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A Note on the Harmonic Quasiconformal Diffeomorphisms of the Unit Disc

Miljan Knežević[a]

[a] *University of Belgrade, Faculty of Mathematics, Studentski trg 16, 11000 Belgrade, Serbia*

Abstract. We analyze the properties of harmonic quasiconformal mappings and by comparing some suitably chosen conformal metrics defined in the unit disc we obtain some geometrically motivated inequalities for those mappings (see for instance [15, 17, 20]). In particular, we obtain the answers to many questions concerning these classes of functions which are related to the determination of different properties that are of essential importance for validity of the results such as those that generalize famous inequalities of the Schwarz-Pick type. The approach used is geometrical in nature, via analyzing the properties of the Gaussian curvature of the conformal metrics we are dealing with. As a consequence of this approach we give a note to the co-Lipschicity of harmonic quasiconformal self mappings of the unit disc at the origin.

1. Introduction

A sense preserving homeomorphism $f : \Omega \to \Omega'$, where Ω and Ω' are subdomains of the complex plane $\mathbb{C}$, is said to be k-quasiconformal if f is absolutely continuous on almost all horizontal and almost all vertical lines in Ω and if there exists a constant $k \in [0, 1)$ such that $|f_{\bar{z}}(z)| \leqslant k|f_z(z)|$, for almost all $z \in \Omega$. The last requirement is equivalent to $|f_x(z)|^2 + |f_y(z)|^2 \leqslant \left(K + \dfrac{1}{K}\right) J_f(z)$, for almost all $z \in \Omega$, where J_f is the Jacobian of f and $K = \dfrac{1+k}{1-k} \geqslant 1$. Note that many authors, instead of k, sometimes use constant K as the quasiconformal constant of some k-quasiconformal mapping f, where $K = \dfrac{1+k}{1-k}$.

Let $\mathbb{D} = \{z \in \mathbf{C} : |z| < 1\}$ denote the unit disc in the complex plane. O. Martio (see [19]) was the first who considered the importance of the Euclidean Lipschitz and co-Lipschitz character of the harmonic quasiconformal mappings. By using the inequality (see Theorem 2.4)

$$|f_z(z)|^2 + |f_{\bar{z}}(z)|^2 \geqslant \frac{1}{\pi^2},\ z \in \mathbb{D},$$

he noticed that every harmonic quasiconformal diffeomorphism f of $\mathbb{D}$ onto itself is also a co-Lipschitz mapping. On the other hand, in joint work of the author of this note with M. Mateljević (see [17]), by comparing some suitably chosen conformal metrics defined on the unit disc $\mathbb{D}$ and using some versions of

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Email address: kmiljan@matf.bg.ac.rs (Miljan Knežević)

Ahlfors-Schwarz lemma, it has been shown that every such a mapping is also a quasi-isometry with respect to the hyperbolic metric. Moreover, the specific bi-Lipschitz constants were obtained. The same holds for harmonic k-quasiconformal diffeomorphisms of the half plane. Those mappings are (K^{-1}, K) bi-Lipschitz with respect to the Euclidean metric too.

Theorem 1.1 ([17]). *Let f be a k-quasiconformal harmonic diffeomorphism of the unit disc $\mathbb{D}$ onto itself. Then f is a quasi-isometry of the unit disc $\mathbb{D}$ with respect to the hyperbolic metric. In addition, f is a (K^{-1}, K) bi-Lipschitz with respect to the hyperbolic metric.*

Theorem 1.2 ([17]). *Let f be a k-quasiconformal harmonic diffeomorphism of the upper half plane $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ onto itself. Then f is a quasi-isometry of $\mathbb{H}$ with respect to the hyperbolic metric on $\mathbb{H}$. More specifically, f is a (K^{-1}, K) bi-Lipschitz with respect to the hyperbolic metric on $\mathbb{H}$, but also with respect to the Euclidean metric, i.e.*

$$|z_1 - z_2|/K \leqslant |f(z_1) - f(z_2)| \leqslant K|z_1 - z_2|,$$

for all z_1, $z_2 \in \mathbb{H}$. The estimates are sharp.

For further properties and characterizations of the harmonic quasiconformal mappings, that act between various subdomains of the complex plane $\mathbb{C}$, we refer to [6, 11–14, 20, 22, 23, 25, 27, 28].

