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Semi-Symmetry of $\delta(2,2)$ Chen Ideal Submanifolds

Anica Pantić[a], Miroslava Petrović-Torgašev[a]

[a]*Faculty of Science, University of Kragujevac, 34000 Kragujevac, Serbia*

Abstract. In this paper we discuss $\delta(2,2)$ Chen ideal submanifolds M^4 in the Euclidean space $\mathbb{E}^6$, and we find the necessary and sufficient conditions under which such a submanifold M^4 is semi-symmetric, i.e. it satisfies the condition $R(X,Y)\cdot R = 0$.

1. Chen ideal submanifolds of Euclidean spaces

Let M^n be an n-dimensional Riemannian submanifold of an $(n+m)$-dimensional *Euclidean space* $\mathbb{E}^{n+m}$, $(n \geq 2, m \geq 1)$ and let g, ∇ and $\widetilde{g}, \widetilde{\nabla}$ be the *Riemannian metric* and the corresponding *Levi-Civita connection* on M^n and on $\mathbb{E}^{n+m}$, respectively. *Tangent vector fields* on M^n will be written as $X, Y, \ldots$ and *normal vector fields* on M^n in $\mathbb{E}^{n+m}$ will be written as $\xi, \eta, \ldots$ The *formulae of Gauss and Weingarten*, concerning the decomposition of the vector fields $\widetilde{\nabla}_X Y$ and $\widetilde{\nabla}_X \xi$, respectively, into their tangential and normal components along M^n in $\mathbb{E}^{n+m}$, are given by $\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y)$ and $\widetilde{\nabla}_X \xi = -A_\xi(X) + \nabla^\perp_X \xi$, respectively, whereby h is the *second fundamental form* and A_ξ is the *shape operator* or *Weingarten map* of M^n with respect to the normal vector field ξ, such that $\widetilde{g}(h(X,Y),\xi) = g(A_\xi(X),Y)$, and $\nabla^\perp$ is the *connection in the normal bundle*. The *mean curvature vector field* $\vec{H}$ is defined by $\vec{H} = \frac{1}{n}\operatorname{tr} h$ and its length $\|\vec{H}\| = H$ is the (extrinsic) *mean curvature* of M^n in $\mathbb{E}^{n+m}$. A submanifold M^n in $\mathbb{E}^{n+m}$ is *totally geodesic* when $h = 0$, *totally umbilical* when $h = g\vec{H}$, *minimal* when $H = 0$ and *pseudo-umbilical* when $\vec{H}$ is an umbilical normal direction [2]. Let $\{E_1, \ldots, E_n, \xi_1, \ldots, \xi_m\}$ be any *adapted orthonormal* local frame field on the submanifold M^n in $\mathbb{E}^{n+m}$, denoted for short also as $\{E_i, \xi_\alpha\}$, whereby $i \in \{1, 2, \ldots, n\}$ and $\alpha \in \{1, 2, \ldots, m\}$. By the *equation of Gauss*, the $(0,4)$ *Riemann-Christoffel curvature tensor* of a submanifold M^n in $\mathbb{E}^{n+m}$ is given by $R(X,Y,Z,W) = \widetilde{g}(h(Y,Z),h(X,W)) - \widetilde{g}(h(X,Z),h(Y,W))$. The $(0,2)$ *Ricci curvature tensor* of M^n is defined by $S(X,Y) = \sum_i R(X,E_i,E_i,Y)$ and the metrically corresponding $(1,1)$ tensor or *Ricci operator* will also be denoted by S : $g(S(X),Y) = S(X,Y)$. The *scalar curvature* of a Riemannian manifold M^n is defined by $\tau = \sum_{i<j} K(E_i \wedge E_j)$ whereby $K(E_i \wedge E_j) = R(E_i,E_j,E_j,E_i)$ is the *sectional curvature* for the plane section $\pi = E_i \wedge E_j$, $(i \neq j)$. By the *equation of Ricci*, the *normal curvature tensor* of a submanifold M^n in $\mathbb{E}^{n+m}$ is given by $R^\perp(X,Y,\xi,\eta) = g([A_\xi, A_\eta](X), Y)$, whereby $[A_\xi, A_\eta] = A_\xi A_\eta - A_\eta A_\xi$, which, as already observed by Cartan [1], implies that *the normal connection is flat or trivial* if and only if *all shape operators A_ξ are simultaneously diagonalisable*.

The function $\inf K : M^n \to \mathbb{R}$ is defined by $(\inf K)(p) = \inf\{K(p,\pi)\,|\,\pi$ is a plane section of $T_p(M^n)\}$. In [3], B.-Y. Chen introduced the $\delta(2)$-*curvature* as $\delta(2) = \tau - \inf K$, which clearly is a *Riemannian scalar invariant*

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Research supported by the project 174012 of the Serbian Ministry of Education, Science and Technological Development
Email addresses: anica.pantic@kg.ac.rs (Anica Pantić), mirapt@kg.ac.rs (Miroslava Petrović-Torgašev)

of the manifold (M^n, g). Later B.-Y. Chen introduced many further new scalar Riemannian invariants, together with $\delta(2)$ called his *delta-curvatures* $\delta(n_1, n_2, \dots, n_k)$; (cfr. [4][5][6][7]). And, for all submanifolds M^n of Euclidean spaces $\mathbb{E}^{n+m}$, or of arbitrary Riemannian ambient spaces $\widetilde{M}^{n+m}$ for that matter, B.-Y. Chen established *optimal pointwise inequalities between these intrinsic delta-curvatures* of M^n *and the squared mean curvature* H^2, and some number determined by the curvature of the ambient space $\widetilde{M}^{n+m}$, which is zero for Euclidean spaces. Such inequalities can be considered as imposing definite lower bounds, basically dictated by these delta-curvatures, to the extrinsic squared mean curvature or surface tension H^2 which results from the kind of shape of the submanifold M^n in the ambient space $\widetilde{M}^{n+m}$. From this point of view, the submanifolds M^n which actually do realise such lower bound for their surface tension are called *Chen ideal submanifolds*.

