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On Completing Triangles in Teichmüller Space

Guowu Yao[a]

[a] *Department of Mathematical Sciences, Tsinghua University*
Beijing, 100084, People's Republic of China

Abstract. Let $T(\Delta)$ be the universal Teichmüller space. Three points $[f]$, $[g]$ and $[h]$ in $T(\Delta)$ are called to form a completing triangle if each pair of them has a unique geodesic joining them. Recently, Z. Zhou and L. Liu constructed two Strebel points $[f]$ and $[g]$ such that $[id]$, $[f]$ and $[g]$ form a non-completing triangle. The computation in their construction is lengthy and complicated. In this note, it is shown that their results can be obtained in much simpler a way. Indeed, the current theory of Teichmüller spaces allows us to give more information on triangles in an infinite-dimensional Teichmüller space. Our method is self-contained and applies for general Teichmüller spaces.

1. Introduction

Let S be a Riemann surface of topological type. The Teichmüller space $T(S)$ is the space of equivalence classes of quasiconformal maps f from S to a variable Riemann surface $f(S)$. Two quasiconformal maps f from S to $f(S)$ and g from S to $g(S)$ are equivalent if there is a *conformal* map c from $f(S)$ onto $g(S)$ and a homotopy through quasiconformal maps h_t mapping S onto $g(S)$ such that $h_0 = c \circ f$, $h_1 = g$ and $h_t(p) = c \circ f(p) = g(p)$ for every $t \in [0, 1]$ and every p in the ideal boundary of S. Denote by $[f]$ the Teichmüller equivalence class of f; also sometimes denote the equivalence class by $[\mu]$ where μ is the Beltrami differential of f. The basepoint of $T(S)$ is denoted by $[id]$ where id is the identity map of S.

The constants

$$K(f) = \frac{1 + \|\mu\|_\infty}{1 - \|\mu\|_\infty}, \ K_0([f]) = \inf\{K(g) : \ g \in [f]\}$$

are called the *maximal dilatation* of f and the *extremal maximal dilatation* of $[f]$ respectively. If $K([f])$ is attained by f, then f is called an *extremal* quasiconformal mapping in $[f]$. f is said to be *uniquely extremal* if it is extremal and if

$$K(f) < K(g)$$

holds for any other $g \in [f]$.

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Email address: gwyao@math.tsinghua.edu.cn (Guowu Yao)

The Teichmüller metric between two points $[f]$ and $[g]$ is defined as follows,

$$d([f],[g]) = \frac{1}{2}\inf_{f_1\in[f],g_1\in[g]}\log K(g_1\circ f_1^{-1}).$$

The boundary dilatation of f is defined as

$$H^*(f) = \inf\{K(f|_{S\setminus E}) : \ E \text{ is a compact subset of } S\},$$

where $K(f|_{S\setminus E})$ is the maximal dilatation of $f|_{S\setminus E}$. The boundary dilatation of $[f]$ is defined as

$$H([f]) = \inf\{H^*(g) : \ g \in [f]\}.$$

It is obvious that $H([f]) \le K_0([f])$. Following [4], a point $[f] \in T(S)$ is called a *Strebel point* if $H([f]) < K_0([f])$; otherwise, it is called a *non-Strebel point*.

Denote by $Bel(S)$ the Banach space of Beltrami differentials $\mu = \mu(z)d\bar{z}/dz$ on S with finite L^∞-norm and by $M(S)$ the open unit ball in $Bel(S)$.

Let $Q(S)$ be the Banach space of integrable holomorphic quadratic differentials on S with L^1-norm

$$\|\varphi\| = \iint_S |\varphi(z)|\, dxdy < \infty.$$

In what follows, let $Q^1(S)$ denote the unit sphere of $Q(S)$.

We shall use some geometric terminologies adapted from [1] by Busemann. Let X and Y be metric spaces. An isometry of X into Y is a distance preserving map. A straight line in Y is a (necessarily closed) subset L that is an isometric image of the real line $\mathbb{R}$. A geodesic in Y is an isometric image of a non-trivial compact interval of $\mathbb{R}$. Its endpoints are the images of the endpoints of the interval, and we say that the geodesic joins its endpoints.