In [27] the authors D. Partyka and K. Sakan obtained important inequalities that determines the bounds for bi-Lipschitz constants for a harmonic k-quasiconformal diffeomorphism f of the unit disc $\mathbb{D}$ that fixes the origin. They find explicit estimates of those constants by means of the quasiconformal constant k and showed that the estimations are asymptotically sharp as $K = (1+k)/(1-k) \to 1_+$, so f behaves almost like a rotation for sufficiently small k. In particular, for a lower bound for the co-Lipschitz constant they obtained a constant $m_1(K)$, for which obviously holds $m_1(K) \leqslant 1/K$ (see below).

Theorem 1.3 ([27]). *Let f be a k-quasiconformal harmonic mapping of the unit disc $\mathbb{D}$ onto itself such that $f(0) = 0$. Then, for all z_1, $z_2 \in \mathbb{D}$ it follows*

$$|f(z_1) - f(z_2)| \geqslant m_1(K)|z_1 - z_2|,$$

where $m_1(K) = \dfrac{2^{\frac{5}{2}(1-K^2)(3+\frac{1}{K})}}{K^{3K+1}(K^2+K+1)^{3K}}$ *and* $K = (1+k)/(1-k)$.

Later, in [14], by using the properties of Mori's constant $M(K)$ for the class of k-quasiconformal self mappings of the unit disc, $K = (1+k)/(1-k)$, that leave the origin invariant, the authors D. Kalaj and M. Pavlović obtained the better co-Lipschitz constant $m_2(K)$.

Theorem 1.4 ([14]). *Let f be a k-quasiconformal harmonic mapping of the unit disc $\mathbb{D}$ onto itself and let $f(0) = 0$. Then, for all z_1, $z_2 \in \mathbb{D}$ it follows*

$$|f(z_1) - f(z_2)| \geqslant m_2(K)|z_1 - z_2|, \tag{1}$$

where $m_2(K) = \dfrac{2^{2K-2}\Gamma(K-\frac{1}{2})}{\sqrt{\pi}K^2\Gamma(K)(M(K))^{2K}}$ *and* $K = (1+k)/(1-k)$.

Motivated by some questions from the Belgrade complex analysis Seminar, the author of this note, as a corollary of the previous result (see Theorem 3.2), with an additional condition that the mapping f leaves the origin fixed, obtained upper and lower bounds for the quantity $\dfrac{|f(z)|}{|z|}$, $z \in \mathbb{D}$, $z \neq 0$, and then concrete constants of Euclidean bi-Lipschicity of such a mapping at the point $z = 0$. In particular, it was shown that

$$\frac{1}{K}|z| \leqslant |f(z)| \leqslant K|z|,$$

whenever $z \in \mathbb{D}$. Otherwise, it was known that if $m(K) \in (0, 1]$ is the best possible constant for which $m(K)|z| \leqslant |f(z)|$, whenever $z \in \mathbb{D}$, then $\lim_{K \to 1_+} m(K) = 1$ (see [14] and [27]), which also follows directly from our results, since $1/K \leqslant m(K) \leqslant 1$. Hence, it seems to us that it is important to get to know the properties of the constant $m(K)$ (see, for example, [5]).

2. Preliminary results

In the classical theory of functions of one complex variable, the next result takes a substantial place.

Lemma 2.1 (Schwarz). *Let $f : \mathbb{D} \to \mathbb{D}$ be an analytic function and $f(0) = 0$. Then $|f(z)| \leqslant |z|$, $z \in \mathbb{D}$, and $|f'(0)| \leqslant 1$. If $|f(z)| = |z|$, for some $z \neq 0$, or $|f'(0)| = 1$, then $f(z) = e^{i\alpha}z$, for some $\alpha \in [0, 2\pi)$.*

The first impression is that the Schwarz lemma has only analytic character, but Pick gives to the Schwarz lemma a geometric interpretation.

Lemma 2.2 (Schwarz-Pick). *Let f be an analytic function from the unit disc $\mathbb{D}$ into itself. Then f does not increase the corresponding hyperbolic (pseudo-hyperbolic) distances.*

Also, there is a harmonic counterpart of Lemma 2.1 (for details see [9]).

Theorem 2.3 (Heinz). *Let f be a harmonic mapping from the unit disc $\mathbb{D}$ into itself. If $f(0) = 0$, then*

$$|f(z)| \leqslant \frac{4}{\pi} \arctan |z|, \tag{2}$$

for all $z \in \mathbb{D}$.