Here we quote the following result of B.-Y. Chen, (for more details, cfr. [7]).

Theorem A. *For any submanifold M^n in $\mathbb{E}^{n+m}$, $\delta(n_1, \dots, n_k) \le c(n_1, \dots, n_k) H^2$, for all $(n_1, \dots, n_k) \in S(n)$, and equality holds at a point p if and only if, with respect to some suitable adapted orthonormal frame $\{E_i, \xi_\alpha\}$ around p along M^n in $\mathbb{E}^{n+m}$, the shape operators of M^n in $\mathbb{E}^{n+m}$ are given by*

$$A_\alpha = \begin{pmatrix} \begin{matrix} A_1^\alpha & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & A_k^\alpha \end{matrix} & 0 \\ 0 & \mu_\alpha I \end{pmatrix} \qquad (\alpha = 1, \dots, m),$$

whereby I is an identity matrix and $A_1^\alpha, \dots, A_k^\alpha$ are symmetric $n_1 \times n_1, \dots, n_k \times n_k$ matrices, respectively, for which $\operatorname{tr} A_1^\alpha = \dots = \operatorname{tr} A_k^\alpha = \mu_\alpha : M^n \to \mathbb{R}$.

The submanifolds M^n of $\mathbb{E}^{n+m}$ for which the above Chen's inequality at all points of M^n actually is an equality are called *Chen ideal submanifolds* [8][9][10].

The next result is the special case of Theorem A for $k = 1$ and $n_1 = 2$ [3].

Theorem B. *For any submanifold M^n in $\mathbb{E}^{n+m}$, $\delta(2) \le \{[n^2(n-2)]/[2(n-1)]\} H^2$, and equality holds at a point p of M^n if and only if, with respect to some suitable adapted orthonormal frame $\{E_i, \xi_\alpha\}$ around p along M^n in $\mathbb{E}^{n+m}$, the shape operators of M^n in $\mathbb{E}^{n+m}$ are given by*

$$A_\alpha = \begin{pmatrix} A_1^\alpha & 0 \\ 0 & \mu_\alpha I \end{pmatrix} \qquad (\alpha = 1, \dots, m),$$

whereby I is an identity matrix and A_1^α is a symmetric 2×2 *matrix for which* $\operatorname{tr} A_1^\alpha = \mu_\alpha : M^n \to \mathbb{R}$.

Such Chen ideal submanifolds are also called $\delta(2)$ *Chen ideal submanifolds* and the frame $\{E_i, \xi_\alpha\}$ is called *an adapted Chen frame on* $\delta(2)$ *Chen ideal submanifolds* [9][10].

The special case of B.-Y. Chen's Theorem A, for $n_1 = n_2 = 2$ and for $k = 2$ is the following.

Theorem C. *For any submanifold M^n in $\mathbb{E}^{n+m}$, $\delta(2,2) \le c(2,2)H^2$, and the equality holds at a point p, if and only if, with respect to some suitable adapted orthonormal frame $\{E_i, \xi_\alpha\}$ around p on M^n in $\mathbb{E}^{n+m}$ the shape operators of M^n in $\mathbb{E}^{n+m}$ are given by*

$$A_\alpha = \begin{pmatrix} A_1^\alpha & 0 & 0 \\ 0 & A_2^\alpha & 0 \\ 0 & 0 & \mu_\alpha I \end{pmatrix} \qquad (\alpha = 1, \dots, m),$$

whereby I is an identity matrix and A_1^α, A_2^α are symmetric 2×2 *matrices for which* $\operatorname{tr} A_1^\alpha = \operatorname{tr} A_2^\alpha = \mu_\alpha : M^n \to \mathbb{R}$.

The submanifolds M^n of $\mathbb{E}^{n+m}$ for which the above Chen's inequality at all points of M^n actually is an equality are called $\delta(2,2)$ *Chen ideal submanifolds*.

Therefore, a submanifold M^4 in $\mathbb{E}^6$ is a $\delta(2,2)$ *Chen ideal submanifold* if and only if there exists some suitable adapted orthonormal frame $\{E_i, \xi_\alpha\}$, $(i = 1,\ldots,4;\ \alpha = 1,2)$ around p on M^4 in $\mathbb{E}^6$ such that the shape operators of M^4 in $\mathbb{E}^6$ are given by

$$A_\alpha = \begin{pmatrix} b_\alpha & c_\alpha & 0 & 0 \\ c_\alpha & d_\alpha & 0 & 0 \\ 0 & 0 & e_\alpha & f_\alpha \\ 0 & 0 & f_\alpha & g_\alpha \end{pmatrix} \qquad (\alpha = 1,2), \tag{$*$}$$

whereby $b_\alpha + d_\alpha = e_\alpha + g_\alpha = \mu_\alpha : M^4 \to \mathbb{R}$.

