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Intrinsic Equations for a Relaxed Elastic Line of Second Kind in Minkowski 3-Space

Ergin Bayram[a], Emin Kasap[a]

[a] *Ondokuz Mayıs University, Faculty of Arts and Sciences, Department of Mathematics, 55139 Kurupelit/Samsun, TURKEY*

Abstract. Let α be an arc on a connected oriented surface S in Minkowski 3-space, parameterized by arc length s, with torsion τ and length l. The total square torsion H of α is defined by $H = \int_0^l \tau^2 ds$. The arc α is called a relaxed elastic line of second kind if it is an extremal for the variational problem of minimizing the value of H within the family of all arcs of length l on S having the same initial point and initial direction as α. In this study, we obtain the differential equation and boundary conditions for a relaxed elastic line of second kind on an oriented surface in Minkowski 3-space. This formulation should give a more direct and more geometric approach to questions concerning relaxed elastic lines of second kind on a surface.

1. Preliminaries and Introduction

In this section, we give some fundamentals required for this paper.

Definition 1.1. $\mathbb{R}^n$ *equipped with the metric*

$$\langle u, w \rangle = -\sum_{i=1}^{v} u_i w_i + \sum_{j=v+1}^{n} u_j w_j, \quad u, w \in \mathbb{R}^n,\ 0 \leq v \leq n,$$

is called semi-Euclidean space and is denoted by $\mathbb{R}^n_v$, where v is called the index of the metric. For $n = 3$, $\mathbb{R}^3_1$ is called Minkowski 3-space [3].

Let $\alpha(s)$ denote an arc on a connected oriented surface S in $\mathbb{R}^3_1$, parameterized by arc length s, $0 \leq s \leq l$, with curvature $\kappa(s)$ and torsion $\tau(s)$. Let the energy density be given as some function of the curvature and torsion, $f(\kappa,\ \tau)$. Then

$$H = \int f(\kappa,\ \tau)\, ds \tag{1}$$

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Email addresses: erginbayram@yahoo.com (Ergin Bayram), kasape@omu.edu.tr (Emin Kasap)

is an Hamiltonian for curves [1]. Thus the following integral can be taken as a special case of Hamiltonians for curves:

$$H = \int \tau^2 ds. \tag{2}$$

Ekici and Görgülü [2] handled the problem of minimization of the integral $\int \kappa^2 \tau ds$ in Minkowski 3-space. In [4] authors defined relaxed elastic line of second kind on an oriented surface in Minkowski space and for a curve of this kind lying on an oriented surface, the Euler-Lagrange equations were derived. In this paper, we give intrinsic equations for a curve of this kind. Particularly, we obtain the differential equation and boundary conditions for a curve to be a relaxed elastic line of second kind on an oriented surface.

Definition 1.2. *The arc α is called a relaxed elastic line of second kind in Minkowski 3-space if it is an extremal for the variational problem of minimizing the value of H within the family of all arcs of length l on S having the same initial point and initial direction as α in Minkowski 3-space [4].*

In this study, we would like to calculate the intrinsic equations for the curve α which is an extremal for (2). We shall require that the coordinate functions of S are smooth enough to have partial derivatives and coordinate functions of α, as functions of s, are smooth enough in these coordinates.

Definition 1.3. *A tangent vector v in $\mathbb{R}^3_1$ is*
spacelike if $\langle v, v\rangle > 0$ or $v = 0$,
null if $\langle v, v\rangle = 0$ and $v \neq 0$,
timelike if $\langle v, v\rangle < 0$.

Definition 1.4. *At a point $\alpha(s)$ of α, let T denote the unit tangent vector to α, n the unit normal to S, and*

$$n \times T = \varepsilon Q(s), \quad \varepsilon = \pm 1, \tag{3}$$

respectively. Then $\{T, Q, n\}$ gives an orthonormal basis in $\mathbb{R}^3_1$. If S is a spacelike surface then $T \times Q = n$, $Q \times n = -T$, $n \times T = -Q$. Similarly, if S is a timelike surface then $T \times Q = -n$, $Q \times n = \pm T$, $n \times T = \mp Q$ [5].

Theorem 1.5. *Let S be a surface in $\mathbb{R}^3_1$ and α be a curve on S. The analogue of the Frenet–Serret formulas is given by*

$$\begin{pmatrix} T' \\ Q' \\ n' \end{pmatrix} = \begin{pmatrix} 0 & \varepsilon_2\kappa_g & \varepsilon_3\kappa_n \\ -\varepsilon_1\kappa_g & 0 & \varepsilon_3\tau_g \\ -\varepsilon_1\kappa_n & -\varepsilon_2\tau_g & 0 \end{pmatrix} \begin{pmatrix} T \\ Q \\ n \end{pmatrix}, \tag{4}$$

where $\varepsilon_1 = \langle T, T\rangle$, $\varepsilon_2 = \langle Q, Q\rangle$, $\varepsilon_3 = \langle n, n\rangle$. Here $k_g(s) = \langle T'(s), Q(s)\rangle$, $\tau_g(s) = \langle Q'(s), n(s)\rangle$ and $k_n(s) = \langle II(T(s), T(s)), n(s)\rangle$ are geodesic curvature, geodesic torsion and normal curvature of α, respectively.