It is well known that if $\tau \in T(S)$ is a Strebel point, then there are a unique geodesic joining the basepoint $[id]$ and τ. There are a lot of non-Strebel points $\tau \in T(S)$ such that there are infinitely many geodesics connecting $[id]$ and τ ([4, 9–11]).

Let τ_i $(i = 1, 2, 3)$ be three distinct points in $T(S)$. According to [5] by F. P. Gardiner, they form a "*completing triangle*", if for each pair of them, there is only one geodesic joining them. Otherwise, they form a "*non-completing triangle*".

In [15], Z. Zhou and L. Liu considered the following question in the universal Teichmüller space $T(\Delta)$ where Δ is the unit disk in the complex plane.

QUESTION $\mathscr{A}$. For arbitrarily given two Strebel points τ_1 and τ_2, do the three points $[id]$, τ_1 and τ_2 always form a completing triangle?

In virtue of a result in [8] and by a lengthy and complicated computation, they gave a negative answer to QUESTION $\mathscr{A}$.

Theorem A. *There are two Strebel points τ_1 and τ_2 with $\tau_1 \neq \tau_2$ such that $[id]$, τ_1 and τ_2 do not form a completing triangle.*

Therefore, they asked the second question.

QUESTION $\mathscr{B}$. Suppose both τ_1 and τ_2 are Strebel points. What are the conditions for the three points $[id]$, τ_1 and τ_2 to form a completing triangle?

With respect to QUESTION $\mathscr{B}$, they gave a sufficient condition for three points to form a completing triangle by following theorem.

Theorem B. *Suppose both $[f]$ and $[g_K]$ are Strebel points. Moreover, g_K is a Teichmüller mapping whose Beltrami differential is*

$$\mu_K = \frac{K-1}{K+1}\frac{\overline{\phi}}{|\phi|}, \ (K > 1),$$

where $\phi \neq 0$ is an integrable holomorphic quadratic differential on Δ. If K is sufficiently closed to 1, then the three points $[id]$, $\tau = [f]$ and $\tau_K = [g_K \circ f]$ form a completing triangle.

In the end of [15], they left the following question unsolved.

QUESTION $\mathscr{C}$. For $[f]$ and $[g_K]$ as in Theorem B, whether or not for all $K > 1$, $[f \circ g_K]$ is always a Strebel point?

Indeed, the current knowledge of Teichmüller space theory allows us to give more information on completing triangles in a Teichmüller space. Their results can be obtained in much simpler a way. We will prove more general results by basic techniques so that a slight computation is done. Certainly, the self-contained argument contains a negative answer to QUESTION $\mathscr{C}$ in general.

In what follows, we always assume $dim T(S) = \infty$. The following theorems are a part of our main results.

Theorem 1.1. *For any given Strebel point $[f]$ in $T(S)$, there exists infinitely many Strebel points $[g]$ such that $[id]$, $[f]$ and $[g]$ do not form a completing triangle.*

Theorem 1.2. *Suppose $[f]$ and $[g]$ are two Strebel points in $T(S)$ and there is a unique geodesic joining them. Then there exists a neighborhood B of the basepoint $[id]$ such that for any point $\tau \in B$, the three points τ, $[f]$ and $[g]$ form a completing triangle.*

Theorem 1.3. *Suppose there is a unique geodesic joining two points $[f]$ and $[g]$ in $T(S)$. Then for any $K > (max\{K_0([f]), K_0([g])\})^2$, there exists a Strebel point $[h]$ and a neighborhood $\mathfrak{B}$ of $[h]$ such that $K_0([h]) = K$ and for any point $\tau \in \mathfrak{B}$, the three points τ, $[f]$ and $[g]$ form a completing triangle.*

In the next section, we introduce the basic notion of asymptotic Teichmüller space and prove several lemmas for our use. The relationship among different Strebel and non-Strebel points will be investigated in Section 3 where QUESTION $\mathscr{C}$ is answered negatively in general. The proofs of Theorems 1.1 ∼ 1.3 will be given in the last section.