2. Semi-symmetric $\delta(2,2)$ Chen ideal submanifolds

For a Riemannian manifold (M^n, g), let R also denote the $(1,1)$ curvature operator $R(X,Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}$, besides the $(0,4)$ curvature tensor, such that, by definition $R(X,Y,Z,W) = g(R(X,Y)Z,W)$, $[.,.]$ denoting the Lie bracket on the differential manifold M^n. By the action of the curvature operator R, working as a derivation on the curvature tensor R, the following $(0,6)$ tensor $R \cdot R$ is obtained:

$$(R\cdot R)(X_1,X_2,X_3,X_4;X,Y) = (R(X,Y)\cdot R)(X_1,X_2,X_3,X_4) =$$

$$= -R(R(X,Y)X_1,X_2,X_3,X_4) - R(X_1,R(X,Y)X_2,X_3,X_4) - R(X_1,X_2,R(X,Y)X_3,X_4) - R(X_1,X_2,X_3,R(X,Y)X_4),$$

whereby X, Y, X_1, X_2, X_3, X_4 are arbitrary tangent vector fields on M^n.

The Riemannian manifolds M^n for which $R \cdot R = 0$ are called *semi-symmetric spaces* or *Szabó-symmetric spaces*. These spaces were classified by Z. Szabó [13] [14]. This condition $R \cdot R = 0$ first appeared as the integrability condition of $\nabla R = 0$ during the study of the *locally symmetric spaces* or *Cartan symmetric spaces*, i.e. of the Riemannian manifolds M for which $\nabla R = 0$ holds, which have been classified by E. Cartan. The locally symmetric or Cartan symmetric spaces constitute a proper subclass of the Szabó-symmetric spaces. As was shown in [11] (see also [12]), the tensor $R \cdot R$ can be geometrically interpreted as giving *the second order measure of the change of the sectional curvatures $K(p,\pi)$ for the tangent 2D-planes π at points p after the parallel transport of π all around infinitesimal coordinate parallelograms in M cornered at p*. Thus the *semi-symmetric spaces* are *the Riemannian manifolds for which all sectional curvatures remain preserved after parallel transport of their planes around infinitesimal coordinate parallelograms in M*.

In [8] the authors classified the semi-symmetric $\delta(2)$ Chen ideal submanifolds as follows.

Theorem D. *A $\delta(2)$ Chen ideal submanifold M^n $(n \geq 3)$ in $\mathbb{E}^{n+m}$ is semi-symmetric if and only if it is minimal (in which case M^n is $(n-2)$-ruled), or M^n is a round hypercone in some totally geodesic subspace $\mathbb{E}^{n+1}$ of $\mathbb{E}^{n+m}$ (including as "degenerate cases" the totally geodesic and the totally umbilical submanifolds).*

In this section we present the necessary and sufficient conditions for $\delta(2,2)$ *Chen ideal submanifolds M^4* in Euclidean space $\mathbb{E}^6$ to be *semi-symmetric*.

Consider a $\delta(2,2)$ Chen ideal submanifold M^4 in $\mathbb{E}^6$. Its Riemann-Christoffel curvature tensor is obtained by inserting the shape operator $(*)$ in the equation of Gauss. Up to the algebraic symmetries of the $(0,4)$ curvature tensor R of such $\delta(2,2)$ Chen ideal submanifolds, all non-zero components of R are the following:

$R_{1212} = \|C\|^2 - \langle B,D\rangle$, $R_{1313} = -\langle B,E\rangle$, $R_{1314} = -\langle B,F\rangle$, $R_{1323} = -\langle C,E\rangle$,

$R_{1324} = -\langle C,F\rangle = R_{1423}$, $R_{1414} = -\langle B,G\rangle$, $R_{1424} = -\langle C,G\rangle$, $R_{2323} = -\langle D,E\rangle$,

$R_{2324} = -\langle D,F\rangle$, $R_{2424} = -\langle D,G\rangle$, $R_{3434} = \|F\|^2 - \langle E,G\rangle$,

where $B,C,D,E,F,G,M \in \mathbb{E}^2$ are defined by $B = (b_1,b_2)$, $C = (c_1,c_2)$, $D = (d_1,d_2)$, $E = (e_1,e_2)$, $F = (f_1,f_2)$, $G = (g_1,g_2)$, $M = (\mu_1,\mu_2)$. Notice that $M = B + D = E + G$. If X is a point of $\mathbb{E}^2$, under the vector X we mean the corresponding radius-vector $\overrightarrow{OX}$. All other components R_{ijkl} are zero, or differ from the previous mostly in sign.

Denote: $R_1 = -R_{1212}$, $R_2 = -R_{1313}$, $R_3 = -R_{1314}$, $R_4 = -R_{1323}$, $R_5 = -R_{1324}$, $R_6 = -R_{1414}$, $R_7 = -R_{1424}$, $R_8 = -R_{2323}$, $R_9 = -R_{2324}$, $R_{10} = -R_{2424}$, $R_{11} = -R_{3434}$. The condition $R \cdot R = 0$ of the semi-symmetry can be expressed in coordinates by a system of equations $(ijklpq) := (R(E_p, E_q) \cdot R)(E_i, E_j, E_k, E_l) = 0$, for all possible combinations $i, j, k, l, p, q = 1, 2, 3, 4$, or equivalently, for all $i < j, k < l, p < q$ $(i, j, k, l, p, q = 1, 2, 3, 4)$. By a straightforward calculations we find that a submanifold $M^4 \subset \mathbb{E}^6$ is semi-symmetric if and only if the components (121313), (131412), (121414), (131334), (132334), (232334), (132312), (132412), (142412), (131434), (132434), (232434), (121314), (121324), (122324), (133423), (143424), (121424), (133414), (122314), (143423), (133424), (233424), (121323) vanish. So we find that the semi-symmetry in this case is characterized by the following system of equations:

(1) $R_1 R_4 = 0$, (2) $R_1 R_5 = 0$, (3) $R_1 R_7 = 0$, (4) $R_3 R_{11} = 0$, (5) $R_5 R_{11} = 0$,

(6) $R_9 R_{11} = 0$, (7) $R_1 (R_2 - R_8) = 0$, (8) $R_1 (R_3 - R_9) = 0$, (9) $R_1 (R_6 - R_{10}) = 0$,

(10) $R_{11} (R_2 - R_6) = 0$, (11) $R_{11} (R_4 - R_7) = 0$, (12) $R_{11} (R_8 - R_{10}) = 0$,

(13) $R_3 (R_4 - R_7) + R_5 (R_6 - R_2) = 0$,

(14) $R_5 (R_4 + R_7) + R_9 (R_1 - R_2) - R_3 R_{10} = 0$,

(15) $R_5 (R_8 - R_{10}) + R_9 (R_7 - R_4) = 0$,

(16) $R_5 (R_8 - R_2) + R_4 (R_3 - R_9) = 0$,

(17) $R_5 (R_6 - R_{10}) + R_7 (R_9 - R_3) = 0$,

(18) $R_5^2 + R_7^2 - R_3 R_9 + R_{10} (R_1 - R_6) = 0$,

(19) $R_6 (R_{11} - R_2) + R_3^2 + R_5^2 - R_4 R_7 = 0$,

(20) $R_3 (R_8 - R_1) - R_5 (R_4 + R_7) + R_6 R_9 = 0$,

(21) $R_4 (R_6 - R_{11}) + R_7 R_8 - R_5 (R_3 + R_9) = 0$,

(22) $R_5 (R_3 + R_9) + R_7 (R_{11} - R_2) - R_4 R_{10} = 0$,

(23) $R_{10} (R_{11} - R_8) + R_5^2 + R_9^2 - R_4 R_7 = 0$,

(24) $R_8 (R_1 - R_2) + R_4^2 + R_5^2 - R_3 R_9 = 0$.

Hence, the property of semi-symmetry is equivalent with solving of the corresponding system of equations (1) – (24).

In the sequel we will distinguish between the following cases: (a) $R_1, R_{11} \neq 0$; (b) $R_1 = 0, R_{11} \neq 0$; (c) $R_1 = R_{11} = 0$. The case $R_{11} = 0, R_1 \neq 0$ is similar to (b) because R_1 and R_{11} are "symmetric", i.e. by the prenumeration of the orthonormal basis $\{E_1, E_2, E_3, E_4\}$ into $\{E_3, E_4, E_1, E_2\}$, R_1 becomes $\widetilde{R}_{11}$, R_{11} becomes $\widetilde{R}_1$, etc.

Next, since the parameter functions (B, C, D, E, F, G) characterize the semi-symmetry of a submanifold $M^4 \subset \mathbb{E}^6$, roughly speaking we can "identify" such 6-tuples of vectors with the corresponding submanifold and call them solutions of the considered system (1) – (24). They characterize the semi-symmetry condition on such Chen ideal submanifolds.

In the sequel, we give some simple assertions about such solutions which can be easily proved, for instance some of them by changing the tangent orthonormal basis vectors $\{E_1, E_2, E_3, E_4\}$ into some other order.

Let $\mathcal{R}$ be the set of all solutions (B, C, D, E, F, G) of the considered system. First, we give some properties of the set $\mathcal{R}$.

Proposition 2.1. *If* $(B, C, D, E, F, G) \in \mathcal{R}$, *then all* $(B, \varepsilon_1 C, D, E, \varepsilon_2 F, G)$ $(\varepsilon_1, \varepsilon_2 = \pm 1)$, (D, C, B, E, F, G), (B, C, D, G, F, E), $(E, F, G, B, C, D) \in \mathcal{R}$.

We give only an instruction for the proof. To prove that $(D, C, B, E, F, G) \in \mathcal{R}$, we will change the orthonormal basis $\{E_1, E_2, E_3, E_4\}$ of the tangent space $T_p(M^4)$ into the basis $\{E_2, E_1, E_3, E_4\}$, and consequently we get that the new parameter functions are: $\widetilde{B} = D$, $\widetilde{C} = C$, $\widetilde{D} = B$, $\widetilde{E} = E$, $\widetilde{F} = F$, $\widetilde{G} = G$, etc. Since the above system remains the same if we replace B and D, we find that $(D, C, B, E, F, G) \in \mathcal{R}$. Similarly, since it remains the same if we replace C by $-C$, we find that $(B, -C, D, E, F, G) \in \mathcal{R}$, too. The remaining part of the proof is similar.

For the solutions described in this proposition, we will say that they are of a similar type.

Proposition 2.2. *A $\delta(2,2)$ Chen ideal submanifold $M^4 \subset \mathbb{E}^6$ is flat if and only if $B = C = D = E = F = G = 0$.*

Next, we will define a particular type of solutions. A 6–tuple (B, C, D, E, F, G) will be called "trivial" if $R_2 = \cdots = R_{10} = 0$. Obviously, any such 6–tuple is a solution of the considered system, and we shall call it – a trivial solution.

