{\pi}(|f'|^2 - |f|^2)dx$.

Suppose that (a1): f is real-valued function defined on $[0, 2\pi]$ and $\int_0^{2\pi} f(t)dt = 0$.

Since f has mean 0, you can write $f = F'$ for F another 2π periodic function and we can consider the curve $C = C_f$ defined by $C(t) = f(t) + iF(t)$, $t \in [0, 2\pi]$. By Cauchy-Schwartz inequality $L^2/2\pi \leqslant \int_0^{2\pi}[f^2 + f'^2]dt$. Hence $\Delta_1(C_f) = L^2/2\pi - 2A \leqslant \underline{W}(f)$. We might summarize the above consideration in the form:

Proposition 4.9. (i) *If γ is a closed rectifiable curve, s natural parameter and $t = 2\pi s/L$, then* $\underline{W}(\gamma) \leqslant \Delta_1(\gamma)$.
(ii) *Under the hypothesis* (a1), $\Delta_1(C_f) \leqslant \underline{W}(f)$.

Hence using Wirtinger's inequality one can prove the Isoperametric inequality and vice versa. Hurwitz used Wirtinger inequality in 1904 to prove the isoperimetric inequality.

Theorem 4.10 (Version 1). *Let $f : \mathbb{R} \to \mathbb{R}$ be a periodic function of period 2π, which is continuous and has a continuous derivative throughout $\mathbb{R}$ (more generally if f is absolutely continuous on $[0, 2\pi]$ and $f' \in L^2(0, 2\pi)$), and such that $\int_0^{2\pi} f(x)\,dx = 0$. Then*

$$\int_0^{2\pi} f'^2(x)\,dx \geqslant \int_0^{2\pi} f^2(x)\,dx$$

with equality if and only if $f(x) = a\sin(x) + b\cos(x)$, *for some a and b* (*or equivalently* $f(x) = c\,sin(x+d)$ *for some c and d*).

This version of the Wirtinger inequality is the one-dimensional Poincaré inequality, with optimal constant.

Before we proceed further note that the hypothesis that f is absolutely continuous function has a important role in correspoding version of previous theorem. Indeed, if f is odd periodic function of period 2π, which satisfies the condition $f(\pi/2 + x) = f(\pi/2 - x)$, $x \in \mathbb{R}$, defined by $f(t) = C(2t/\pi)$, $0 \leqslant t \leqslant \pi/2$, where C is Cantor function on $[0,1]$, then f is continuous on $\mathbb{R}$, $\int_0^{2\pi} f(x)\,dx = 0$, and $f' = 0$ a.e. on $\mathbb{R}$, but we can not apply the theorem on f.

The following related inequality is also called Wirtinger's inequality (see Dym & McKean 1985). We will call it Wirtinger's inequality version 2.

Theorem 4.11 (Version 2). *If* $a > 0$ *and* f *is a* C^1 *function such that* $f(0) = f(a) = 0$, *then*

$$\pi^2 \int_0^a |f|^2 \leqslant a^2 \int_0^a |f'|^2 .$$

In this form, Wirtinger's inequality is seen as the one-dimensional version of Friedrich's inequality.

Proof. The proof of the two versions are similar. The first version can be scaled to give the second version. Here is a proof of the first version of the inequality. Since Dirichlet's conditions are met, we can write

$$f(x) = \frac{1}{2}a_0 + \sum_{n\geqslant 1}\left(a_n \frac{\sin nx}{\sqrt{\pi}} + b_n \frac{\cos nx}{\sqrt{\pi}}\right),$$

and moreover $a_0 = 0$ since the integral of f vanishes. By Parseval's identity,

$$\int_0^{2\pi} f^2(x)dx = \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

and

$$\int_0^{2\pi} f'^2(x)\,dx = \sum_{n=1}^{\infty} n^2(a_n^2 + b_n^2)$$

and since the summands are all $\geqslant 0$, we get the desired inequality, with equality if and only if $a_n = b_n = 0$ for all $n \geqslant 2$. □

The version of (Poincaré-) Wirtinger inequality on $[0,\pi]$ is often appears in the practice. Although it is formally a corollary of Version 2, in fact it is equivalent statement and since it has independent significance we will state it as a separate result.

Theorem 4.12 (Version 3). $f : [0,\pi] \to \mathbb{C}$ *is* C^1 *and* $f(0) = f(\pi) = 0$ *then*

$$\int_0^{\pi} |f(t)|^2 dt \leqslant \int_0^{\pi} |f'(t)|^2 dt .$$

More generally the theorem holds if f is absolutely continuous on $[0,2\pi]$ *and* $f' \in L^2(0,\pi)$ *and* $\int_0^{\pi} f(x)\,dx = 0$.

The following proof is found in section 7.7 of Hardy-Littlewood-Polya Inequalities, it is motivated by Hilbert's investigations into calculus of variations, especially Hilbert's method of invariant integrals.

Proposition 4.13. *If* $y : [0,\pi] \to \mathbb{R}$ *is absolutely continuous function,* $y' \in L^2(0,\pi)$ *and* $y(0) = y(\pi) = 0$, *then* $\underline{I}(y) := \int_0^{\pi}(y'^2 - y^2)dx = \int_0^{\pi}(y' - y\cot x)^2 dx \geqslant 0$
with equality only if $y' = y\cot x$, *which is when* $y = k\sin x$.

Proof. Note that from the hypothesis that y is absolutely continuous function follows that $y(x) = \int_0^x y'(s)ds$, $x \in [0, \pi]$. Consider the expression $A(x) = (y'^2 - y^2) - (y' - y\cot x)^2 = -(1 + \cot^2 x)y^2 + 2yy'\cot x$ and $B(x) = y^2 \cot x$. One can check that $dB = A(x)dx$.
Now, since $y' \in L^2$, we have that $y^2(x) = (\int_0^x y'(s)ds)^2 \leqslant (\int_0^x y'^2(s)ds)\int_0^x 1ds = x\int_0^x y'^2(s)ds$ and therefore $y^2(x) = o(1)x$, when $x \to 0$. In a similar way, for every $a \in [0, \pi]$, we have $y^2(x) = o(1)(x - a)$, when $x \to a$.
Hence $\lim_{x\to 0} B(x) = \lim_{x\to\pi} B(x) = 0$ and
$I = \int_0^\pi A(x)dx = \int_0^\pi dB = B|_0^\pi = B(\pi) - B(0) = 0 - 0 = 0$, and therefore
$\underline{I}(y) = \int_0^\pi (y'^2 - y^2)dx = \int_0^\pi (y' - y\cot x)^2 dx \geqslant 0$
with equality only if $y' = y\cot x$, which is when $y = k\sin x$. □

4.3. Lax's Proof of the Isoperimetric Inequality

Let α be a closed curve. Recall then area $A = A(\alpha)$ enclosed by α is $A = \int_\alpha xdy = \frac{i}{2}\int_\alpha zd\bar{z} = -\frac{1}{2}\mathrm{Im}(\int_\alpha zd\bar{z})$.

Peter Lax constructed what is currently considered to be the shortest and most elementary of all existing proofs. Tapia [58] presents a short, elementary, and teachable solution of the isoperimetric problem. He demonstrates that Euler's approach can be extended to give a sufficiency proof which is short and elementary and therefore for which he believes that is competitive with the Lax proof from this point of view. The following proof is due to Peter Lax and is taken from a paper presented in American Mathematical Monthly [27]. This journal is generally a good source for short, readable papers.