Theorem 2.4 (Heinz). *Let f be a univalent harmonic mapping of the unit disc $\mathbb{D}$ onto itself and let $f(0) = 0$. Then*

$$|f_x(z)|^2 + |f_y(z)|^2 \geqslant \frac{2}{\pi^2}, \tag{3}$$

for all $z \in \mathbb{D}$. In addition, $|f_z(z)|^2 + |f_{\bar{z}}(z)|^2 \geqslant \dfrac{1}{\pi^2}$, $z \in \mathbb{D}$.

Otherwise, in the classical theory of quasiconformal self mappings of the unit disc, which leave the origin fixed, a result known as Mori's theorem is important (see [18] and [26]).

Theorem 2.5 ([26]). *Let f be a k-quasiconformal mapping from the unit disc $\mathbb{D}$ onto itself. If $f(0) = 0$, then*

$$|f(z_1) - f(z_2)| \leqslant 16|z_1 - z_2|^{\frac{1}{K}},$$

for all $z_1, z_2 \in \mathbb{D}$, where $K = (1 + k)/(1 - k)$. The constant 16 is optimal as an absolute constant, i.e. it cannot be replaced by any smaller bound if the inequality is to hold for all K.

Note that this estimate is meaningful if $|z_1 - z_2|$ is small.

Let $QC_K(\mathbb{D})$, $K = \dfrac{1+k}{1-k} \geqslant 1$, be the family of all k-quasiconformal mappings of the unit disc $\mathbb{D}$ onto itself fixing the origin. For such a family we define the constant

$$M(K) = \sup \left\{ \frac{|f(z_1) - f(z_2)|}{|z_1 - z_2|^{\frac{1}{K}}} : f \in QC_K(\mathbb{D}),\, z_1, z_2 \in \mathbb{D},\, z_1 \neq z_2 \right\},$$

which is sometimes known as Mori's constant. Ahlfors (see [1]) was the first who noticed that this constant is finite, but later, in 1956, Mori proved that $M(K) \leqslant 16$ and that the constant 16 cannot be replaced by a smaller one independent of K. He also conjectured that $M(K) = 16^{1-\frac{1}{K}}$. In [18] it was shown that $M(K) \geqslant 16^{1-\frac{1}{K}}$. On the other hand, there are many upper bounds for that constant. Here we mention only the following upper bound.

Theorem 2.6 ([30]). *If $K \geqslant 1$, then $M(K) \leqslant 16^{1-\frac{1}{K}} \min \left\{ \left(\frac{23}{8}\right)^{1-\frac{1}{K}}, \left(1 + 2^{3-2K}\right)^{\frac{1}{K}} \right\}$.*

For the properties of Mori's constant of the quasiconformal mappings in higher dimensions we refer to [8].

3. Properties of harmonic quasiconformal quasi-isometries

Let f be a harmonic k-quasiconformal mapping of the unit disc $\mathbb{D}$ into itself, with complex dilatation $\mu : z \mapsto \mu(z) = \dfrac{f_{\bar{z}}(z)}{f_z(z)}$, $z \in \mathbb{D}$, and let $\sigma(z) = \lambda(f(z))|f_z(z)|^2$, $z \in \mathbb{D}$, be the density of a conformal metric $ds^2 = \sigma(z)|dz|^2$ on $\mathbb{D}$, where $\lambda(z) = \dfrac{4}{(1-|z|^2)^2}$, $z \in \mathbb{D}$, is the well known hyperbolic metric density on $\mathbb{D}$. In [17] we showed that

$$K_\sigma(z) = -\left(1 + |\mu(z)|^2 + 2\mathrm{Re}\left(\frac{(f(z))^2\,\overline{f_{\bar{z}}(z)}}{f_z(z)}\right)\right),$$

and therefore,

$$-(1+|\mu(z)|)^2 \leqslant K_\sigma(z) \leqslant -(1-|\mu(z)|)^2,$$

for all $z \in \mathbb{D}$, where K_σ denotes the Gaussian curvature of the conformal metric $ds^2 = \sigma(z)|dz|^2$ defined on $\mathbb{D}$. Hence, by applying the Ahlfors-Schwarz lemma, we obtained the following result (see [15, 17]).