The above condition means that each of B, C, D is orthogonal to each of E, F, G, i.e. that $B, C, D \perp E, F, G$. As is easily seen, for any such solution we have that $M = 0$, i.e. $D = -B$ and $G = -E$. So, for instance, any flat submanifold $M^4 \subset \mathbb{E}^6$ is a trivial solution of the system. It is also easy to prove the following facts.

(1°) Any trivial solution in the case (a) is of the form $(\alpha P, \beta P, -\alpha P, \gamma Q, \delta Q, -\gamma Q)$, where $\{P, Q\}$ is an arbitrary orthonormal basis in $\mathbb{E}^2$, $\alpha^2 + \beta^2 > 0$ and $\gamma^2 + \delta^2 > 0$.

(2°) Any trivial solution in the case (b) is of the form $(0, 0, 0, E, F, -E)$ where $E \neq 0$ or $F \neq 0$.

(3°) Any trivial solution in the case (c) is a flat submanifold.

Next, we will discuss the general system (1) – (24). In solving the considered system, we will distinguish between the cases (a), (b) and (c).

Case (a): $R_1, R_{11} \neq 0$. The whole system is reduced to the following equations:

$R_3 = R_4 = R_5 = R_7 = R_9 = 0$, $R_2 = R_6 = R_8 = R_{10}$, $\quad R_2 (R_1 - R_2) = 0$, $\quad R_2 (R_2 - R_{11}) = 0$.

First, assume that $R_2 \neq 0$, and consequently $R_1 = R_2 = R_{11} \neq 0$. Therefore, all $B, D, E, G \neq 0$. Next, it can be also seen that $C, F \neq 0$ leads to contradiction, and $C = 0, F \neq 0$ (and the similar case $C \neq 0, F = 0$) do the same. Hence, $C = F = 0$. Since now $M = 0$ also leads to the contradiction, we obtain that $M \neq 0$. But then, by decomposing $B = \frac{M}{2} + \frac{P}{2}, D = \frac{M}{2} - \frac{P}{2}, E = \frac{M}{2} + \frac{Q}{2}, G = \frac{M}{2} - \frac{Q}{2}$, it can be easily found that $B = D = E = G = \frac{M}{2}$ $(M \neq 0)$.

Next, assume that $R_2 = 0$, thus $R_i = 0$ $(i = 2, \ldots, 10)$. Then we get the trivial solution (case (a)), and its structure is known. So, we get the following proposition.

Proposition 2.3. *In case* (a) *there are two series of solutions:*

(1°) $(\frac{M}{2}, 0, \frac{M}{2}, \frac{M}{2}, 0, \frac{M}{2})$, *whereby $M \neq 0$ is an arbitrary vector in $\mathbb{E}^2$.*

(2°) *The series of trivial solutions $(\alpha P, \beta P, -\alpha P, \gamma Q, \delta Q, -\gamma Q)$ where $\{P, Q\}$ is an arbitrary orthonormal basis of the plane $\mathbb{E}^2$, $\alpha^2 + \beta^2 > 0$ and $\gamma^2 + \delta^2 > 0$.*

In the case (1°) $R_1 = \langle B, D\rangle - ||C||^2 = \frac{||M||^2}{4}$, $R_{11} = \langle E, G\rangle - ||F||^2 = \frac{||M||^2}{4}$.

Case (b): $R_1 = 0, R_{11} \neq 0$, i.e. $||C||^2 = \langle B, D\rangle, ||F||^2 \neq \langle E, G\rangle$. By the straightforward calculations, and eliminating some other subcases which lead to contradictions, we find that the only possible solution of the system in this case is $(0, 0, 0, E, F, -E)$, where $||E|| + ||F|| > 0$. Thus we have the following.

Proposition 2.4. *In case* (b) *the only possible solution of the considered system of equations is the trivial solution* $(0, 0, 0, E, F, -E)$, *where $||E|| + ||F|| > 0$, i.e. $E \neq 0$ or $F \neq 0$.*

Case (c): $R_1 = R_{11} = 0$, i.e. $||C||^2 = \langle B, D\rangle$, $||F||^2 = \langle E, G\rangle$. Obviously, the whole considered system is reduced to equations (13) – (24) only.

Next it is easy to see the following.

Proposition 2.5. *The only solution in the case* (c) *with the property $M = 0$ is* $(0, 0, 0, 0, 0, 0)$, *i.e. a flat submanifold M^4 in $\mathbb{E}^6$.*

Hence, in the sequel, we search only for the solutions with the property $M \neq 0$.

Lemma 2.1. *If in the case* (c) *$(M \neq 0)$ we have $\langle M, F\rangle^2 = \langle M, E\rangle\langle M, G\rangle$, then there are $\alpha \in [0, 1]$ and $\beta \in \mathbb{R}$, such that $E = \alpha M$, $G = (1 - \alpha)M$, $F = \beta M, \beta^2 = \alpha(1 - \alpha)$.*

This can be easily seen by using an orthogonal basis $\{M, N\}$ in $\mathbb{E}^2$, then by decomposing vectors E, F, G in this basis, and using the equation $||F||^2 = \langle E, G\rangle$ and the equation from the Lemma.

Next, we discuss the particular case $B = 0$.