Proof. Let $\alpha(s)$ be parameterized by arc length, and $\alpha(s) = (x(s), y(s))$. In this proof, we will assume that the perimeter of the image of α is equal to 2π, and show that the area is less than π. We will later show that the results can be scaled to give the formula proposed in the theorem. Since α is unit speed, $(x')^2 + (y')^2 = 1$. Also, we may assume that $y(0) = y(\pi) = 0$. If this is not the case, then rotate the coordinate axes to make it so.

Area is a positive quantity, but the integrals used to compute the area bounded by the image of α carry an ambiguity due to the orientation of α and in general $A(\alpha) = \int_\alpha ydx$ can have negative value.

Set $|A| = \pm\int_\alpha ydx = \pm(B + C)$, where $B = \int_0^\pi yx'ds$ and $C = \int_\pi^{2\pi} yx'ds$. Since $(x')^2 + (y')^2 = 1$,

$|B| \leqslant \frac{1}{2}\int_0^\pi [y^2 + (x')^2]ds = \frac{1}{2}\int_0^\pi (y^2 + 1 - (y')^2)ds$. Set $a(s) = y^2 + 1 - (y')^2$ and $\underline{J}(y) := \int_0^\pi a(s)ds$. Since $\underline{J}(y) = \pi - \underline{I}(y)$, by the version 3 of (Poincare-) Wirtinger inequality, we find $\underline{J}(y) \leqslant \pi$. Lax gave direct proof of this inequality. Write $y(s) = u(s)\sin(s)$ for $0 \leqslant s \leqslant \pi$ and show that $\underline{J} = \int_0^\pi c(s)ds$, where $c(s) = 1 - (u')^2\sin^2 s$.

Now we outline his argument which uses a bit of cleverness.. Using $y' = u'\sin s + u\cos s$, we find $a(s) = u^2(\sin^2 s - cos^2 s) + 1 - (u')^2\sin^2 s - 2uu'\sin s\cos s$. Set $b(s) = u^2(\sin^2 s - cos^2 s) - 2uu'\sin s\cos s$, $b_1(s) = u^2(\sin^2 s - cos^2 s)$, $b_3(s) = -2uu'\sin s\cos s$, $c(s) = b_2(s) = 1 - (u')^2\sin^2 s$ and $B_k = \int_0^\pi b_k(s)ds$.

Since we know very little about the function u, we know even less about its derivative u'. A standard mathematical technique is to remove unwanted derivatives using integration by parts. Note that $2uu' = (u^2)'$, and so we take $t = u^2$ and $v = \sin s\cos s$, we have $B_1 = -\int_0^\pi u^2 dv$. Since $v(0) = v(\pi) = 0$, by the formula for integration by parts, we find $B_3 = -\int_0^\pi d(u^2)v = \int_0^\pi u^2 dv$ and therefore $B_1 + B_3 = 0$. Hence $\underline{J} = \int_0^\pi c(s)ds$ and since $c(s) \leqslant 1$, we get $\underline{J} \leqslant \pi$ and $|B| \leqslant \pi/2$.

A similar proof can be used to show that the second integral is also less than or equal to $\pi/2$. Hence $A \leqslant 2\cdot\pi/2 = \pi$, as claimed. □

4.4. Goodman -Papus theorem

In geometry the term barycenter is a synonym for "centroid", in physics "barycenter" may also mean the physical center of mass or the center of gravity, depending on the context. The center of mass (and center of gravity in a uniform gravitational field) is the arithmetic mean of all points weighted by the local density or specific weight. If a physical object has uniform density, then its center of mass is the same as the centroid of its shape.

The centroid of a triangle is the intersection of the three medians of the triangle (each median connecting a vertex with the midpoint of the opposite side). It lies on the triangle's Euler line, which also goes through various other key points including the orthocenter and the circumcente. The geometric centroid of a convex object always lies in the object. A non-convex object might have a centroid that is outside the figure itself. The centroid of a ring or a bowl, for example, lies in the object's central void.

If the centroid is defined, it is a fixed point of all isometries in its symmetry group. In particular, the geometric centroid of an object lies in the intersection of all its hyperplanes of symmetry. The centroid of many figures (regular polygon, regular polyhedron, cylinder, rectangle, rhombus, circle, sphere, ellipse, ellipsoid, superellipse, superellipsoid, etc.) can be determined by this principle alone.

In particular, the centroid of a parallelogram is the meeting point of its two diagonals. This is not true for other quadrilaterals.

For the same reason, the centroid of an object with translational symmetry is undefined (or lies outside the enclosing space), because a translation has no fixed point.

For a plane figure X, in particular, the barycenter coordinates are

$$C_{\mathrm{x}} = \frac{\int x S_{\mathrm{y}}(x)\, dx}{A} \; and \tag{20}$$

$$C_{\mathrm{y}} = \frac{\int y S_{\mathrm{x}}(y)\, dy}{A}, \tag{21}$$

where A is the area of the figure X; $S_y(x)$ is the length of the intersection of X with the vertical line at abscissa x; and $S_x(y)$ is the analogous quantity for the swapped axes. Pappus' centroid theorem (also known as the Guldinus theorem, Pappus-Guldinus theorem or Pappus' theorem) is either of two related theorems dealing with the surface areas and volumes of surfaces and solids of revolution, cf. [66].

The first theorem states that the surface area A of a surface of revolution generated by rotating a plane curve C about an axis external to C and on the same plane is equal to the product of the arc length L of C and the distance d traveled by its geometric centroid: $A = Ld$.

The second theorem states that the volume V of a solid of revolution generated by rotating a plane figure F about an external axis is equal to the product of the area A of F and the distance d traveled by its geometric centroid:

$$V = Ad.$$

For example, the volume of the torus with minor radius r and major radius R is

$$V = (\pi r^2)(2\pi R) = 2\pi^2 R r^2.$$

The theorem can be generalized for arbitrary curves and shapes, under appropriate conditions, cf. [21].

5. Isoperimetric inequality in space

A condenser in $\mathbb{R}^n$ is a pair $E = (A, F)$, where A is open set in $\mathbb{R}^n$ and F is a compact subset of A. The p-capacity of E is defined by

$$\mathrm{cap}_p E = \inf \int_A |\nabla u|^p dm, \quad 1 \leqslant p < \infty, \tag{22}$$

where the infimum is taken over all nonnegative functions u in $ACL^p(A)$ (for ACL propery see subsection 5.4) with compact support in A and $u|F \geqslant 1$. The n-capacity (respectively 2-capacity) of E is called the conformal capacity (respectively electrostatic capacity) of E and denoted by cap$E$.

For all condenser $E = (A, F)$ in $\mathbb{R}^n$

$$\mathrm{cap}(A, F) = Mod(\triangle(f, \partial A; A)).$$

If $R = R(C_0, C_1)$ is a ring and if C_0^* and C_1^* are the spherical symmetrization of C_0 and C_1 in opposite rays L_0,L_1, then cap$R^* \leqslant$ cap$R$, where $R^* = R(C_0^*, C_1^*)$. The n-dimensional generalization of planar isoperimetric inequality is

$$(mes_n D)^{\frac{n-1}{n}} \leqslant c_n H_{n-1}(\partial D), \tag{23}$$

where D is a domain with smooth boundary ∂D and compact closure, and H_{n-1} is the (n - 1)-dimensional area. The constant c_n is such that (23) becomes equality for any ball, that is $c_n = n^{-1}\omega_n^{1/n}$ with ω_n standing for the volume of the unit ball. Inequality (23) holds for arbitrary measurable sets with H_{n-1} replaced by the so called perimeter in the sense of De Giorgi (1954-1955).