2. Asymptotic Teichmüller space and some lemmas

The asymptotic Teichmüller space is the space of a larger equivalence classes. The definition of the new equivalence classes is exactly the same as that of Teichmüller equivalence classes with one exception; the word *conformal* is replaced by *asymptotically conformal*. A quasiconformal map f is asymptotically conformal if for every $\epsilon > 0$, there is a compact subset E of S, such that the dilatation of f outside of E is less than $1 + \epsilon$. Accordingly, denote by $[[f]]$ the asymptotic equivalence class of f. There is a canonical projection π from $T(S)$ onto $AT(S)$ defined by $\pi([[f]]) = [f]$.

The boundary dilatation of $[[f]]$ is defined by

$$H([[f]]) = \inf\{H^*(g) : \ g \in [[f]]\}.$$

In fact, the definition of asymptotic equivalence classes implies that $H([[f]]) = H([f])$. The asymptotic Teichmüller distance between two points $[[f]]$ and $[[g]]$ is defined as follows,

$$\bar{d}([[f]], [[g]]) = \frac{1}{2} \inf_{f_1 \in [[f]], g_1 \in [[g]]} \log H(g_1 \circ f_1^{-1}).$$

For more knowledge of asymptotic Teichmüller space, the reader may refer to [2, 3, 6, 12].

Lemma 2.1. *Suppose $[f] \in T(S)$. Then $K_0([f]) = K_0([f^{-1}])$ and $H([f]) = H([f^{-1}])$ where we regard $[f^{-1}]$ as a point in $T(f(S))$. Moreover, f is extremal if and only if f^{-1} is extremal.*

Proof. Let z, w be the local coordinates on S and $f(S)$. Then $w = f(z)$ in the local parameters. For a quasiconformal mapping $g \in [f]$, by the definition of Teichmüller equivalence class there is a conformal mapping c from $f(S)$ onto $g(S)$ such that $g = c \circ f$ on the boundary ∂S. Therefore, we have $\widetilde{g} = g^{-1} \circ c \in [f^{-1}]$. A simple computation yields

$$\mu_{\widetilde{g}}(w) = \mu_{g^{-1} \circ c}(w) = \mu_{g^{-1}}(c(w)) = \mu_{g^{-1}}(\zeta) = \frac{-1}{\omega} \mu_g(z),$$

where $\zeta\,(= c(w))$, $z\,(= g^{-1}(\zeta))$ are the local coordinates on $g(S)$ and S respectively and $\omega = \overline{\partial_z g}/\partial_z g$. It is easy to see that

$$K(g) = K(g^{-1}) = K(\widetilde{g}),\ H^*(g) = H^*(g^{-1}) = H^*(\widetilde{g}).$$

This implies that $K_0([f^{-1}]) \le K_0([f])$ and $H([f^{-1}]) \le H([f])$ when g varies over $[f]$. Symmetrically, it holds that $K_0([f^{-1}]) \ge K_0([f])$ and $H([f^{-1}]) \ge H([f])$. Therefore, we get $K_0([f^{-1}]) = K_0([f])$ and $H([f^{-1}]) = H([f])$. It is clear that f is extremal if and only if f^{-1} is extremal.
□

Lemma 2.2. *$[f]$ is a Strebel point in $T(S)$ if and only if $[f^{-1}]$ is a Strebel point in $T(f(S))$.*

Proof. By the definition of Strebel point, this lemma is a direct consequence of Lemma 2.1. □

Generally, if $[f]$ is a Strebel point and f is extremal, then both f and f^{-1} are Teichmüller mappings. In particular, the Beltrami differential μ of f can be written as

$$\mu(z) = k\frac{\overline{\varphi(z)}}{|\varphi(z)|},\ k \in (0,1),$$

where $\varphi \in Q^1(S)$ is a holomorphic quadratic differential on S; and the Beltrami differential $\mu_{f^{-1}}$ of f^{-1} has the form

$$\mu_{f^{-1}}(w) = k\frac{\overline{\psi(w)}}{|\psi(w)|},$$

where $\psi \in Q^1(f(S))$ is a holomorphic quadratic differential on $f(S)$.