Theorem 1.6. *Let α be any regular curve on a surface in $\mathbb{R}^3_1$. Then we have*

$$\kappa = \frac{\|\alpha' \times \alpha''\|}{\|\alpha'\|^3}, \quad \tau = -\frac{\langle \alpha' \times \alpha'', \alpha''' \rangle}{\|\alpha' \times \alpha''\|^2}.$$

2. Derivation of equations

Suppose that α lies in a coordinate patch $(u, v) \rightarrow x(u, v)$ of a surface S, and let $x_u = \partial x/\partial u$, $x_v = \partial x/\partial v$. Then α is expressed as

$$\alpha(s) = x(u(s), v(s)),\ 0 \leq s \leq l,$$

with

$$T(s) = \alpha'(s) = \frac{du}{ds}x_u + \frac{dv}{ds}x_v$$

and

$$Q(s) = p(s)x_u + q(s)x_v$$

for suitable scalar functions $p(s)$ and $q(s)$.

Now we will define variational fields for our problem. In order to obtain variational arcs of length l, we need to extend α to an arc $\alpha^*(s)$ defined for $0 \leq s \leq l^*$, with $l^* \geq l$ but sufficiently close to l so that α^* lies in the coordinate patch. Let $\mu(s)$, $0 \leq s \leq l^*$, be a scalar function of class C^2, not vanishing identically. Define

$$\eta(s) = \mu(s)p^*(s),\quad \zeta(s) = \mu(s)q^*(s).$$

Then

$$\eta(s)x_u + \zeta(s)x_v = \mu(s)Q(s) \tag{5}$$

along α. Also assume that

$$\mu(0) = 0,\ \mu'(0) = 0, \mu''(0) = 0. \tag{6}$$

Now define

$$\beta(\sigma; t) = x(u(\sigma) + t\eta(\sigma), v(\sigma) + t\zeta(\sigma)), \tag{7}$$

for $0 \leq \sigma \leq l^*$. For $|t| < \varepsilon_1$ (where $\varepsilon_1 > 0$ depends upon the choice of α^* and of μ), the point $\beta(\sigma; t)$ lies in the coordinate patch. For fixed t, $\beta(\sigma; t)$ gives an arc with the same initial point and initial direction as α, because of (6). For $t = 0$, $\beta(\sigma; 0)$ is the same as α^* and σ is arc length. For $t \neq 0$, the parameter σ is not arc length in general.

For fixed t, $|t| < \varepsilon_1$, let $L^*(t)$ denote the length of the arc $\beta(\sigma; t)$, $0 \leq \sigma \leq l^*$. Then

$$L^*(t) = \int_0^{l^*} \sqrt{\left|\left\langle \frac{\partial \beta}{\partial \sigma}, \frac{\partial \beta}{\partial \sigma} \right\rangle\right|} d\sigma \tag{8}$$

with

$$L^*(0) = l^* > l. \tag{9}$$

By (7) and (8) $L^*(t)$ is continuous and differentiable in t. Particularly, it follows from (9) that

$$L^*(t) > \frac{l+l^*}{2} > l \quad \text{for} \quad |t| < \varepsilon \tag{10}$$

for a suitable ε satisfying $0 < \varepsilon \leq \varepsilon_1$. Because of (10) one can restrict $\beta(\sigma;t)$, $0 \leq |t| < \varepsilon$, to an arc of length l by restricting the parameter σ to an interval $0 \leq \sigma \leq \lambda(t) \leq l^*$ by requiring

$$\int_0^{\lambda(t)} \sqrt{\left|\left\langle \frac{\partial\beta}{\partial\sigma}, \frac{\partial\beta}{\partial\sigma} \right\rangle\right|} d\sigma = l. \tag{11}$$

Note that $\lambda(0) = l$. The function $\lambda(t)$ need not be determined explicitly, but we shall need

$$\left.\frac{d\lambda}{dt}\right|_{t=0} = \varepsilon_1 \int_0^l \mu\kappa_g ds. \tag{12}$$

The proof of (12) and of other results will depend on calculations from (7) such as

$$\left.\frac{\partial\beta}{\partial\sigma}\right|_{t=0} = T, \ \ 0 \leq s \leq l, \tag{13}$$

which gives

$$\left.\frac{\partial^2\beta}{\partial\sigma^2}\right|_{t=0} = T' = \varepsilon_2\kappa_g Q + \varepsilon_3\kappa_n n. \tag{14}$$

Also

$$\left.\frac{\partial\beta}{\partial t}\right|_{t=0} = \mu Q \tag{15}$$

because of (5). Further differentiation of (15) gives

$$\left.\frac{\partial^2\beta}{\partial t\partial\sigma}\right|_{t=0} = \left.\frac{\partial^2\beta}{\partial\sigma\partial t}\right|_{t=0} = \mu' Q + \mu Q' = -\varepsilon_1\mu\kappa_g T + \mu' Q + \varepsilon_3\mu\tau_g n \tag{16}$$

and using (4),

$$\begin{aligned}\left.\frac{\partial^3\beta}{\partial t\partial\sigma^2}\right|_{t=0} = & \left(-2\varepsilon_1\mu'\kappa_g - \varepsilon_1\mu\kappa_g' - \varepsilon_1\varepsilon_3\mu\kappa_n\tau_g\right)T + \left(\mu'' - \varepsilon_1\varepsilon_2\mu\kappa_g^2 - \varepsilon_2\varepsilon_3\mu\tau_g^2\right)Q \\ & + \left(2\varepsilon_3\mu'\tau_g + \varepsilon_3\mu\tau_g' - \varepsilon_1\varepsilon_3\mu\kappa_g\kappa_n\right)n.\end{aligned} \tag{17}$$