Proposition 2.6. *If, in the case* (c) *($M \neq 0$) $B = 0$ then $C = 0, D = M$, and there are $\alpha \in [0,1]$ and $\beta \in \mathbb{R}$, such that $E = \alpha M$, $G = (1-\alpha)M$, $F = \beta M, \beta^2 = \alpha(1-\alpha)$.*

Proof. If $B = 0$, then obviously $C = 0, D = M$, and the entire considered system reduces only to one equation (23) $\langle D, F\rangle^2 = \langle D, E\rangle\langle D, G\rangle$. Now, it is enough to use the previous Lemma. □

A quite analogous result is obtained if $D = 0$, or $E = 0$, or $G = 0$. Therefore, in the sequel we will assume that all $B, D, E, G \neq 0$.

Next, we will distinguish between the subcases $F = 0$ and $F \neq 0$.

Subcase $F = 0$ ($M, B, D, E, G \neq 0$). Then $E \perp G$, and the entire system reduces to equations (18), (19), (21), (22), (23), (24). Next, we distinguish between the following possibilities: (1.1°) $R_4 = R_7 = 0$; (1.2°) $R_4 = 0, R_7 \neq 0$; (1.3°) $R_4, R_7 \neq 0$. The case $R_4 \neq 0, R_7 = 0$ is analogous to (1.2°).

It is easy to see that in case (1.1°) $C = 0$, and we get two solutions of the system: $(E, 0, G, E, 0, G)$, $(G, 0, E, E, 0, G)$, $(E, G \neq 0, E \perp G)$, which are of the same type. The case (1.2°) leads to contradiction $E = 0$.

Finally, in the subcase (1.3°), the entire system reduces to three equations (24), (18), (19), which read:

$$\langle C, E\rangle^2 = \langle B, E\rangle\langle D, E\rangle \neq 0\,, \qquad \langle C, G\rangle^2 = \langle B, G\rangle\langle D, G\rangle \neq 0, \quad \langle B, E\rangle\langle B, G\rangle + \langle C, E\rangle\langle C, G\rangle = 0.$$

Decomposing $B = pE + qG$, $D = (1-p)E + (1-q)G$ ($p, q \in \mathbb{R}$), $C = xE + yG$ ($x, y \neq 0$), we easily find $x^2 = p(1-p)$, $y^2 = q(1-q)$, where $p, q \in (0, 1)$. Next we also find that $q = 1 - p$, and hence $x = \epsilon\sqrt{p(1-p)}$, $y = -x$ ($\epsilon = \pm 1$), so that $B = pE + (1-p)G$, $D = (1-p)E + pG$, $C = \epsilon\sqrt{p(1-p)}\{E - G\}$ ($0 < p < 1, \epsilon = \pm 1$). Geometrically, this means that both B, D lie on the line EG (between E and G), and are symmetrically situated with respect to the point $M/2 = E/2 + G/2$. The equality $C = \pm\sqrt{p(1-p)}\{E - G\}$ means that vector C is parallel with $E - G$ and $\|C\|^2 = \langle B, D\rangle$.

Also notice that previous two series of solutions in the case (1.1°) can be obtained from (1.3°) taking $p = 1$ and $p = 0$.

Thus we have the following.

Proposition 2.7. *If in the case* (c) *($M, B, D, E, G \neq 0$) $F = 0$, then $E \perp G$ and there is a $p \in [0,1]$ such that $B = pE + (1-p)G$, $D = (1-p)E + pG$, $C = \pm\sqrt{p(1-p)}\{E - G\}$.*

Similarly holds if $C = 0$. Hence, in the sequel, we shall assume that both $C, F \neq 0$ (together with $M, B, D, E, G \neq 0$). First, we will discuss a particular case when $B, C, D \neq 0$ and are collinear with M.

Proposition 2.8. *If $B = pM$, $D = (1-p)M$, $C = qM$ ($0 < p < 1, q = \pm\sqrt{p(1-p)}$), then E, F, G are also collinear with M, i.e. $E = \rho M$, $G = (1-\rho)M$, $F = \nu M$ ($0 < \rho < 1, \nu = \pm\sqrt{\rho(1-\rho)}$).*

Proof. Under the above assumptions the whole system reduces to only one equation (19) $\langle M, F\rangle^2 = \langle M, E\rangle\langle M, G\rangle$. Then it is enough to apply Lemma 2.1. □

A quite similar result holds if $E, F, G, M \neq 0$ and E, F, G are collinear with M. Hence, in the sequel we will assume that B, C, D are not collinear (thus not collinear with M), and that E, F, G are also not collinear. Next observe that equation (20) can be replaced with equation

(20′) $R_3(R_8 - R_{10}) + R_9(R_6 - R_2) = 0$,

obtained by subtracting equations (14) and (20), and equation (22) can be replaced with equation