Lemma 2.3. *Suppose $[f]$ is a Strebel point in $T(S)$. Let $[\widetilde{h}] \in T(f(S))$. If $K_0([\widetilde{h}]) < \sqrt{K_0([f])/H([f])}$, then $[\widetilde{h} \circ f]$ is a Strebel point in $T(S)$.*

Proof. Let $g = \widetilde{h} \circ f$. Then $\widetilde{h} = g \circ f^{-1}$. By the distance property, we have

$$\begin{aligned}\frac{1}{2}\log K_0([\widetilde{h}]) &= d([f],[g]) \ge d([f],[id]) - d([g],[id])\\ &= \frac{1}{2}\log K_0([f]) - \frac{1}{2}\log K_0([g]).\end{aligned}$$

Therefore,

$$K_0([g]) \ge \frac{K_0([f])}{K_0([\widetilde{h}])} > \frac{K_0([f])}{\sqrt{K_0([f])/H([f])}} = \sqrt{K_0([f])H([f])}.$$

On the other hand, the property of the asymptotic Teichmüller distance implies

$$\begin{aligned}\frac{1}{2}\log H([[\widetilde{h}]]) &= \bar{d}([[f]],[[g]]) \ge \bar{d}([[g]],[[id]]) - \bar{d}([[f]],[[id]])\\ &= \frac{1}{2}\log H([[g]]) - \frac{1}{2}\log H([[f]]).\end{aligned}$$

Since $H([[f]]) = H([f])$ and $H([[g]]) = H([g])$, we have,

$$\begin{aligned}H([g]) &= H([[g]]) \le H([[f]])H([[\widetilde{h}]]) = H([f])H([\widetilde{h}])\\ &\le H([f])K_0([\widetilde{h}]) < H([f])\sqrt{K_0([f])/H([f])} = \sqrt{K_0([f])H([f])}.\end{aligned}$$

Thus, we have proved that $K_0([g]) > H([g])$ which implies that $[g]$ is a Strebel point.
□

Lemma 2.4. *Suppose* $[f]$ *is a Strebel point in* $T(S)$*. Let* $[h] \in T(S)$*. If* $K_0([h]) < \sqrt{K_0([f])/H([f])}$*, then* $[h \circ f^{-1}]$ *is a Strebel point in* $T(f(S))$*.*

Proof. By Lemma 2.2, $[f^{-1}]$ is a Strebel point in $T(\widetilde{S})$ where $\widetilde{S} = f(S)$. Replace the roles of S and f in Lemma 2.3 by $\widetilde{S}$ and f^{-1} respectively. Combining Lemmas 2.1 and 2.3, it yields that, if $K_0([h]) < \sqrt{K_0([f^{-1}])/H([f^{-1}])} = \sqrt{K_0([f])/H([f])}$, then $[h \circ f^{-1}]$ is a Strebel point in $T(S)$. □

Lemma 2.5. *Let* $[f]$ *and* $[g]$ *be two points in* $T(S)$*. Then the situation of geodesic between* $[f]$ *and* $[g]$ *is identical with that between* $[id]$ *and* $[g \circ f^{-1}]$ *in* $T(f(S))$ *where id is viewed as the identity map of* $f(S)$*.*

Proof. It is easy to see that the map

$$\sigma : T(S) \to T(f(S))$$
$$[h] \to [h \circ f^{-1}],$$

is an isometry between $T(S)$ and $T(f(S))$ with respect to the corresponding Teichmüller metrics. Therefore, the geodesic configuration between $[f]$ and $[g]$ is determined by that between $\sigma([f]) = [id]$ and $\sigma([g]) = [g \circ f^{-1}]$ in $T(f(S))$ and vice versa. □

3. Relationship among Strebel and non-Strebel points

In this section, we discuss the relationship among Strebel and non-Strebel points as well as the geodesic configuration among them. A natural question is ask whether $[g \circ f^{-1}]$ is a Strebel point if $[f]$ and $[g]$ are Strebel points. Also, the question whether $[g \circ f^{-1}]$ is a non-Strebel point if $[f]$ and $[g]$ are non-Strebel points, should be asked. In general, the answers to these two questions are negative.