Also using (14) we have

$$\left.\frac{\partial^3\beta}{\partial\sigma^3}\right|_{t=0} = -\left(\varepsilon_1\varepsilon_2\kappa_g^2 + \varepsilon_1\varepsilon_3\kappa_n^2\right)T + \left(\varepsilon_2\kappa_g' - \varepsilon_2\varepsilon_3\kappa_n\tau_g\right)Q + \left(\varepsilon_3\kappa_n' + \varepsilon_2\varepsilon_3\kappa_g\tau_g\right)n \tag{18}$$

and by (15)

$$\left.\frac{\partial^4\beta}{\partial t\partial\sigma^3}\right|_{t=0} = \Big(-3\varepsilon_1\mu''\kappa_g - 3\varepsilon_1\mu'\kappa_g' - \varepsilon_1\mu\kappa_g'' - \varepsilon_1\varepsilon_3\mu\kappa_n'\tau_g - 2\varepsilon_1\varepsilon_3\mu\kappa_n\tau_g' - 3\varepsilon_1\varepsilon_3\mu'\kappa_n\tau_g + \varepsilon_3\mu\kappa_g\kappa_n^2 + \varepsilon_2\mu\kappa_g^3 + \varepsilon_1\varepsilon_2\varepsilon_3\mu\kappa_g\tau_g^2\Big)T + \Big(\mu''' - 3\varepsilon_1\varepsilon_2\mu'\kappa_g^2 - 3\varepsilon_2\varepsilon_3\mu'\tau_g^2 - 3\varepsilon_1\varepsilon_2\mu\kappa_g\kappa_g' - 3\varepsilon_2\varepsilon_3\mu\tau_g\tau_g'\Big)Q + \Big(-3\varepsilon_1\varepsilon_3\mu'\kappa_g\kappa_n - 2\varepsilon_1\varepsilon_3\mu\kappa_n\kappa_g' - \varepsilon_1\varepsilon_2\varepsilon_3\mu\kappa_g^2\tau_g + 3\varepsilon_3\mu''\tau_g + 3\varepsilon_3\mu'\tau_g' + \varepsilon_3\mu\tau_g'' - \varepsilon_1\varepsilon_3\mu\kappa_g\kappa_n' - \varepsilon_2\mu\tau_g^3 - \varepsilon_1\mu\kappa_n^2\tau_g\Big)n. \tag{19}$$

Now, let $H(t)$ denote the functional of a relaxed elastic line of second kind for the arc $\beta(\sigma;t)$, $0 \le \sigma \le \lambda(t)$, $|t| < \varepsilon$. Since, in general, σ is not the arc length for $t \neq 0$ functional (2) can be calculated as follows:

$$H(t) = \int_0^{\lambda(t)} \left(\frac{\left\langle \frac{\partial\beta}{\partial\sigma} \times \frac{\partial^2\beta}{\partial\sigma^2}, \frac{\partial^3\beta}{\partial\sigma^3} \right\rangle}{\left\langle \frac{\partial\beta}{\partial\sigma}, \frac{\partial\beta}{\partial\sigma} \right\rangle \left\langle \frac{\partial^2\beta}{\partial\sigma^2}, \frac{\partial^2\beta}{\partial\sigma^2} \right\rangle - \left\langle \frac{\partial^2\beta}{\partial\sigma^2}, \frac{\partial\beta}{\partial\sigma} \right\rangle^2} \right)^2 d\sigma.$$

A necessary condition for α to be an extremal is that

$$\left.\frac{dH}{dt}\right|_{t=0} = 0$$

for arbitrary μ satisfying (6). We have

$$H'(t) = \frac{d\lambda}{dt} \left\{ \left(\frac{\left\langle \frac{\partial\beta}{\partial\sigma} \times \frac{\partial^2\beta}{\partial\sigma^2}, \frac{\partial^3\beta}{\partial\sigma^3} \right\rangle}{\left\langle \frac{\partial\beta}{\partial\sigma}, \frac{\partial\beta}{\partial\sigma} \right\rangle \left\langle \frac{\partial^2\beta}{\partial\sigma^2}, \frac{\partial^2\beta}{\partial\sigma^2} \right\rangle - \left\langle \frac{\partial^2\beta}{\partial\sigma^2}, \frac{\partial\beta}{\partial\sigma} \right\rangle^2} \right)^2 \right\}_{\sigma=\lambda(t)} + \int_0^{\lambda(t)} \frac{\partial}{\partial t} \left(\frac{\left\langle \frac{\partial\beta}{\partial\sigma} \times \frac{\partial^2\beta}{\partial\sigma^2}, \frac{\partial^3\beta}{\partial\sigma^3} \right\rangle}{\left\langle \frac{\partial\beta}{\partial\sigma}, \frac{\partial\beta}{\partial\sigma} \right\rangle \left\langle \frac{\partial^2\beta}{\partial\sigma^2}, \frac{\partial^2\beta}{\partial\sigma^2} \right\rangle - \left\langle \frac{\partial^2\beta}{\partial\sigma^2}, \frac{\partial\beta}{\partial\sigma} \right\rangle^2} \right)^2 d\sigma.$$