The isoperimetric inequality in n-dimensions can be quickly proven by the Brunn-Minkowski inequality (Osserman [48](1978); Federer (1969, 3.2.43)).

The Minkowski-Steiner formula is used, together with the BrunnMinkowski theorem, to prove the isoperimetric inequality.

The Minkowski-Steiner formula is a formula relating the surface area and volume of compact subsets of Euclidean space. More precisely, it defines the surface area as the "derivative" of enclosed volume in an appropriate sense. Define the quantity $\lambda(\partial A)$ by the Minkowski-Steiner formula

$$\lambda(\partial A) := \liminf_{\delta \to 0} \frac{\mu\left(A + \overline{B_\delta}\right) - \mu(A)}{\delta},$$

where $\overline{B_\delta} = \{x : |x| \leqslant \delta\}$.

The Brunn-Minkowski inequality: Let $n \geqslant 1$ and let μ denote the Lebesgue measure on $\mathbb{R}^n$. Let A and B be two nonempty convex compact subsets of $\mathbb{R}^n$. Then the following inequality holds:

$$[\mu(A + B)]^{1/n} \geqslant [\mu(A)]^{1/n} + [\mu(B)]^{1/n},$$

where $A + B$ denotes the Minkowski sum: $A + B := \{a + b \in \mathbb{R}^n \mid a \in A,\ b \in B\}$.

In general, no reverse bound is possible, since one can find convex bodies A and B of unit volume so that the volume of their Minkowski sum is arbitrarily large. Milman's theorem states that one can replace one of the bodies by its image under a properly chosen volume-preserving linear map so that the left-hand side of the BrunnMinkowski inequality is bounded by a constant multiple of the right-hand side.

The n-dimensional isoperimetric inequality is equivalent (for sufficiently smooth domains) to the Sobolev inequality on $\mathbb{R}^n$ with optimal constant:

$$\left(\int_{\mathbb{R}^n} |u|^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}} \leqslant n^{-1}\omega_n^{-1/n} \int_{\mathbb{R}^n} |\nabla u|$$

for all $u \in W^{1,1}(R^n)$. For recent development see [15].

If F is compact set in a domain $D \subset \mathbb{R}^n$ and $E = (D, F)$, then [43]

$$cap_n E \geqslant n^n v_n \left[\ln \frac{mes_n(D)}{mes_n(F)}\right]^{1-n}. \tag{24}$$

5.1. *Recent developments concerning isoperimetric inequality in space*

There are new ideas concerning isoperimetric inequality in space; see for example [15, 20] and the literature cited there. Fusco's notes [20] contains five lectures given by the author at the CNA Summer School held at Carnegie Mellon University in Pittsburgh from May 30 to June 7, 2013. The course gives a self contained introduction to the classical isoperimetric inequality and to various stability results proved in recent years for this inequality and other related geometric and analytic inequalities. Druet paper [15] yields an overview of what is known about isoperimetric inequalities on nonpositively curved spaces, more precisely on Cartan-Hadamard manifolds. The whole course turns around a conjecture, which is still open, which asserts that the Euclidean isoperimetric inequality should hold on complete, simply-connected Riemannian manifolds of nonpositive sectional curvature (Cartan-Hadamard manifolds). A natural way to prove the isoperimetric inequality is to find a map from a domain into the ball of same volume which preserves the volume and decreases the area. In particular a proof is given due to Gromov. Although exact reference for this proof is not given in [15] [1)] Duret follows M. Berger, who in one of his books on differential geometry, writes that this proof is due to Gromov and we will follow the same. Note that according McCann-Gullien [45], McCann and Trudinger observed independently that a solution to the second boundary value problem for the Monge-Ampère equation (M-A) (see subsection 5.3) yields a simple proof of the isoperimetric inequality (with its sharp constant).

The proof of Gromov uses the Knothe map. There is another nice proof in in the same spirit using the Brenier map coming from optimal transport. The course gives some backgrounds to understand why this conjecture should be true and to realize the outlines of the proofs of some of the main results on the subject, leaving details to the reader who can refer to the original papers. If E is a subset of $\mathbb{R}^n$ we denote by $|E| = m_n(E)$ the Lebesgue n- dimensional measure and by $H^k(E)$, $1 \leqslant k \leqslant n$, Hausdorff k-dimensional measure. We also use notation $|\partial E|$ for $(n-1)$- dimensional surface measure of ∂E. If $\omega_n = |\mathbb{B}^n|$ and $\sigma_{n-1} = |\mathbb{S}^{n-1}|$, it is known that $\sigma_{n-1} = n\omega_n$.

One can use the general Stokes' Theorem to equate the n-dimensional volume integral of the divergence of a vector field F over a region V to the $(n-1)$-dimensional surface integral of F over the boundary of V:

$$\int_V \nabla \cdot \mathbf{F}\, dV_n = \oint_{\partial V} \mathbf{F} \cdot \mathbf{n}\, dS_{n-1}\,.$$

The left side is a volume integral over the domain V, the right side is the surface integral over the boundary of the domain V. The closed manifold ∂V is quite generally the boundary of V oriented by outward-pointing normals, and $\mathbf{n}$ is the outward pointing unit normal field of the boundary ∂V. In terms of the intuitive description above, the left-hand side of the equation represents the total of the sources in the volume V, and the right-hand side represents the total flow across the boundary ∂V. This equation is also known as the Divergence theorem. When $n = 2$, this is equivalent to Green's theorem. When $n = 1$, it reduces to the Fundamental theorem of calculus.

We now outline a formal proof of the isoprimetric inequality. By scaling invariance of isoperimetric inequality, we can assume that the volume of E is the volume of unit ball ω_n. Let $M(x) = (M_1(x_1), M_2(x_1, x_2), ..., M_n(x))$ be the Knothe map which maps E onto $\mathbb{B}$, see for example [18] and subsection 5.2. This map has several interesting properties, that are easily checked at a formal level. Its gradient ∇M is upper triangular, its diagonal entries (the partial derivatives $d_k = \partial M_k/\partial x_k$) are positive on E, and their product, the Jacobian of M, $J(M)$ is constantly equal to $1 := |K|/|E|$, i.e.,

$$J(M) = det\,\nabla M = \prod_{k=1}^{n} d_k = 1\,. \tag{25}$$

By the arithmetic-geometric mean inequality $n = nJ(M)^{1/n} \leqslant div(M)$, that is $n \leqslant div(M)$, and therefore $\sigma_{n-1} = n\omega_n = n\int_E 1dx \leqslant \int_E div(M)dx$. Hence we first conclude that $\sigma_{n-1} \leqslant \int_E div(M)dx$ and if the boundary of E is oriented by outward-pointing normals, and ν is the outward pointing unit normal field of the boundary, by a formal application of the Divergence Theorem [2)],

$$\sigma_{n-1} \leqslant \int_E div(M)dx = \int_{\partial E} M \cdot \nu\, dS\,. \tag{26}$$

Since $|M| = 1$ on ∂E we find $|M \cdot \nu| \leqslant 1$ on ∂E. Hence $\int_{\partial E} M \cdot \nu dS \leqslant |\partial E|$, and we obtain that $n|E| \leqslant |\partial E|$ which is exactly the isoprimetric inequality.