Theorem 3.1. *Suppose* $[f]$ *is a Strebel point in* $T(S)$ *and* $[\widetilde{h}]$ *is a Strebel point in* $T(f(S))$*. Then* $[\widetilde{h} \circ f]$ *can be either a Strebel point or a non-Strebel point in* $T(S)$*, and the geodesic joining* $[id]$ *and* $[\widetilde{h} \circ f]$ *in* $T(S)$ *can be either unique or non-unique.*

Proof. By Lemma 2.3, when $K_0([\widetilde{h}]) < \sqrt{K_0([f])/H([f])}$, $[\widetilde{h} \circ f]$ is a Strebel point and then the geodesic joining $[id]$ and $[\widetilde{h} \circ f]$ is unique.

To prove that $[\widetilde{h} \circ f]$ can be a non-Strebel point, we need to choose $[\widetilde{h}]$ suitably. Let $g = \widetilde{h} \circ f$ and then $\widetilde{h} = g \circ f^{-1}$. By Lemma 2.2, $[f^{-1}]$ is a Strebel point in $T(\widetilde{S})$ where $\widetilde{S} = f(S)$. Applying Lemma 2.4, we see that, when $K_0([g]) < \sqrt{K_0([f])/H([f])}$, $[g \circ f^{-1}]$ is a Strebel point in $T(\widetilde{S})$. Choose g such that $[g]$ is not a Strebel point and the extremal maximal dilatation $K_0([g])$ satisfies the inequality, then the corresponding $[\widetilde{h}]$ is the desired Strebel point in $T(f(S))$. In such case, the geodesic joining $[id]$ and $[\widetilde{h} \circ f] = [g]$ can be non-unique. In fact, if the Beltrami differential of the extremal quasiconformal mapping in $[g]$ is not of constant modulus, then there are infinitely geodesics connecting $[id]$ and $[g]$ (see [4, 14]).

□

Theorem 3.2. *Suppose* $[f]$ *and* $[g]$ *are two Strebel points in* $T(S)$*. Then* $[g \circ f^{-1}]$ *can be either a Strebel point or a non-Strebel point in* $T(f(S))$*, and the geodesic joining* $[f]$ *and* $[g]$ *can be either unique or non-unique.*

Proof. Let $\widetilde{h} = g \circ f^{-1}$, then $g = \widetilde{h} \circ f$. By the foregoing reason, when $K_0([\widetilde{h}]) < \sqrt{K_0([f])/H([f])}$, $[g] = [\widetilde{h} \circ f]$ is a Strebel point which conversely indicates that $[\widetilde{h}] = [g \circ f^{-1}]$ can be either a Strebel point or a non-Strebel point in $T(f(S))$. Naturally, by Lemma 2.5, the geodesic between $[f]$ and $[g]$ is determined by that between $[id]$ and $[\widetilde{h}]$ in $T(f(S))$ which can be either unique or non-unique. □

The universal Teichmüller space $T(\Delta)$ can be viewed as the set of the equivalence classes $[f]$ of quasiconformal mappings f from Δ onto itself. So, for any quasiconformal mapping from Δ onto itself, there is no difference between $T(\Delta)$ and $T(f(\Delta))$. Both Theorems 3.1 and 3.2 answer QUESTION $\mathscr{C}$ negatively in general.

In [15], the following Proposition is obtained by a complicated construction.

Proposition 3.3. *There exist two non-Strebel points τ_1 and τ_2 in the universal Teichmüller space $T(\Delta)$ such that there is only a geodesic joining them.*

This proposition is actually trivial by the following fact. Let $\mu \in M(S)$ be uniquely extremal with $|\mu| = constant$ $(\neq 0)$ a.e. on S such that $[\mu]$ is a non-Strebel point. Then the geodesic disk $\mathfrak{D} = \{[t\mu/\|\mu\|_\infty] : t \in \Delta\}$ has the property: for any two points $[\mu_1]$ and $[\mu_2]$ in $\mathfrak{D}$, the geodesic connecting them is unique. In particular, the point $[g \circ f^{-1}]$ is a non-Strebel point, where f and g are the quasiconformal mappings with the Beltrami differentials μ_1 and μ_2 respectively. It should be noted here that any three points in $\mathfrak{D}$ form a completing triangle although they are all non-Strebel points.