In calculating dH/dt; we give explicitly only terms that do not vanish for $t = 0$. The omitted terms are those with factor

$$\left\langle \frac{\partial^2\beta}{\partial\sigma^2}, \frac{\partial\beta}{\partial\sigma} \right\rangle$$

which vanishes at $t = 0$ because $\langle T, T' \rangle = 0$. Thus, using $(11-14)$ and $(16-19)$ we get

$$\begin{aligned}
H'(0) &= \varepsilon_1\tau^2(l)\int_0^l \mu\kappa_g ds + 2\int_0^l \frac{\tau(s)}{\left(\varepsilon_2\kappa_g^2+\varepsilon_3\kappa_n^2\right)^2}\Big\{\left(\varepsilon_1\varepsilon_2\kappa_g^2+\varepsilon_1\varepsilon_3\kappa_n^2\right)\Big[\mu\Big(-3\kappa_g^2\kappa_n' - 2\varepsilon_2\kappa_g^3\tau_g + 3\kappa_g\kappa_n\kappa_g' \\
&\quad +\varepsilon_2\tau_g^2\kappa_n' + 2\kappa_g\tau_g^3 - \varepsilon_3\kappa_g\kappa_n^2\tau_g + \varepsilon_2\varepsilon_3\kappa_g'\tau_g' - 4\varepsilon_2\kappa_n\tau_g\tau_g' - \varepsilon_2\varepsilon_3\kappa_g\tau_g'' + \varepsilon_1\varepsilon_2\kappa_g\kappa_n^2\tau_g\Big) \\
&\quad +\mu'\left(-\kappa_g^2\kappa_n + \varepsilon_1\kappa_n^3 + 2\varepsilon_2\varepsilon_3\tau_g\kappa_g' - 5\varepsilon_2\kappa_n\tau_g^2 - 3\varepsilon_2\varepsilon_3\kappa_g\tau_g'\right) - \mu''\left(\varepsilon_3\kappa_n' + 4\varepsilon_2\varepsilon_3\kappa_g\tau_g\right) + \varepsilon_3\mu'''\kappa_n\Big] \\
&\quad +\Big[\varepsilon_2\kappa_g\left(\varepsilon_3\kappa_n' + \varepsilon_2\varepsilon_3\kappa_g\tau_g\right) - \varepsilon_3\kappa_n\left(\varepsilon_2\kappa_g' - \varepsilon_2\varepsilon_3\kappa_n\tau_g\right)\Big]\Big[-4\varepsilon_2\mu\kappa_g^3 - 4\varepsilon_3\mu\kappa_g\kappa_n^2 + 2\mu\kappa_g\tau_g^2 \\
&\quad +2\varepsilon_1\varepsilon_3\mu\kappa_n\tau_g' + 4\varepsilon_1\varepsilon_3\mu'\kappa_n\tau_g + 2\varepsilon_1\mu''\kappa_g\Big]\Big\} ds \\
&= \int_0^l \mu\left(\varepsilon_1\tau^2(l)\kappa_g\right) ds + \int_0^l \mu\left\{\frac{2\tau(s)}{\left(\varepsilon_2\kappa_g^2+\varepsilon_3\kappa_n^2\right)^2}\Big[\left(\varepsilon_1\varepsilon_2\kappa_g^2+\varepsilon_1\varepsilon_3\kappa_n^2\right)\Big(-3\kappa_g^2\kappa_n' - 2\varepsilon_2\kappa_g^3\tau_g + 3\kappa_g\kappa_n\kappa_g' \right.\\
&\quad +\varepsilon_2\tau_g^2\kappa_n' + 2\kappa_g\tau_g^3 - \varepsilon_3\kappa_g\kappa_n^2\tau_g + \varepsilon_2\varepsilon_3\kappa_g'\tau_g' - 4\varepsilon_2\kappa_n\tau_g\tau_g' - \varepsilon_2\varepsilon_3\kappa_g\tau_g'' + \varepsilon_1\varepsilon_2\kappa_g\kappa_n^2\tau_g\Big) \\
&\quad \left.+\Big[\varepsilon_2\kappa_g\left(\varepsilon_3\kappa_n' + \varepsilon_2\varepsilon_3\kappa_g\tau_g\right) - \varepsilon_3\kappa_n\left(\varepsilon_2\kappa_g' - \varepsilon_2\varepsilon_3\kappa_n\tau_g\right)\Big]\left(-4\varepsilon_2\kappa_g^3 - 4\varepsilon_3\kappa_g\kappa_n^2 + 2\kappa_g\tau_g^2 + 2\varepsilon_1\varepsilon_3\kappa_n\tau_g'\right)\Big]\right\} ds \\
&\quad + \int_0^l \mu'\left\{\frac{2\tau(s)}{\left(\varepsilon_2\kappa_g^2+\varepsilon_3\kappa_n^2\right)^2}\Big[\left(\varepsilon_1\varepsilon_2\kappa_g^2+\varepsilon_1\varepsilon_3\kappa_n^2\right)\Big(-\kappa_g^2\kappa_n + \varepsilon_1\kappa_n^3 + 2\varepsilon_2\varepsilon_3\tau_g\kappa_g' - 5\varepsilon_2\kappa_n\tau_g^2 \right.\\
&\quad \left.-3\varepsilon_2\varepsilon_3\kappa_g\tau_g'\Big) + 4\varepsilon_1\varepsilon_3\kappa_n\tau_g\Big[\varepsilon_2\kappa_g\left(\varepsilon_3\kappa_n' + \varepsilon_2\varepsilon_3\kappa_g\tau_g\right) - \varepsilon_3\kappa_n\left(\varepsilon_2\kappa_g' - \varepsilon_2\varepsilon_3\kappa_n\tau_g\right)\Big]\Big]\right\} ds \\
&\quad + \int_0^l \mu''\left\{\frac{2\tau(s)}{\left(\varepsilon_2\kappa_g^2+\varepsilon_3\kappa_n^2\right)^2}\Big[\left(\varepsilon_1\varepsilon_2\kappa_g^2+\varepsilon_1\varepsilon_3\kappa_n^2\right)\left(-\varepsilon_3\kappa_n' - 4\varepsilon_2\varepsilon_3\kappa_g\tau_g\right)\right.\\
&\quad \left.+2\varepsilon_1\kappa_g\Big[\varepsilon_2\kappa_g\left(\varepsilon_3\kappa_n' + \varepsilon_2\varepsilon_3\kappa_g\tau_g\right) - \varepsilon_3\kappa_n\left(\varepsilon_2\kappa_g' - \varepsilon_2\varepsilon_3\kappa_n\tau_g\right)\Big]\Big]\right\} ds + \int_0^l \mu'''\left\{\frac{2\varepsilon_1\varepsilon_3\kappa_n\tau(s)}{\varepsilon_2\kappa_g^2+\varepsilon_3\kappa_n^2}\right\} ds.
\end{aligned}$$