Proposition 3.1. *Let f be a harmonic k-quasiconformal mapping from the unit disc $\mathbb{D}$ into itself. Then for any two points z_1 and z_2 in $\mathbb{D}$ we have*

$$d_{\mathbb{D}}(f(z_1), f(z_2)) \leqslant K\, d_{\mathbb{D}}(z_1, z_2),$$

where by $d_{\mathbb{D}}$ we denoted the hyperbolic distance, induced by the hyperbolic metric $ds^2 = \lambda(z)|dz|^2$, on the unit disc $\mathbb{D}$ and $K = (1+k)/(1-k)$.

In order to show the opposite inequality, and hence Theorem 1.1, we had to suppose that the mapping f is onto. Thus, according to the completeness of the conformal metric $ds^2 = \sigma(z)|dz|^2$ and by applying opposite version of Ahlfors-Schwarz lemma (see [21]), we conclude the next proposition to be valid.

Proposition 3.2. *Let f be a harmonic k-quasiconformal mapping from the unit disc $\mathbb{D}$ onto itself. Then for any two points z_1 and z_2 in $\mathbb{D}$ we have*

$$d_{\mathbb{D}}(f(z_1), f(z_2)) \geqslant \frac{1}{K}\, d_{\mathbb{D}}(z_1, z_2),$$

where $d_{\mathbb{D}}$ and K are as before.

As a corollary we get a new result.

Corollary 3.3. *Let $w = f(z)$ be a harmonic k-quasiconformal mapping of the unit disc $\mathbb{D}$ onto itself and let $f(0) = 0$. Then,*

$$|f(z)| \leqslant \frac{(1+|z|)^K - (1-|z|)^K}{(1+|z|)^K + (1-|z|)^K} \tag{4}$$

and

$$|f(z)| \geqslant \frac{(1+|z|)^{\frac{1}{K}} - (1-|z|)^{\frac{1}{K}}}{(1+|z|)^{\frac{1}{K}} + (1-|z|)^{\frac{1}{K}}}, \tag{5}$$

where $K = (1+k)/(1-k)$.

Proof. In this case if $w = f(z)$ and if $d_{\mathbb{D}}$ is corresponding hyperbolic distance function on $\mathbb{D}$, we have

$$d_{\mathbb{D}}(w, 0) \leqslant K d_{\mathbb{D}}(z, 0) \quad \text{and} \quad d_{\mathbb{D}}(w, 0) \geqslant \frac{1}{K} d_{\mathbb{D}}(z, 0),$$

as we proved earlier. Since, $d_{\mathbb{D}}(r, 0) = \ln \dfrac{1+r}{1-r}$, $0 \leqslant r < 1$, we get the estimates. □

To obtain the main result of this note, we have to prove a lemma below (see also [16]).

Lemma 3.4. *Let $a > 0$, $a \neq 1$, be a real number. Then the function*

$$s : x \mapsto s(x) = \frac{(1+x)^a - (1-x)^a}{(1+x)^a + (1-x)^a}, \ 0 < x < 1,$$

is strictly increasing on the interval $(0,1)$. In addition, if $a > 1$, then $s(x) < ax$, for all $0 < x < 1$, whereas, if $0 < a < 1$, then $s(x) > ax$, for all $0 < x < 1$.

Proof. By easy calculation we get

$$s'(x) = \frac{4a(1-x^2)^{a-1}}{((1-x)^a + (1+x)^a)^2} > 0,$$

for all $0 < x < 1$. On the other hand,

$$s''(x) = \frac{8a(1-x^2)^{a-2}\left((1+x)^a(x-a) + (1-x)^a(a+x)\right)}{((1-x)^a + (1+x)^a)^3}, \ 0 < x < 1.$$

Therefore, since for $a > 1$, $\dfrac{a-x}{a+x} > \dfrac{1-x}{1+x} > \left(\dfrac{1-x}{1+x}\right)^a$, for all $0 < x < 1$, we obtain that in this case the function s is concave on $(0,1)$. Otherwise, if $0 < a < 1$, we have $\dfrac{a-x}{a+x} < \dfrac{1-x}{1+x} < \left(\dfrac{1-x}{1+x}\right)^a$, whenever $0 < x < 1$, and the function s is then convex on $(0,1)$. Now, the statement easily follows from the fact that $s'_+(0) = a$, where $s'_+(0)$ is the right derivative of the function s at the point $x = 0$. □

We are now ready to prove the main result.