(22′) $R_7(R_2 - R_8) + R_4(R_{10} - R_6) = 0$,

obtained by subtracting equations (21) and (22). Hence, the considered system becomes (13), (14), (15), (16), (17), (18), (19), (20′), (21), (22′), (23), (24). Next, we discuss equations (13), (15), (20′), as well as equations (16), (17), (22′). First notice that equation (13) geometrically means that $E - G \perp \langle F, B\rangle C - \langle F, C\rangle B$. Since $F \perp \langle F, B\rangle C - \langle F, C\rangle B$, we find that equation (13) $\Leftrightarrow$ $E - G = \lambda F$ ($\lambda \in \mathbb{R}$) $\vee$ $B = \beta C$ ($\beta \neq 0$). Similarly, equation (15) $\Leftrightarrow$ $E - G = \lambda F$ ($\lambda \in \mathbb{R}$) $\vee$ $D = \delta C$ ($\delta \neq 0$), and equation (20′) $\Leftrightarrow$ $E - G = \lambda F$ ($\lambda \in \mathbb{R}$) $\vee$ $B = \gamma D$ ($\gamma \neq 0$). Therefore, obviously (13), (15) $\Rightarrow$ (20′), so the equation (20′) can be removed from the above system. Analogously, equation (16) $\Leftrightarrow$ $B - D = \mu C$ ($\mu \in \mathbb{R}$) $\vee$ $E = \varepsilon F$ ($\varepsilon \neq 0$), equation (17) $\Leftrightarrow$ $B - D = \mu C$

$(\mu \in \mathbb{R}) \vee G = \omega F$ $(\omega \neq 0)$, and equation (22′) $\Leftrightarrow B - D = \mu C$ $(\mu \in \mathbb{R}) \vee G = \nu E$ $(\nu \neq 0)$. Hence, equations (16) and (17) $\Rightarrow$ (22′), so that equation (22′) can be removed from the considered system. Thus, the entire system becomes: (13), (14), (15), (16), (17), (18), (19), (21), (23), (24). Besides, (13) $\wedge$ (15) $\Leftrightarrow E - G = \lambda F$ $(\lambda \in \mathbb{R}) \vee B = \beta C, D = \delta C$ $(\beta, \delta \neq 0)$, and equations (16) $\wedge$ (17) $\Leftrightarrow B - D = \mu C$ $(\mu \in \mathbb{R}) \vee E = \varepsilon F, G = \omega F$ $(\varepsilon, \omega \neq 0)$.

If $B = \beta C$, $D = \delta C$ $(\beta, \delta \neq 0)$, then all vectors B, C, D are collinear (and collinear with M), so we have the previously discussed case from Proposition 2.8. Similarly holds if $E = \varepsilon F$, $G = \omega F$ $(\varepsilon, \omega \neq 0)$. Therefore, in the sequel we will assume that $B, C, D \neq 0$ are not collinear, and $E, F, G \neq 0$ so do. Then by equations (13), (15), (16), (17) we have

(25) $\quad E - G = \lambda F\ (\lambda \in \mathbb{R}), \quad B - D = \mu C\ (\mu \in \mathbb{R}),$

and it remains to discuss the system of equations (14), (18), (19), (21), (23), (24).

Next we make the following decompositions:

$$E = \tfrac{M}{2} + \tfrac{E-G}{2} = \tfrac{M}{2} + \tfrac{\lambda}{2}F, \quad G = \tfrac{M}{2} - \tfrac{E-G}{2} = \tfrac{M}{2} - \tfrac{\lambda}{2}F,$$

$$B = \tfrac{M}{2} + \tfrac{B-D}{2} = \tfrac{M}{2} + \tfrac{\mu}{2}C, \quad D = \tfrac{M}{2} - \tfrac{B-D}{2} = \tfrac{M}{2} - \tfrac{\mu}{2}C \quad (M \neq 0, \lambda, \mu \in \mathbb{R}).$$

Since $\|C\|^2 = \langle B, D\rangle$, we easily find that $\|C\| = \|M\|/\sqrt{4+\mu^2}$. Similarly, $\|F\| = \|M\|/\sqrt{4+\lambda^2}$. Besides, we have that M, C are not collinear, and M, F so do. By straightforward calculation one can see that equations (14), (21), (18), (19), (23), (24) can be respectively put into the form:

$$\text{(14.1)} \qquad \langle C, F\rangle\langle C, M\rangle = \tfrac{\|M\|^2}{4+\mu^2}\langle M, F\rangle,$$

$$\text{(21.1)} \qquad \langle C, F\rangle\langle M, F\rangle = \tfrac{\|M\|^2}{4+\lambda^2}\langle M, C\rangle,$$

$$\text{(18.1)} \qquad (4+\lambda^2)(4+\mu^2)\langle C, F\rangle^2 + (4+\mu^2)\langle C, M\rangle^2 - (4+\lambda^2)\langle M, F\rangle^2 +$$
$$+2\lambda\|M\|^2\langle M, F\rangle - 2\lambda(4+\mu^2)\langle C, M\rangle\langle C, F\rangle - \|M\|^4 = 0,$$

$$\text{(19.1)} \qquad (4+\lambda^2)(4+\mu^2)\langle C, F\rangle^2 + (4+\lambda^2)\langle M, F\rangle^2 - (4+\mu^2)\langle C, M\rangle^2$$
$$-2\mu\|M\|^2\langle C, M\rangle + 2\mu(4+\lambda^2)\langle C, F\rangle\langle M, F\rangle - \|M\|^4 = 0,$$

$$\text{(23.1)} \qquad (4+\lambda^2)(4+\mu^2)\langle C, F\rangle^2 + (4+\lambda^2)\langle M, F\rangle^2 - (4+\mu^2)\langle C, M\rangle^2$$
$$+2\mu\|M\|^2\langle C, M\rangle - 2\mu(4+\lambda^2)\langle C, F\rangle\langle M, F\rangle - \|M\|^4 = 0,$$

$$\text{(24.1)} \qquad (4+\lambda^2)(4+\mu^2)\langle C, F\rangle^2 + (4+\mu^2)\langle C, M\rangle^2 - (4+\lambda^2)\langle M, F\rangle^2$$
$$-2\lambda\|M\|^2\langle M, F\rangle + 2\lambda(4+\mu^2)\langle C, M\rangle\langle C, F\rangle - \|M\|^4 = 0.$$