However, using integration by parts and (6) we get

$$\begin{aligned}
&\int_0^l \mu'\left\{\frac{2\tau(s)}{\left(\varepsilon_2\kappa_g^2+\varepsilon_3\kappa_n^2\right)^2}\Big[\left(\varepsilon_1\varepsilon_2\kappa_g^2+\varepsilon_1\varepsilon_3\kappa_n^2\right)\Big(-\kappa_g^2\kappa_n + \varepsilon_1\kappa_n^3 + 2\varepsilon_2\varepsilon_3\tau_g\kappa_g' \right.\\
&-5\varepsilon_2\kappa_n\tau_g^2 - 3\varepsilon_2\varepsilon_3\kappa_g\tau_g'\Big) + 4\varepsilon_1\varepsilon_3\kappa_n\tau_g\Big[\varepsilon_2\kappa_g\left(\varepsilon_3\kappa_n' + \varepsilon_2\varepsilon_3\kappa_g\tau_g\right) \\
&\left.-\varepsilon_3\kappa_n\left(\varepsilon_2\kappa_g' - \varepsilon_2\varepsilon_3\kappa_n\tau_g\right)\Big)\Big]\right\} ds = \mu(l)\left\{\frac{2\tau(s)}{\left(\varepsilon_2\kappa_g^2+\varepsilon_3\kappa_n^2\right)^2}\Big[\left(\varepsilon_1\varepsilon_2\kappa_g^2+\varepsilon_1\varepsilon_3\kappa_n^2\right)\Big(-\kappa_g^2\kappa_n \right.\\
&+\varepsilon_1\kappa_n^3 + 2\varepsilon_2\varepsilon_3\tau_g\kappa_g' - 5\varepsilon_2\kappa_n\tau_g^2 - 3\varepsilon_2\varepsilon_3\kappa_g\tau_g'\Big) \\
&\left.+4\varepsilon_1\varepsilon_3\kappa_n\tau_g\left(\varepsilon_2\kappa_g\left(\varepsilon_3\kappa_n' + \varepsilon_2\varepsilon_3\kappa_g\tau_g\right) - \varepsilon_3\kappa_n\left(\varepsilon_2\kappa_g' - \varepsilon_2\varepsilon_3\kappa_n\tau_g\right)\right)\Big]\right\}_{s=l} \\
&-\int_0^l \mu\left\{\frac{2\tau(s)}{\left(\varepsilon_2\kappa_g^2+\varepsilon_3\kappa_n^2\right)^2}\Big[\left(\varepsilon_1\varepsilon_2\kappa_g^2+\varepsilon_1\varepsilon_3\kappa_n^2\right)\Big(-\kappa_g^2\kappa_n + \varepsilon_1\kappa_n^3 + 2\varepsilon_2\varepsilon_3\tau_g\kappa_g' \right.\\
&-5\varepsilon_2\kappa_n\tau_g^2 - 3\varepsilon_2\varepsilon_3\kappa_g\tau_g'\Big) + 4\varepsilon_1\varepsilon_3\kappa_n\tau_g\Big[\varepsilon_2\kappa_g\left(\varepsilon_3\kappa_n' + \varepsilon_2\varepsilon_3\kappa_g\tau_g\right) \\
&\left.-\varepsilon_3\kappa_n\left(\varepsilon_2\kappa_g' - \varepsilon_2\varepsilon_3\kappa_n\tau_g\right)\Big)\Big]\right\}' ds
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^l \mu'' \left\{ \frac{2\tau(s)}{\left(\varepsilon_2\kappa_g^2+\varepsilon_3\kappa_n^2\right)^2} \left[\left(\varepsilon_1\varepsilon_2\kappa_g^2+\varepsilon_1\varepsilon_3\kappa_n^2\right)\left(-\varepsilon_3\kappa_n'-4\varepsilon_2\varepsilon_3\kappa_g\tau_g\right) \right.\right.\\
&\left.\left. +2\varepsilon_1\kappa_g\left[\varepsilon_2\kappa_g\left(\varepsilon_3\kappa_n'+\varepsilon_2\varepsilon_3\kappa_g\tau_g\right)-\varepsilon_3\kappa_n\left(\varepsilon_2\kappa_g'-\varepsilon_2\varepsilon_3\kappa_n\tau_g\right)\right]\right]\right\} ds =\\