[1)] the reference may be the appendix by M. Gromov in [46].

[2)] we suppose enough regularity to apply Stokes theorem

For further discussion we borrow a few details from Villani [62], see p.28 and p.362. Apart from the Euclidean one, the most famous isoperimetric inequality in differential geometry is certainly the Levy-Gromov inequality, which states that if A is a reasonable set in a manifold (M, g) with dimension n and Ricci curvature bounded below by K, then

$$\frac{|\partial A|}{|A|^{\frac{n-1}{n}}} \geqslant \frac{|\partial B|}{|B|^{\frac{n-1}{n}}},$$

where B is a spherical cap in the model sphere S (that is, the sphere with dimension n and Ricci curvature K) such that $|B|/|S| = |A|/|M|$. In other words, isoperimetry in M is at least as strong as isoperimetry in the model sphere.

According Villani it is not known if the Levy-Gromov inequality can be retrieved from optimal transport, and this is one of the most exciting open problems in the field. He writes: "Indeed, there is to my knowledge no reasonable proof of the Levy-Gromov inequality, in the sense that the only known arguments rely on subtle results from geometric measure theory, about the rectifiability of certain extremal sets. A softer argument would be conceptually very satisfactory". He records this in the form of a loosely formulated open problem:

Open Problem : Find a transport-based, soft proof of the Levy-Gromov isoperimetric inequality (see 21.16[62]).

Next, in [18], a sharp quantitative version of the anisotropic isoperimetric inequality is established, corresponding to a stability estimate for the Wulff shape of a given surface tension energy. This is achieved by exploiting mass transportation theory, especially Gromovs proof of the isoperimetric inequality and the Brenier-McCann Theorem. A sharp quantitative version of the Brunn-Minkowski inequality for convex sets is proved as a corollary. In [22], motivated by Carlemans proof of isoperimetric inequality in the plane, the authors study some sharp integral inequalities for harmonic functions on the upper halfspace. They also derive the regularity for nonnegative solutions of the associated integral system and some Liouville type theorems, [22].

Anisotropic perimeter. The anisotropic isoperimetric inequality arises in connection with a natural generalization of the Euclidean notion of perimeter. In dimension $n \geqslant 2$, we consider an open, bounded, convex set K of $\mathbb{R}^n$ containing the origin. Starting from K, we define for every $x \in \mathbb{R}^n$, $|x|_K = \inf\{\lambda > 0 : x \in K\}$, and a weight function on directions through the Euclidean scalar product $|v|_* := \sup\{x \cdot v : x \in K\}$, $v \in S^{n-1}$, where $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$, and $|x| = |x|_e$ is the Euclidean norm of $x \in \mathbb{R}^n$. Let E be an open subset of $\mathbb{R}^n$, with smooth or polyhedral boundary ∂E oriented by its outer unit normal vector v_E, and let H^{n-1} stand for the $(n-1)$-dimensional Hausdorff measure on $\mathbb{R}^n$. The anisotropic perimeter of E is defined as

$$P_K(E) := \int_{\partial E} |v_E(x)|_* \, dH^{n-1}(x)\,.$$

This notion of perimeter obeys the scaling law $P_K(\lambda E) = \lambda^{n-1} P_K(E), \lambda > 0$, and it is invariant under translations. However, by contrast to the Euclidean perimeter, P_K is not invariant by the action of $O(n)$, or even of $SO(n)$.

When K is the Euclidean unit ball $\mathbb{B} = \{x \in R^n : |x|_e < 1\}$ of R^n then $|v|_* = 1$ for every $v \in S^{n-1}$, and therefore $P_K(E)$ coincides with the Euclidean perimeter (surface area) of E. Set $n' = n/(n-1)$ and denote by $|E|$ the Lebesgue measure of E. The notion of volume obeys the scaling law $|\lambda E| = \lambda^n |E|, \lambda > 0$. Hence

$$\frac{P_K(\lambda E)}{|\lambda E|^{1/n'}} = \frac{P_K(E)}{|E|^{1/n'}}\,. \tag{27}$$

Apart from its intrinsic geometric interest, the anisotropic perimeter P_K arises in applications as a model for surface tension in the study of equilibrium configurations of solid crystals. If a liquid drop or a crystal of mass m is subject to the action of a potential, at equilibrium, its shape minimizes (under a volume constraint) the free energy, that consists of a (possibly anisotropic) interfacial surface energy $P_K(E)$ plus a bulk potential energy, see [17] and literature cited there and posted on Figalli site [17](b).

In both settings, one is naturally led to minimize $P_K(E)$ under a volume constraint. By (27), this is, of course, equivalent to study the isoperimetric problem

$$\inf\{\frac{P_K(E)}{|E|^{1/n'}} : 0 < |E| < \infty\}.$$

As conjectured by Wulff back to 1901, the unique minimizer (modulo the invariance group of the functional, which consists of translations and scalings) is the set K itself. In particular the anisotropic isoperimetric inequality holds,

$$(A1) \qquad P_K(E) \geqslant n|K|^{1/n}|E|^{1/n'}.$$

Dinghas showed how to derive (A1) from the Brunn-Minkowski inequality:

$$(A2) \qquad |E+F|^{1/n} \geqslant |E|^{1/n} + |F|^{1/n}$$

for every $E, F \subset \mathbb{R}^n$. The formal argument is well known. (A2) implies $|E+\varepsilon K| \geqslant (|E|^{1/n} + \varepsilon|K|^{1/n})^n$ and therefore

$$\frac{|E+\varepsilon K| - |E|}{\varepsilon} \geqslant \frac{(|E|^{1/n} + \varepsilon|K|^{1/n})^n - |E|}{\varepsilon}.$$

As $\varepsilon \to 0$, the right hand side converges to $n|K|^{1/n}|E|^{1/n'}$, while, if E is regular enough, the left hand side has $P_K(E)$ as its limit. For $y \in \mathbb{R}^n$, we define a weight function on directions through the Euclidean scalar product $|y|_* = |y|_{*,K} := \sup\{x \cdot v : x \in K\}$, which gives the following Cauchy-Schwarz type inequality

$$(A3) \qquad x \cdot y \leqslant |x||y|_*.$$

Gromov [46] deals with the functional version of (A1), proving the anisotropic Sobolev inequality

$$\int_{\mathbb{R}^n} |-\nabla f|_* \, dx \geqslant n|K|^{1/n} \mid f \mid_{L^{n'}(\mathbb{R}^n)} \tag{28}$$

for every $f \in C_c^1(\mathbb{R}^n)$.