Theorem 3.5. *Let f be a harmonic k-quasiconformal mapping of the unit disc $\mathbb{D}$ onto itself and let $f(0) = 0$. Then, for all $z \in \mathbb{D}$ we have*

$$\frac{1}{K}|z| \leqslant |f(z)| \leqslant K|z|, \tag{6}$$

where $K = \dfrac{1+k}{1-k}$.

Proof. According to the inequalities (4) and (5), the proof easily follows by applying the Lemma 3.4. □

Corollary 3.6. *If $m(K)$, $K = (1+k)/(1-k) \geqslant 1$, is the best possible constant for which $m(K)|z| \leqslant |f(z)|$, for all $z \in \mathbb{D}$, where f is a harmonic k-quasiconformal self diffeomorphism of the unit disc $\mathbb{D}$, that leaves the origin invariant, then $\lim\limits_{K \to 1_+} m(K) = 1$.*

Remark 3.7. *In [16] we obtained a result, similar to the result stated in the Theorem 3.5, for the class of k-quasiconformal hyperbolic harmonic self diffeomorphisms of the unit disc $\mathbb{D}$, which fix the point $z = 0$. In particular, for a mapping f that belongs to such a class we get $2|z|/(K+1) \leqslant |f(z)| \leqslant \sqrt{K}\,|z|$, for all $z \in \mathbb{D}$, where $K = (1+k)/(1-k)$.*

4. Further results

Lemma 4.1. *The function $\psi(x) = \dfrac{\Gamma'(x)}{\Gamma(x)}$, $x > 0$, is strictly increasing on $(0, +\infty)$, where $\Gamma : x \mapsto \Gamma(x)$, $x > 0$, is the Gamma function.*

Proof. The proof follows immediately from the log-convexity of the Gamma function on $(0, +\infty)$. □

Lemma 4.2. *For all $x \geqslant 1$ the inequality $\dfrac{\Gamma(x - \frac{1}{2})}{\sqrt{\pi}\Gamma(x)} \leqslant 1$ holds.*

Proof. By Lemma 4.1 we have $\left(\frac{\Gamma(x-\frac{1}{2})}{\sqrt{\pi}\Gamma(x)}\right)' = \frac{\Gamma(x-\frac{1}{2})}{\sqrt{\pi}\Gamma(x)}(\psi(x-\frac{1}{2})-\psi(x)) < 0$, for $x > \frac{1}{2}$. Thus, the function $x \mapsto g(x) = \frac{\Gamma(x-\frac{1}{2})}{\sqrt{\pi}\Gamma(x)}$, $x \in (\frac{1}{2}, +\infty)$, is strictly decreasing, so $g(x) = \frac{\Gamma(x-\frac{1}{2})}{\sqrt{\pi}\Gamma(x)} \leqslant g(1) = 1$, for all $x \geqslant 1$. □

Lemma 4.3. $h(x) = -\frac{1}{x} + \frac{4^{3(1-x)}\Gamma(x-\frac{1}{2})}{\sqrt{\pi}x^2\Gamma(x)} \leqslant 0$, *for all* $x \geqslant 1$.

Proof. According to the Lemma 4.2, $h(x) \leqslant \frac{1}{x^2}\left(-x + 4^{3(1-x)}\right) \leqslant 0$, for $x \geqslant 1$. □

Theorem 4.4. *If* $M(K)$, $K \geqslant 1$, *is Mori's constant, then*

$$m_2(K) = \frac{2^{2K-2}\Gamma(K-\frac{1}{2})}{\sqrt{\pi}K^2\Gamma(K)(M(K))^{2K}} \leqslant \frac{1}{K}.$$

Proof. The proof is immediate consequence of the fact that $M(K) \geqslant 16^{1-\frac{1}{K}}$ (see [18]) and of the previous lemma. □

Remark 4.5. *It is obvious that for* $K \geqslant \frac{4}{\pi}$ *the right hand side of the inequality (6) we obtained is certainly not better than the well known inequality (2) stated in the Schwarz Lemma 2.3 for harmonic mappings. On the other hand, for* $1 \leqslant K < \frac{4}{\pi}$, *that inequality gives better bound only for* $|z|$ *small. But, if we put* $z_2 = 0$ *in the inequality (1), we get* $|f(z)| \geqslant \frac{1}{K}|z| \geqslant m_2(K)|z| = \frac{2^{2K-2}\Gamma(K-\frac{1}{2})}{\sqrt{\pi}K^2\Gamma(K)(M(K))^{2K}}|z|$, *for all* $z \in \mathbb{D}$.

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