&\mu'(l) \left\{ \frac{2\tau(s)}{\left(\varepsilon_2\kappa_g^2+\varepsilon_3\kappa_n^2\right)^2} \left[\left(\varepsilon_1\varepsilon_2\kappa_g^2+\varepsilon_1\varepsilon_3\kappa_n^2\right)\left(-\varepsilon_3\kappa_n'-4\varepsilon_2\varepsilon_3\kappa_g\tau_g\right) \right.\right.\\
&\left.\left. +2\varepsilon_1\kappa_g\left[\varepsilon_2\kappa_g\left(\varepsilon_3\kappa_n'+\varepsilon_2\varepsilon_3\kappa_g\tau_g\right)-\varepsilon_3\kappa_n\left(\varepsilon_2\kappa_g'-\varepsilon_2\varepsilon_3\kappa_n\tau_g\right)\right]\right]\right\}_{s=l}\\
&-\mu(l) \left\{ \frac{2\tau(s)}{\left(\varepsilon_2\kappa_g^2+\varepsilon_3\kappa_n^2\right)^2} \left[\left(\varepsilon_1\varepsilon_2\kappa_g^2+\varepsilon_1\varepsilon_3\kappa_n^2\right)\left(-\varepsilon_3\kappa_n'-4\varepsilon_2\varepsilon_3\kappa_g\tau_g\right) \right.\right.\\
&\left.\left. +2\varepsilon_1\kappa_g\left[\varepsilon_2\kappa_g\left(\varepsilon_3\kappa_n'+\varepsilon_2\varepsilon_3\kappa_g\tau_g\right)-\varepsilon_3\kappa_n\left(\varepsilon_2\kappa_g'-\varepsilon_2\varepsilon_3\kappa_n\tau_g\right)\right]\right]\right\}'_{s=l}\\
&+\int_0^l \mu \left\{ \frac{2\tau(s)}{\left(\varepsilon_2\kappa_g^2+\varepsilon_3\kappa_n^2\right)^2} \left[\left(\varepsilon_1\varepsilon_2\kappa_g^2+\varepsilon_1\varepsilon_3\kappa_n^2\right)\left(-\varepsilon_3\kappa_n'-4\varepsilon_2\varepsilon_3\kappa_g\tau_g\right) \right.\right.\\
&\left.\left. +2\varepsilon_1\kappa_g\left[\varepsilon_2\kappa_g\left(\varepsilon_3\kappa_n'+\varepsilon_2\varepsilon_3\kappa_g\tau_g\right)-\varepsilon_3\kappa_n\left(\varepsilon_2\kappa_g'-\varepsilon_2\varepsilon_3\kappa_n\tau_g\right)\right]\right]\right\}'' ds
\end{aligned}$$

and

$$\begin{aligned}
\int_0^l \mu''' \left\{\frac{2\varepsilon_1\varepsilon_3\kappa_n\tau(s)}{\varepsilon_2\kappa_g^2+\varepsilon_3\kappa_n^2}\right\} ds = \mu''(l)\left\{\frac{2\varepsilon_1\varepsilon_3\kappa_n\tau(s)}{\varepsilon_2\kappa_g^2+\varepsilon_3\kappa_n^2}\right\}_{s=l} &\quad -\mu'(l)\left\{\frac{2\varepsilon_1\varepsilon_3\kappa_n\tau(s)}{\varepsilon_2\kappa_g^2+\varepsilon_3\kappa_n^2}\right\}'_{s=l}\\
+\mu(l)\left\{\frac{2\varepsilon_1\varepsilon_3\kappa_n\tau(s)}{\varepsilon_2\kappa_g^2+\varepsilon_3\kappa_n^2}\right\}''_{s=l} &\quad -\int_0^l \mu\left\{\frac{2\varepsilon_1\varepsilon_3\kappa_n\tau(s)}{\varepsilon_2\kappa_g^2+\varepsilon_3\kappa_n^2}\right\}''' ds.
\end{aligned}$$