We now prove the following stronger result in a simple way.

Theorem 3.4. *For any given non-Strebel point $[f]$ in $T(S)\backslash\{[id]\}$, there exists infinitely many non-Strebel points $[g] \in T(S)$ such that $[g \circ f^{-1}]$ is a Strebel point in $T(f(S))$ and hence there is a unique geodesic joining $[f]$ and $[g]$.*

Proof. Let $[f]$ $(\neq [id])$ be a non-Strebel point. Assume that f is an extremal quasiconformal mapping with the Beltrami differential $\mu \in M(S)$. Choose a small disk D in S such that $\overline{D} \subset S$ and a Beltrami differential χ defined on D such that χ and μ are not (Teichmüller) equivalent restricted on D. Moreover, χ can be chosen to satisfy $\|\chi\|_\infty \leq \|\mu\|_\infty$. In local parameter, put

$$\nu(z) = \begin{cases} \mu(z), & z \in S\backslash D, \\ \chi(z), & z \in D. \end{cases}$$

Let g be the quasiconformal mapping with the Beltrami differential ν. Then it is clear that g is extremal and $[g]$ is a non-Strebel point. We show that $[g \circ f^{-1}]$ is a Strebel point. By a simple computation,

$$\mu_{g\circ f^{-1}}(w) = \mu_{g\circ f^{-1}}(f(z)) = \frac{\nu(z) - \mu(z)}{1 - \nu(z)\overline{\mu(z)}} \frac{\partial_z f}{\overline{\partial_z f}}.$$

Therefore, $\mu_{g\circ f^{-1}}(w) = 0$ for $w \in f(S\backslash D)$. Hence we have $H([g\circ f^{-1}]) = 1$. On the other hand, since $[\chi] \neq [\mu|_D]$ in $T(D)$, it yields that $[g \circ f^{-1}] \neq [id]$ in $T(f(S))$ and therefore $K_0([g \circ f^{-1}]) > 1$. This indicates that$[g \circ f^{-1}]$ is a Strebel point and hence there is a unique geodesic joining $[f]$ and $[g]$ by Lemma 2.5. Clearly, such non-Strebel points $[g]$ are innumberable.

□

Theorem 3.5. *Suppose $[f]$ and $[g]$ are two non-Strebel points in $T(S)\backslash\{[id]\}$. Then $[g \circ f^{-1}]$ can be either a Strebel point or a non-Strebel point in $T(f(S))$, and the geodesic joining $[f]$ and $[g]$ can be either unique or non-unique.*

Proof. By Theorem 3.4, we only need to show that $[g \circ f^{-1}]$ can be a non-Strebel point and the geodesic joining $[f]$ and $[g]$ can be non-unique. In fact, this can be done in an easy way. Suppose f is such an extremal quasiconformal mapping that its Beltrami differential μ vanishes on an open subset of S. Let g be the extremal quasiconformal mapping with the Beltrami differential $t\mu$, $t \in \Delta$. Then the rest proof follows standard channels in the Teichmüller theory.

□

4. How to form a completing triangle?

In this section, we are concerned with the condition for three points to form a completing triangle. Since the set of Srebel points is open and dense in $T(S)$ [7, 13], the completing triangles can be obtained with the vertices varying over certain open subsets of $T(S)$. Theorems 1.2 and 1.3 give sufficient conditions for such construction while Theorem 1.1 disproves some cases which reproduces Theorem A. The following are the proofs of them.