5.2. Knothe map

In this subsection we outline a proof of anisotropic isoperimetric inequality (A1) using the Knothe map. Let E and K be two subset in $\mathbb{R}^n$. We first construct the Knothe map M between E and K. The Knothe construction depends on the choice of an ordered orthonormal basis of $\mathbb{R}^n$. Let us use, for example, the canonical basis of $\mathbb{R}^n$, with coordinates $x = (x_1, x_2, ..., x_n)$, and for every $x \in E$, $y \in K$ and $1 \leqslant k \leqslant n-1$, let us define the corresponding $(n-k)$-dimensional sections of E and K as
$E_{(x_1,x_2,\cdots,x_k)} = \{z \in E : z_1 = x_1, \cdots, z_k = x_k\}$ and
$K_{(y_1,y_2,\cdots,y_k)} = \{z \in K : z_1 = y_1, \cdots, z_k = y_k\}$.
We will define $M(x) = (M_1(x_1), M_2(x_1, x_2), ..., M_n(x))$ as follows. The vertical section E_{x_1} of E is sent into the vertical section $K_{M_1(x_1)}$ of K, where $M_1(x_1)$ is chosen so that the relative measure of $\hat{E}_{x_1} = \{z \in E : z_1 < x_1\}$ in E equals the relative measure of $\{z \in K : z_1 < M_1(x_1)\}$ in K. The same idea is used to displace E_{x_1} along $K_{M_1(x_1)}$: the point $x = (x_1, x_2)$ is placed in $K_{M_1(x_1)}$ at the height $M_2(x)$ such that the relative $H^{(n-1)}$-measure of $\hat{E}_{(x_1,x_2)} = \{z \in E_{x_1} : z_2 < x_2\}$ in E_{x_1} equals the relative $H^{(n-1)}$-measure of $\{z \in K_{M_1(x_1)} : z_2 < M_2(x)\}$ in $K_{M_1(x_1)}$. We proceed by induction. Set $A_k := \hat{E}_{(x_1,x_2,\cdots,x_k,x_{k+1})} = \{z \in E_{(x_1,x_2,\cdots,x_k)} : z_{k+1} < x_{k+1}\}$ and $B_k := \hat{K}_{(M_1,M_2,\cdots,M_k,M_{k+1})} = \{z \in K_{(M_1,M_2,\cdots,M_k)}) : z_{k+1} < M_{k+1}\}$. For given $(x_1, x_2, \cdots, x_k)$ if the functions $M_1(x_1)$, $M_2(x_1, x_2)$, ..., and $M_k(x_1, x_2, \cdots, x_k)$ are determined, then for given x_{k+1}, we define M_{k+1} such that

$$\frac{H^{n-k}(A_k)}{H^{n-k}(E_{(x_1,x_2,\cdots,x_k)})} = \frac{H^{n-k}(B_k)}{H^{n-k}(E_{(M_1,M_2,\cdots,M_k,M_{k+1})})}.$$

By the construction M is volume-preserving and M_k does depend only on $x_1, ..., x_k$. Thus the matrix $D_i M_j$ is upper diagonal. The resulting map has several interesting properties, that are easily checked at a formal level. Its gradient ∇M is upper triangular, its diagonal entries (the partial derivatives $d_k = \partial M_k / \partial x_k$) are positive on E, and their product, the Jacobian of M, $J(M)$ is constantly equal to $a := |K|/|E|$, i.e.,

$$J(M) = det\, \nabla M = \prod_{k=1}^{n} d_k = a\,. \tag{29}$$

By the arithmetic-geometric mean inequality, we find

$$nJ(M)^{1/n} \leqslant div\, M \tag{30}$$

on E. By (29), (30) and a formal application of the Divergence Theorem,
$n|K|^{1/n}|E|^{1/n'} = \int_E nJ(M)^{1/n} \leqslant \int_E div\, M = \int_{\partial E} M \cdot v_E(x) dH^{n-1}(x)$.
Hence

$$n|K|^{1/n}|E|^{1/n'} \leqslant \int_{\partial E} M \cdot v_E(x) dH^{n-1}(x)\,. \tag{31}$$

¿From (31) and (A3), we find

$$(A4) \qquad n|K|^{1/n}|E|^{1/n'} \leqslant \int_{\partial E} |M||v_E(x)|_*\, dH^{n-1}(x) \leqslant P_K(E)$$

and the isoperimetric inequality is proved.

5.3. Brenier map

The above argument could be repeated verbatim if the Knothe map is replaced by the Brenier map. The Brenier-McCann Theorem furnishes a transport map between E and K, which is analogous to the Knothe map, but enjoys a much more rigid structure.

Theorem 5.1 (A version of Brenier's theorem). *If $\mu \ll dx$ and ν are Borel probability measures on $X = Y = \mathbb{R}^d$, then there exists a unique convex, Lipschitz continuous function $L : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ such that its gradient $T = \nabla L$ pushes μ forward to ν. Apart from changes on a set of measure zero, g is unique.*

A construction of Brenier's map for some special measures in a summary An Elementary Introduction to Monotone Transportation, written by Ivanisvili, [67], p.44-48. First, we consider the case when the measure ν is atomic, i.e. with finite support $\{u_j : 1 \leqslant j \leqslant m\}$.

For such a measure we find a convex function of the form $L(x) = \max_j\{(u_j, x) - s_j\}$ with some apropriate nambers s_j, such that it satisfyes the required property. In general, we approximate measure ν weakly by atomic measures ν_k. It turns out that we can choose corresponding convex functions L_k so that they converge locally uniformly to some convex function L, moreover $\nabla L_k \to \nabla L$ except for some set. Finally, by standard weak limit arguments we can see that the map L transports the measure μ to ν.

For connections between Optimal Transport and isoperimetric problem see [67] and literature cited there. Postponing a rigorous discussion to the proof of (A1), we recall that Brenier-McCann Theorem ensures the existence of a convex, continuous function. Let (X, μ) and (Y, ν) be two Borel measurable space and a cost funtion $c : X \times Y \to \mathbb{R}$. Any Borel map $S : X \to Y$ defines an image or push-forward measure $\nu = S_* \mu$ on Y by $\nu[B] = \mu[S^{-1}(B)]$.

The optimal transport problem (The Monge problem) seeks a measurable map $T : X \to Y$ such that $T_* \mu = \nu$, and

$$(M) \qquad \int c(x, Tx)\mu(dx) = \inf_{S_*\mu = \nu} \int c(x, Sx)\mu(dx)\,.$$

Original Monge cost function is $c(x,y) = |x-y|$ in $\mathbb{R}^3$. For this cost, existence of a minimizer proven around 1998 – 2003!! (Ambrosio, Caffarelli, Evans, Feldman, Gangbo, McCann, Sudakov, Trudinger, Wang). Easier solution when the cost is "strictly convex".

Suppose now that $X, Y \subset R^n$ $d\mu = \rho(x)dx$ with measures $d\nu = \tilde{\rho}(x)dx$ absolutely continuous with respect to Lebesgue measure. In $\mathbb{R}^n$ if S is 1-to-1, using changes of variables $\tilde{\rho}(y)dy = \tilde{\rho}(Sx)|det(dS)(x)|$, $\nu[B] = \mu[S^{-1}(B)]$ yields

(M1) $\rho(x) = \tilde{\rho}(Sx)|det(dS)(x)|$.

Hence it is clear that solutions T to (M) satisfy $\rho(x) = \tilde{\rho}(Tx)|det(dT)(x)|$. Now due to the characterization $T = Du$, it can be seen that solutions to (M) satisfy an equation of Monge-Ampère type:

(M-A) $\rho(x) = \tilde{\rho}(Du(x))|det(D^2u(x)|$.