Thus $H'(0)$ can be written as

$$
\begin{aligned}
H'(0) \quad = \quad & \int_0^l \mu \left\{ \varepsilon_1 \kappa_g \tau^2 (l) + \frac{2\tau(s)}{\left(\varepsilon_2 \kappa_g^2 + \varepsilon_3 \kappa_n^2\right)^2} \left(\left(\varepsilon_1 \varepsilon_2 \kappa_g^2 + \varepsilon_1 \varepsilon_3 \kappa_n^2 \right) \left[\left(-3\kappa_g^2 \kappa_n' - 2\varepsilon_2 \kappa_g^3 \tau_g + 3\kappa_g \kappa_n \kappa_g' \right. \right. \right. \right. \\
& + \varepsilon_2 \tau_g^2 \kappa_n' + 2\kappa_g \tau_g^3 - \varepsilon_3 \kappa_g \kappa_n^2 \tau_g + \varepsilon_2 \varepsilon_3 \kappa_g' \tau_g' - 4\varepsilon_2 \kappa_n \tau_g \tau_g' - \varepsilon_2 \varepsilon_3 \kappa_g \tau_g'' + \varepsilon_1 \varepsilon_2 \kappa_g \kappa_n^2 \tau_g \Big) \\
& + \left[\varepsilon_2 \kappa_g \left(\varepsilon_3 \kappa_n' + \varepsilon_2 \varepsilon_3 \kappa_g \tau_g \right) - \varepsilon_3 \kappa_n \left(\varepsilon_2 \kappa_g' - \varepsilon_2 \varepsilon_3 \kappa_n \tau_g \right) \right] \left(-4\varepsilon_2 \kappa_g^3 - 4\varepsilon_3 \kappa_g \kappa_n^2 + 2\kappa_g \tau_g^2 + 2\varepsilon_1 \varepsilon_3 \kappa_n \tau_g' \right) \Big) \\
& - \left[\frac{2\tau(s)}{\left(\varepsilon_2 \kappa_g^2 + \varepsilon_3 \kappa_n^2\right)^2} \left[\left(\varepsilon_1 \varepsilon_2 \kappa_g^2 + \varepsilon_1 \varepsilon_3 \kappa_n^2 \right) \left(-\kappa_g^2 \kappa_n + \varepsilon_1 \kappa_n^3 + 2\varepsilon_2 \varepsilon_3 \tau_g \kappa_g' - 5\varepsilon_2 \kappa_n \tau_g^2 - 3\varepsilon_2 \varepsilon_3 \kappa_g \tau_g' \right) \right. \right. \\
& \left. \left. + 4\varepsilon_1 \varepsilon_3 \kappa_n \tau_g \left(\varepsilon_2 \kappa_g \left(\varepsilon_3 \kappa_n' + \varepsilon_2 \varepsilon_3 \kappa_g \tau_g \right) - \varepsilon_3 \kappa_n \left(\varepsilon_2 \kappa_g' - \varepsilon_2 \varepsilon_3 \kappa_n \tau_g \right) \right) \right] \right]' \\
& + \left[\frac{2\tau(s)}{\left(\varepsilon_2 \kappa_g^2 + \varepsilon_3 \kappa_n^2\right)^2} \left[\left(\varepsilon_1 \varepsilon_2 \kappa_g^2 + \varepsilon_1 \varepsilon_3 \kappa_n^2 \right) \left(-\varepsilon_3 \kappa_n' - 4\varepsilon_2 \varepsilon_3 \kappa_g \tau_g \right) \right. \right. \\
& \left. \left. + 2\varepsilon_1 \kappa_g \left(\varepsilon_2 \kappa_g \left(\varepsilon_3 \kappa_n' + \varepsilon_2 \varepsilon_3 \kappa_g \tau_g \right) - \varepsilon_3 \kappa_n \left(\varepsilon_2 \kappa_g' - \varepsilon_2 \varepsilon_3 \kappa_n \tau_g \right) \right) \right] \right]'' - \left(\frac{2\varepsilon_1 \varepsilon_3 \kappa_n \tau(s)}{\varepsilon_2 \kappa_g^2 + \varepsilon_3 \kappa_n^2} \right)''' \Bigg\} ds \\
& + \mu(l) \left\{ \frac{2\tau(l)}{\left(\varepsilon_2 \kappa_g^2 (l) + \varepsilon_3 \kappa_n^2 (l)\right)^2} \left[\left(\varepsilon_1 \varepsilon_2 \kappa_g^2 + \varepsilon_1 \varepsilon_3 \kappa_n^2 \right) \left(-\kappa_g^2 \kappa_n + \varepsilon_1 \kappa_n^3 + 2\varepsilon_2 \varepsilon_3 \tau_g \kappa_g' \right. \right. \right. \\
& \left. \left. - 5\varepsilon_2 \kappa_n \tau_g^2 - 3\varepsilon_2 \varepsilon_3 \kappa_g \tau_g' \right) + 4\varepsilon_1 \varepsilon_3 \kappa_n \tau_g \left(\varepsilon_2 \kappa_g \left(\varepsilon_3 \kappa_n' + \varepsilon_2 \varepsilon_3 \kappa_g \tau_g \right) - \varepsilon_3 \kappa_n \left(\varepsilon_2 \kappa_g' - \varepsilon_2 \varepsilon_3 \kappa_n \tau_g \right) \right) \right]_{s=l} \\
& - \left(\frac{2\tau(s)}{\left(\varepsilon_2 \kappa_g^2 + \varepsilon_3 \kappa_n^2\right)^2} \left[\left(\varepsilon_1 \varepsilon_2 \kappa_g^2 + \varepsilon_1 \varepsilon_3 \kappa_n^2 \right) \left(-\varepsilon_3 \kappa_n' - 4\varepsilon_2 \varepsilon_3 \kappa_g \tau_g \right) \right. \right. \\
& \left. \left. + 2\varepsilon_1 \kappa_g \left(\varepsilon_2 \kappa_g \left(\varepsilon_3 \kappa_n' + \varepsilon_2 \varepsilon_3 \kappa_g \tau_g \right) - \varepsilon_3 \kappa_n \left(\varepsilon_2 \kappa_g' - \varepsilon_2 \varepsilon_3 \kappa_n \tau_g \right) \right) \right] \right)'_{s=l} + \left(\frac{2\varepsilon_1 \varepsilon_3 \kappa_n \tau(s)}{\varepsilon_2 \kappa_g^2 + \varepsilon_3 \kappa_n^2} \right)''_{s=l} \Bigg\} \\
& + \mu'(l) \left\{ \left(\frac{2\tau(s)}{\left(\varepsilon_2 \kappa_g^2 + \varepsilon_3 \kappa_n^2\right)^2} \left[\left(\varepsilon_1 \varepsilon_2 \kappa_g^2 + \varepsilon_1 \varepsilon_3 \kappa_n^2 \right) \left(-\varepsilon_3 \kappa_n' - 4\varepsilon_2 \varepsilon_3 \kappa_g \tau_g \right) \right. \right. \right. \\
& \left. + 2\varepsilon_1 \kappa_g \left(\varepsilon_2 \kappa_g \left(\varepsilon_3 \kappa_n' + \varepsilon_2 \varepsilon_3 \kappa_g \tau_g \right) - \varepsilon_3 \kappa_n \left(\varepsilon_2 \kappa_g' - \varepsilon_2 \varepsilon_3 \kappa_n \tau_g \right) \right) \right] \\
& \left. \left. - \left(\frac{2\varepsilon_1 \varepsilon_3 \kappa_n \tau(s)}{\varepsilon_2 \kappa_g^2 + \varepsilon_3 \kappa_n^2} \right)' \right\} \right|_{s=l} + \mu''(l) \left(\frac{2\varepsilon_1 \varepsilon_3 \kappa_n \tau(s)}{\varepsilon_2 \kappa_g^2 + \varepsilon_3 \kappa_n^2} \right)_{s=l} .
\end{aligned}
$$