Next, define the unit vectors $C_0 = \frac{\sqrt{4+\mu^2}}{\|M\|}C$, $F_0 = \frac{\sqrt{4+\lambda^2}}{\|M\|}F$. Then equations (14.1) and (21.1) can be transformed into

(14.2) $\quad \langle C_0, F_0\rangle\langle C_0, M_0\rangle = \langle M_0, F_0\rangle, \quad$ (21.2) $\quad \langle C_0, F_0\rangle\langle M_0, F_0\rangle = \langle M_0, C_0\rangle.$

By (14.2) and (21.2) we find that equations (18.1) and (24.1) become:

(18.2) $\quad \langle C_0, F_0\rangle^2 + \langle C_0, M_0\rangle^2 - \langle M_0, F_0\rangle^2 = 1,$

and equations (19.1) and (23.1) become:

(19.2) $\quad \langle C_0, F_0\rangle^2 + \langle M_0, F_0\rangle^2 - \langle C_0, M_0\rangle^2 = 1.$

So the whole system is reduced to equations (14.2), (21.2), (18.2), (19.2). By equations (18.2) and (19.2) we find that $\langle C_0, F_0\rangle^2 = 1, \langle C_0, M_0\rangle = \pm\langle M_0, F_0\rangle$. By $\langle C_0, F_0\rangle = \rho = \pm 1$, we have that $F_0 = \rho C_0$.

If conversely, $F_0 = \rho C_0$ $(\rho = \pm 1)$, then all equations (14.2), (21.2), (18.2), (19.2) are true, so the entire system holds true. Therefore, a $\delta(2,2)$ Chen ideal submanifold M^4 in $\mathbb{E}^6$ is semi-symmetric if and only if $F = \pm\sqrt{\frac{4+\mu^2}{4+\lambda^2}}C$. Hence, we have the following.

Proposition 2.9. *If in the case* (c) $(M, B, D, E, G, F, C \neq 0)$ *not all B, C, D are collinear with M, and not all E, F, G are collinear with M, then a* $\delta(2,2)$ *Chen ideal submanifold* M^4 *is semi-symmetric if and only if none of F, C is collinear with M, F is collinear with C,* $E - G = \lambda F$, $B - D = \mu C$ *and* $F = \pm\sqrt{\frac{4+\mu^2}{4+\lambda^2}}C$ $(\lambda, \mu \in \mathbb{R})$.

Take in particular $\lambda = \mu = 0$. Then $B = D = E = G = M/2$, C is not collinear with M, $\|C\| = \|M\|/2$, F is not collinear with M, $F = \pm C, (C \neq 0)$, we obtain the solutions $(M/2, C, M/2, M/2, \pm C, M/2)$.

Summarizing all Propositions 2.2 – 2.9, we get the following main Theorem.

Theorem 2.1. *Let* M^4 *be a* $\delta(2,2)$ *Chen ideal submanifold in the Euclidean space* $\mathbb{E}^6$*. Then it is semi-symmetric if and only if* M^4 *has one of the following types:*

(1^0) $(\frac{M}{2}, 0, \frac{M}{2}, \frac{M}{2}, 0, \frac{M}{2})$, *where M is the arbitrary vector in* $\mathbb{E}^2$*;*

(2^0) $(0, 0, 0, E, F, -E)$, *where* $E \neq 0$ *or* $F \neq 0$*;*

(3^0) $(\alpha P, \beta P, -\alpha P, \gamma Q, \delta Q, -\gamma Q)$ *where* $\{P, Q\}$ *is an orthonormal basis in* $\mathbb{E}^2$, $\alpha^2 + \beta^2 > 0$ *and* $\gamma^2 + \delta^2 > 0$.

(4^0) $(\frac{M+\alpha N}{2}, \beta N, \frac{M-\alpha N}{2}, \frac{M+\mu N}{2}, \nu N, \frac{M-\mu N}{2})$, *for some vectors* $M, N \in \mathbb{E}^2$, $\|N\| = \|M\|$, *and some* $\alpha, \mu \in [-1, 1]$ $(\beta = \pm\frac{1}{2}\sqrt{1-\alpha^2}, \nu = \pm\frac{1}{2}\sqrt{1-\mu^2})$.

We can also reformulate this theorem in terms of the shape operators A_1, A_2 of $\delta(2,2)$ Chen ideal submanifold M^4 in $\mathbb{E}^6$.

Corollary 2.2. *Let* M^4 *be a* $\delta(2,2)$ *Chen ideal submanifold in the Euclidean space* $\mathbb{E}^6$*. Then* M^4 *is semi-symmetric if and only if one of the following statements hold true:*

(i) M^4 *is a minimal submanifold;*

(ii) M^4 *is a totally umbilical submanifold (including the totally geodesic one);*

(iii) M^4 *is a hypersphere in the Euclidean space* $\mathbb{E}^5 \subset \mathbb{E}^6$*;*

(iv) the shape operators of M^4 *in* $\mathbb{E}^6$ *have the form* $A_1 = \mathrm{diag}(a, a, b, b)$, $A_2 = \mathrm{diag}(c, c, d, d)$, *where* $ab + cd = 0$ $(a, b, c, d \in \mathbb{R})$.

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