For the quadratic cost function $c(x,y) = -(x,y)$ it was shown by Brenier that there exists a unique solution $T = Du$ which is the gradient of some convex potential function u. This result was extended to more general cost functions by Gangbo and McCann. Thus solution of the Euclidean optimal transport $T = \nabla L$, L is convex. In particular T is monotone. Monotone changes of variables have applications .

Recall that Euclidean isoperimetric problem states that:

(A) Among all domains of fixed volume the sphere has minimal surface. We now outline Gromov's approach. Let D be a domain and denote by S the boundary of D. Set $f(x) = 1/|D|$, $g(y) = 1/|\mathbb{B}|$, and consider $\mu = f dx$ and $\nu = g dy$. By Brenier's theorem there is $T : D \to \mathbb{B}$ such that T pushes uniform measure forward μ to uniform measure ν and $T = \nabla u$, where u is convex. Then dT has nonnegative eigenvalues λ_i. Hence, by Proposition 5.3,

$$\left(\frac{|B|}{|D|}\right)^{1/n} = (det\, dT)^{1/n} = (\prod \lambda_i)^{1/n} \leqslant \frac{\sum \lambda_i}{n} = \frac{divT}{n}$$

and

$$A = A(D,B) =:= \int_D (det\, dT)^{1/n} = |D|\left(\frac{|B|}{|D|}\right)^{1/n} = |B|^{1/n}|D|^{1/n'}$$

and therefore

$$A = |D|\left(\frac{|B|}{|D|}\right)^{1/n} \leqslant \int_D \frac{divT}{n}.$$

Using the divergence theorem

$$\int_D \frac{divT}{n} = \frac{1}{n}\int_S T\cdot n,$$

we find

$$A \leqslant \frac{1}{n}\int_S |T| = \frac{|S|}{n}.$$

Hence

$$n|D|^{(n-1)/n}|B|^{1/n} \leqslant |S|.$$

Using the Brenier map, it is (formally) easy to characterize the minimizers of the inequality. Indeed, equality implies that $n(det\, dT)^{1/n} = divT$ on D. Thus, we have equality in the arithmetic-geometric mean inequality which in turn implies that $\lambda_1 = \lambda_2 = \cdots \lambda_n$. Therefore $T = Id$ and consequently, D must be a translate of B.

If K is a convex set, in a similar way we can prove (A1) $\quad P_K(E) \geqslant n|K|^{1/n}|E|^{1/n'}$. Equality holds if D is a translate of K. A rigorous discussion requires more details. Without loss of generality, we can assume $E \subset \mathbb{R}^n$ bounded and smooth. Let $\mathbb{B}_r = B(0,r)$ be the ball centered at the origin with radius $r > 0$. By Brenier's Theorem 5.1, there exists a unique convex, Lipschitz continuous function $L : \mathbb{R}^n \to \mathbb{R}$ such that its gradient $T = \nabla L$ pushes forward the probability density $\frac{1}{|E|}\chi_E(x)dx$ onto the probability density

$$\frac{1}{\omega_n r^n}\chi_{B_r}(y)dy\,,$$

where $|E|$ denotes the volume of E. By Caffarelli's regularity result we can assume $T \in C^\infty(E, B_r)$. Moreover L solves the following Monge-Ampere equation

$$det\nabla^2 L = \frac{\omega_n r^n}{|E|}$$

on E, where $\nabla^2 L$ is the Hessian matrix of L. As L is convex, the Hessian matrix $\nabla^2 L$ is a positive definite symmetric matrix, and so by the arithmetic-geometric inequality we get $n(det\,\nabla^2 L)^{1/n} \leqslant \Delta L$. Now we can proceed as in subsections 5.2 and 5.3 by application of the Divergence Theorem.

5.4. Appendix 1, Hessian matrix, Trace of matrix & ACL propery

Given the real-valued function $f(x_1, x_2, \dots, x_n)$, if all second partial derivatives of f exist and are continuous over the domain of the function, then the Hessian matrix of f is

$$H(f)_{ij}(\mathbf{x}) = D_i D_j f(\mathbf{x})$$

where $x = (x_1, x_2, ..., x_n)$ and D_i is the differentiation operator with respect to the i-th argument.

If u is harmonic in a domain in $\mathbb{R}^n$, $n \geqslant 3$, such that at every point x, the Hessian matrix H has at most one negative eigenvalue, then either $detH$ never vanishes, or it vanishes identically.

Because f is often clear from context, $H(f)(\mathbf{x})$ is frequently abbreviated to $H(\mathbf{x})$.

The Hessian matrix is related to the Jacobian matrix by $H(f)(\mathbf{x}) = J(\nabla f)(\mathbf{x})$.

The determinant of the above matrix is also sometimes referred to as the Hessian. Hessian matrices are used in large-scale optimization problems within Newton-type methods because they are the coefficient of the quadratic term of a local Taylor expansion of a function. Assuming still that the function f is twice continuously differentiable on a domain $G \subset \mathbb{R}^n$, by using Taylor's Theorem, we have for $\mathbf{x}, \mathbf{x} + \Delta\mathbf{x} \in G$ that

$$y = f(\mathbf{x} + \Delta\mathbf{x}) = f(\mathbf{x}) + f'(\mathbf{x})\Delta\mathbf{x} + \frac{1}{2}\Delta\mathbf{x}^{\mathrm{T}} H(\mathbf{x})\Delta\mathbf{x} + o|\Delta\mathbf{x}|^2, \quad \Delta\mathbf{x} \to 0\,. \tag{32}$$

Sometimes in the literature the above formula is written in the form:

$$y = f(\mathbf{x} + \Delta\mathbf{x}) \approx f(\mathbf{x}) + J(\mathbf{x})\Delta\mathbf{x} + \frac{1}{2}\Delta\mathbf{x}^{\mathrm{T}} H(\mathbf{x})\Delta\mathbf{x}, \tag{33}$$

where J is the Jacobian matrix, which is a vector (the gradient) for scalar-valued functions: $J = \nabla f = (D_1 f, D_2 f, ..., D_n f)$.

The following test can be applied at a non-degenerate critical point x. If the Hessian is positive definite at x(we write $H(x) > 0$), then f attains a local minimum at x. If the Hessian is negative definite at x(we write $H(x) < 0$), then f attains a local maximum at x. If the Hessian has both positive and negative eigenvalues then x is a saddle point for f (this is true even if x is degenerate).

Let X be a convex set in a real vector space and let $f : X \to \mathbb{R}$ be a function.

f is called convex if:

(B1) $\forall x_1, x_2 \in X, \forall t \in [0,1]: \qquad f(tx_1 + (1-t)x_2) \leqslant tf(x_1) + (1-t)f(x_2).$

f is called strictly convex if:

(B2) $\forall x_1 \neq x_2 \in X, \forall t \in (0,1): \qquad f(tx_1 + (1-t)x_2) < tf(x_1) + (1-t)f(x_2).$

A function f is said to be (strictly) concave if $-f$ is (strictly) convex.

We can determine the concavity/convexity of a function by determining whether the Hessian is negative or positive semidefinite, as follows.