Proof of Theorem 1.1. Suppose that $[f]$ is a Strebel point. By Theorem 3.2, we can choose another Strebel point $[g]$ such that $[g \circ f^{-1}]$ is a non-Strebel and the geodesic joining $[f]$ and $[g]$ in not unique. In fact, by Lemma 2.3, if $[\widetilde{h}] \in T(f(S))$ satisfies the condition $K_0([\widetilde{h}]) < \sqrt{K_0([f])/H([f])}$, $[g] = [\widetilde{h \circ f}]$ is a Strebel point. The geodesic configuration between $[f]$ and $[g]$ is identical with that between $[id]$ and $[\widetilde{h}]$ in $T(f(S))$. Therefore, we can get infinitely many Strebel points $[g]$ such that $[id]$, $[f]$ and $[g]$ do not form a completing triangle by the foregoing reason.

Proof of Theorem 1.2. Suppose $[f]$ and $[g]$ are two Strebel points in $T(S)$ and there is a unique geodesic joining them. By Lemma 2.4, when

$$K_0([h]) < K := \min\{\sqrt{K_0([f])/H([f])},\ \sqrt{K_0([g])/H([g])}\},$$

both $[h \circ f^{-1}]$ and $[h \circ g^{-1}]$ are Strebel points (in $T(f(S))$ and $T(g(S))$ respectively). Therefore, if set

$$B = \{\tau \in T(S) :\ d(\tau, [id]) < \frac{1}{2} \log K\},$$

then for any point $\tau \in B$, the three points τ, $[f]$ and $[g]$ form a completing triangle.

Proof of Theorem 1.3. Suppose $[f]$ and $[g]$ are connected by a unique geodesic. For any sufficiently large $K > (max\{K_0([f]), K_0([g])\})^2$, there is a Strebel point $[h]$ such that $K_0([h]) = K$ and

$$\sqrt{K_0([h])/H([h])} = \sqrt{K/H([h])} > M := \max\{K_0([f]), K_0([g])\}. \tag{4. 1}$$

By Lemma 2.4, both $[h \circ f^{-1}]$ and $[h \circ g^{-1}]$ are Strebel points. It is easy to prove that if $\tau \in T(S)$ satisfies

$$d([h], \tau) < \delta := \frac{1}{4} \log \frac{K_0([h])}{H([h])M^2}, \tag{4. 2}$$

then the inequality (4. 1) still holds for τ, that is,

$$\sqrt{K_0(\tau)/H(\tau)} = \sqrt{K/H(\tau)} > M. \tag{4. 3}$$

In fact, by the property of the Teichmüller and asymptotic Teichmüller distances, we have

$$\frac{1}{2} \log K_0(\tau) \geq \frac{1}{2} \log K([h]) - \delta, \tag{4. 4}$$

$$\frac{1}{2} \log H(\tau) \leq \frac{1}{2} \log H([h]) + \delta. \tag{4. 5}$$

The inequality (4. 3) follows from (4. 2), (4. 4) and (4. 5) immediately.

Set

$$\mathfrak{B} = \{\tau \in T(S) :\ d([h], \tau) < \delta\}.$$

Then for any point $\tau \in \mathfrak{B}$, the three points τ, $[f]$ and $[g]$ form a completing triangle. The completes the proof of Theorem 1.3.

By use of Lemma 2.4, one can furher prove that there exist three neighborhoods B_i ($i = 1, 2, 3$) of $[h]$, $[f]$ and $[g]$ separately such that any triangle with vertices in these three neighborhoods respectively is completing. Also, we can even choose $[h]$ such that $K_0([h]) = K$ together with $H([h]) = 1$. Such a point $[h]$ is generally called a T_0-class [6].

At last, we give a corollary of Theorem 1.3.

Corollary 4.1. *Suppose there is a unique geodesic joining* $[id]$ *and* $[f]$ *in* $T(S)$*. Then for any* $K > (K_0([f]))^2$*, there exists a Strebel point* $[h]$ *and a neighborhood* $\mathfrak{B}$ *of* $[h]$ *such that* $K_0([h]) = K$ *and for any point* $\tau \in \mathfrak{B}$*, the three points* $[id]$*,* $[f]$ *and* τ *form a completing triangle.*

Note that $[f]$ or $[g]$ can be non-Strebel points in Theorem 1.3.

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