Proposition 5.2. *Let f be a function of many variables with continuous partial derivatives of first and second order on the convex open set S and denote the Hessian of f at the point x by $H(x)$. Then*

(B3) *f is concave if and only if $H(x)$ is negative semidefinite for all $x \in S$;*

(B4) *if $H(x)$ is negative definite for all $x \in S$ then f is strictly concave;*

(B5) *f is convex if and only if $H(x)$ is positive semidefinite for all $x \in S$;*

(B6) *if $H(x)$ is positive definite for all $x \in S$ then f is strictly convex.*

The concept of strong convexity extends and parametrizes the notion of strict convexity. A strongly convex function is also strictly convex, but not vice versa.

A differentiable function f is called strongly convex with parameter $m > 0$ if the following inequality holds for all points x, y in its domain:

(B7) $(\nabla f(x) - \nabla f(y))^T(x - y) \geqslant m\|x - y\|_2^2$.

A function f is strongly convex with parameter m if and only if the function $x \mapsto f(x) - \frac{m}{2}\|x\|^2$ is convex.

If the function f is twice continuously differentiable, then f is strongly convex with parameter m if and only if $\nabla^2 f(x) \succeq mI$ for all x in the domain, where I is the identity and $\nabla^2 f$ is the Hessian matrix, and the inequality $\succeq$ means that

(B8) $\nabla^2 f(x) - mI$ is positive semi-definite.

If the domain is just an interval $I = (a, b)$ in the real line $\mathbb{R}$, then $\nabla^2 f(x)$ is just the second derivative $f''(x)$, so the condition becomes $f''(x) \geqslant m$, $x \in I$. In particular if $m = 0$ it means that $f''(x) \geqslant 0$), which implies the function is convex, and perhaps strictly convex, but not strongly convex if $f''(x_0) = 0$ for some $x_0 \in I$.

The condition (B8) is equivalent to requiring that

(B'8): the minimum eigenvalue of $\nabla^2 f(x)$ be at least m for all x.

If $m = 0$, then this means the Hessian is positive semidefinite.

Assuming still that the function is twice continuously differentiable, one can show that the lower bound of $\nabla^2 f(x)$ implies that it is strongly convex. Start by using Taylor's Theorem:

$$f(y) = f(x) + \nabla f(x)^T(y - x) + \frac{1}{2}(y - x)^T \nabla^2 f(z)(y - x)$$

for some (unknown) $z \in \{tx + (1 - t)y : t \in [0, 1]\}$. Then by the assumption about the eigenvalues (B'8),

$$(y - x)^T \nabla^2 f(z)(y - x) \geqslant m(y - x)^T(y - x)$$

and therefore

(B9) $f(y) - f(x) + \nabla f(x)^T(x - y) \geqslant \frac{m}{2}|y - x|^2$.

If x and y change their roles in (B9), we find

(B'9) $f(x) - f(y) - \nabla f(y)^T(x - y) \geqslant \frac{m}{2}|y - x|^2$.

Hence, summing (B9) and (B'9), we recover the second strong convexity equation (B7) above.

The trace of a matrix is the sum of the (complex) eigenvalues, and it is invariant with respect to a change of basis. If A is a square n-by-n matrix with real or complex entries and if $\lambda_1, ..., \lambda_n$ are the eigenvalues of A (listed according to their algebraic multiplicities), then

$$\operatorname{tr}(A) = \sum_i \lambda_i.$$

This follows from the fact that A is always similar to its Jordan form, an upper triangular matrix having $\lambda_1, ..., \lambda_n$ on the main diagonal. In contrast, the determinant of A is the product of its eigenvalues; i.e.,

$$\det(A) = \prod_i \lambda_i.$$

Geometrically, the trace can be interpreted as the infinitesimal change in volume (as the derivative of the determinant), which is made precise in Jacobi's formula.

Proposition 5.3. *If u is convex,then dT has nonnegative eigenvalues and*

$$(det\, dT)^{1/n} \leqslant \frac{divT}{n}.$$

If u is convex and $T = \nabla u$, then by Proposition 5.2, dT has nonnegative eigenvalues. Hence, since $divT = \operatorname{tr}(T) = \sum \lambda_i$, we get

$$(det\, dT)^{1/n} = (\prod \lambda_i)^{1/n} \leqslant \frac{\sum \lambda_i}{n} = \frac{divT}{n}.$$

We denote $R^{n-1}_k = \{x \in R^n : x_k = 0\}$. The projection P_k, given by $P_k x = x - x_k e_k$, is the orthogonal projection of R^n onto R^{n-1}_k.

We now define ACL propery.

Let $I = \{x \in \mathbb{R}^n : a_k \leqslant x_k \leqslant b_k\}$ be a closed n-interval.

A mapping $f : I \to \mathbb{R}^m$ is said to be absolutely continuous on lines (ACL) if f is continuous and if f is absolutely continuous on almost every line segment in I, parallel to the coordinate axes.

More precisely, if E_k is the set of all $x \in P_k I$ such that the functions $t \to u(x + te_k)$ is not absolutely continuous on $[a_k, b_k]$, then $m_{n-1}(E_k) = 0$ for $1 \leqslant k \leqslant n$.

If Ω is an open set in $\mathbb{R}^n$, a mapping $f : \Omega \to \mathbb{R}^m$ is ACL (absolutely continuous on lines) if $f|I$ is ACL for every closed interval $I \subset \Omega$.

If $f : \Omega \to \mathbb{R}$ is continuous we say that $f \in W^{1,p}$ if f is ACL and $D_k f \in L^p$.

Absolutely Continuous on Lines (ACL) characterization of Sobolev functions

Let Ω be an open set in $\mathbb{R}^n$ and $1 \leqslant p \leqslant \infty$. If a function is in $W^{1,p}(\Omega)$, then, possibly after modifying the function on a set of measure zero, the restriction to almost every line parallel to the coordinate directions in $\mathbb{R}^n$ is absolutely continuous; what's more, the classical derivative along the lines that are parallel to the coordinate directions are in $L^p(\Omega)$. Conversely, if the restriction of f to almost every line parallel to the coordinate directions is absolutely continuous, then the pointwise gradient ∇f exists almost everywhere, and f is in $W^{1,p}(\Omega)$ provided f and $|\nabla f|$ are both in $L^p(\Omega)$. In particular, in this case the weak partial derivatives of f and pointwise partial derivatives of f agree almost everywhere.

5.5. *Appendix 2, Isoperimetry for Euclidean polyhedra*

In this section we follow [5]. The polytopes are, by definition, the convex envelopes of finite sets of points of an affine space. When this space is of dimension 2 (a plane), we speak of polygons; if the dimension is 3, we speak of polyhedra, and from then on- or from the very beginning-of polytopes. We are thus dealing with objects that are simplest after triangles. Now a detailed study of polyhedra is very recent. If we exclude the fundamental book of Steinitz from 1934 and his papers from between 1906 and 1928, we find practically nothing on polyhedra before the 1960s. We can read an interesting analysis of Steinitz's book in Tucker (1935-2000), but even though the analysis is very enthusiastic, the polyhedra are qualified as Steinitz's „hobby".

For each polyhedron with f faces, we always have

$$A^3/V^2 \geqslant 54(f-2)(4\sin^2\omega_f - 1)\tan^2\omega_f, \tag{34}$$

where $\omega_f = \frac{\pi f}{6(f-2)}$, equality holding only for the regular tetrahedron, cube and dodecahedron. The proof of (VIII.7.1) is difficult and was preceded by several incomplete proofs. But it is also interesting to know that for $f = 8$ and $f = 20$ we can do better,for A^3/V^2, than the value for the regular octahedron and icosahedron. Still with a fixed number of faces, we know since Minkowski (1897) that there always exists at least one optimal polyhedron; Lindelöf (1869) showed that such an optimal polyhedron is always circumscribed about a sphere and that, moreover, the faces touch this sphere at their centers of gravity. Conjecture: for all polyhedra with v vertices we have

$$A^3/V^2 \geqslant \frac{27\sqrt{3}}{2}(v-2)(3\tan^2\omega_v - 1)\tan^2\omega_v, \tag{35}$$

where ω is defined as above, equality holding only for regular tetrahedra, octahedra and icosahedra.

We also know that the cube and dodecahedron are not the best for $v = 8$ and $v = 20$. Finally, in the case where the number a of edges is fixed, we know since (Steinitz, 1927) that there exists at least one polyhedron realizing the minimum and since Fejes Tóth (1948) that it is always simplicial.

Despite of the appearance of simplicity for polyhedra, the isoperimetric questions are far from being resolved for them. In fact, there are several types of problems, depending on how the combinatorics are imposed, either the number of vertices, or faces or edges. See the Florian report in 1.6 of Gruber and

Wills (1993). A first question of Steiner in 1842: is it the case that, in their combinatorial class, the regular polyhedra are those having the best isoperimetric ratio? It has to do with the ratio A^3/V^2 between the area of the boundary and the enclosed volume. The question is natural in the sense that the regular polyhedra play the role of the regular polygons. Steiner's question has not been completely resolved, although many have attacked it. Contrary to the case of polygons, we can not proceed by compactness, because a limit will have to have a different combinatoric. Another difficulty (see the case of polygons) is the fact that not all the combinatorial types are inscribable (or circumscribable) in a sphere; see the following section. Still another difficulty (see Sect. VII.9) is that polarity, which can make it possible to treat just half the cases, does not respect volumes. And above all we lack a conceptual tool, even imagining that any exist. It is of course possible that one day we will be able to reduce isoperi metric problems to a computer program; but here is where we are today, to the best knowledge of the author. Steiners conjecture is true for the combinatorial types of the regular polyhedra, with the exception of the icosahedron, for which the problem still remains open. For the tetrahedron, it is classic (the proof is left to readers); for the octahedron Steiner proved it as early as 1842, by symmetrization. For the cube and the octahedron, its a consequence of a general result of Fejes Tóth (1948). To find the convex polyhedra in Euclidean 3-space $\mathbb{R}^3$, with a given number of faces and with minimal isoperimetric quotient, is a centuries old question of geometry. We know from Marcel Berger's book [5] that it is not yet established which polyhedron in $\mathbb{R}^3$ on 8 vertices achieves the optimal isoperimetric ratio A^3/V^2, where A is the surface area and V the volume. Berger says "We also know that the cube is not the best for $v = 8$" (where v is the number of vertices).

Let P and P' be two polyhedra (convex as usual) and f a mapping between their boundaries that preserves combinatorics, i.e. sends faces onto faces while respecting incidence relations. Suppose that, restricted to each face, f is a Euclidean isometry. Then there exists an isometry f^* of all space such that f is the restriction of f^* to the boundary of P.

6. Appendix 3, Abel summability and Tauber's theorem

Abel summability is a generalized convergence criterion for power series. It extends the usual definition of the sum of a series, and gives a way of summing up certain divergent series. Let us start with a series $\sum_{n=0}^{\infty} a_n$, convergent or not, and use that series to define a power series

$$f(r) = \sum_{n=0}^{\infty} a_n r^n.$$

Note that for $|r| < 1$ the summability of $f(r)$ is easier to achieve than the summability of the original series. Starting with this observation we say that the series $\sum a_n$ is *Abel summable* if the defining series for $f(r)$ is convergent for all $|r| < 1$, and if $f(r)$ converges to some limit L as $r \to 1^-$. If this is so, we shall say that $\sum a_n$ Abel converges to L.

Of course it is important to ask whether an ordinary convergent series is also Abel summable, and whether it converges to the same limit? This is true, and the result is known as Abel's limit theorem, or simply as Abel's theorem.

Let $a = \{a_k : k \geqslant 0\}$ be any sequence of real or complex numbers and let

$$F_a(z) = \sum_{k=0}^{\infty} a_k z^k$$

be the power series with coefficients a. Suppose that the series $\sum_{k=0}^{\infty} a_k$converges. Then

$$\lim_{z\to 1^-} F_a(z) = \sum_{k=0}^{\infty} a_k, \qquad (*)$$

where the variable z is supposed to be real, or, more generally, to lie within any Stolz angle, that is, a region of the open unit disk where $|1 - z| \leqslant M(1 - |z|)$.

Theorem 6.1 (Abel). *Let $\sum_{n=0}^{\infty} a_n$ be a series; let*

$$s_n = a_0 + \cdots + a_k, \quad k \in \mathbb{N},$$

denote the corresponding partial sums; and let $f(r)$ be the corresponding power series defined as above. If $\sum a_n$ is convergent, in the usual sense that the s_n converge to some limit L as $n \to \infty$, then the series is also Abel summable and $f(r) \to L$ as $r \to 1^-$.

The standard example of a divergent series that is nonetheless Abel summable is the alternating series

$$\sum_{n=0}^{\infty} (-1)^n.$$

The corresponding power series is

$$\frac{1}{1+r} = \sum_{n=0}^{\infty} (-1)^n r^n.$$

Since

$$\frac{1}{1+r} \to \frac{1}{2} \quad \text{as} \quad r \to 1^-,$$

this otherwise divergent series Abel converges to $\frac{1}{2}$.

That theorem has its main interest in the case that the power series has radius of convergence exactly 1: if the radius of convergence is greater than one, the convergence of the power series is uniform for r in $[0, 1]$ so that the sum is automatically continuous and it follows directly that the limit as r tends up to 1 is simply the sum of the an. When the radius is 1 the power series will have some singularity on $|z| = 1$; the assertion is that, nonetheless, if the sum of the an exists, it is equal to the limit over r. This therefore fits exactly into the abstract picture.

Abel's theorem is the prototype for a number of other theorems about convergence, which are collectively known in analysis as Abelian theorems. An important class of associated results are the so-called Tauberian theorems. These describe various convergence criteria, and sometimes provide partial converses for the various Abelian theorems.

The general converse to Abel's theorem is false, as the example above illustrates[3)]. However, in the 1890's it is proved the following partial converse.

Theorem 6.2 (Tauber). *Suppose that $\sum a_n$ is an Abel summable series and that $na_n \to 0$ as $n \to \infty$. Then, $\sum_k a_k$ is convergent in the ordinary sense as well.*

We also need some results concerning magnitude of Fourier coefficients.

In applications, it is often useful to know the size of the Fourier coefficient.

If f is an absolutely continuous function,

$$\left|\widehat{f}(n)\right| \leqslant \frac{K}{|n|}$$

for a constant K that only depends on f.

Theorem 6.3 ([57]). *If f is a bounded variation function,*

$$\left|\widehat{f}(n)\right| \leqslant \frac{\mathrm{var}(f)}{2\pi|n|}.$$

[3)]We want the converse to be false; the whole idea is to describe a method of summing certain divergent series!

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