2.1. Intrinsic equations for a relaxed elastic line of second kind on a spacelike surface

On a spacelike surface n is timelike. So, T and Q are spacelike and $\varepsilon_1 = \langle T, T\rangle = 1$, $\varepsilon_2 = \langle Q, Q\rangle = 1$, $\varepsilon_3 = \langle n, n\rangle = -1$. Hence $H'(0)$ can be written as

$$\begin{aligned} H'(0) \;=\; & \int_0^l \mu \Bigg\{ \kappa_g \tau^2(l) + \frac{2\tau(s)}{\left(\kappa_g^2-\kappa_n^2\right)^2}\Big[\left(\kappa_g^2-\kappa_n^2\right)\left(-3\kappa_g^2\kappa_n' - 2\kappa_g^3\tau_g + 3\kappa_g\kappa_n\kappa_g' + \tau_g^2\kappa_n' + 2\kappa_g\tau_g^3 + 2\kappa_g\kappa_n^2\tau_g \right.\\ & \left. -\kappa_g'\tau_g' - 4\kappa_n\tau_g\tau_g' + \kappa_g\tau_g''\right) + \left[-\kappa_g\left(\kappa_n' + \kappa_g\tau_g\right) + \kappa_n\left(\kappa_g' + \kappa_n\tau_g\right)\right]\left(-4\kappa_g^3 + 4\kappa_g\kappa_n^2 + 2\kappa_g\tau_g^2 - 2\kappa_n\tau_g'\right)\Big] \\ & -\Bigg[\frac{2\tau(s)}{\left(\kappa_g^2-\kappa_n^2\right)^2}\Big[\left(\kappa_g^2-\kappa_n^2\right)\left(-\kappa_g^2\kappa_n + \kappa_n^3 - 2\tau_g\kappa_g' - 5\kappa_n\tau_g^2 + 3\kappa_g\tau_g'\right) - 4\kappa_n\tau_g\left(-\kappa_g\left(\kappa_n' + \kappa_g\tau_g\right)\right.\\ & \left.\left. + \kappa_n\left(\kappa_g' + \kappa_n\tau_g\right)\right)\Big]\Bigg]' + \Bigg[\frac{2\tau(s)}{\left(\kappa_g^2-\kappa_n^2\right)^2}\Big[\left(\kappa_g^2-\kappa_n^2\right)\left(\kappa_n' + 4\kappa_g\tau_g\right) + 2\kappa_g\left(-\kappa_g\left(\kappa_n' + \kappa_g\tau_g\right)\right.\\ & \left.\left. + \kappa_n\left(\kappa_g' + \kappa_n\tau_g\right)\right)\Big]\Bigg]'' + \left(\frac{2\kappa_n\tau(s)}{\kappa_g^2-\kappa_n^2}\right)'''\Bigg\} ds + \mu(l)\Bigg\{\frac{2\tau(l)}{\left(\kappa_g^2(l)-\kappa_n^2(l)\right)^2}\Big[\left(\kappa_g^2-\kappa_n^2\right)\left(-\kappa_g^2\kappa_n + \kappa_n^3\right.\\ & \left. -2\tau_g\kappa_g' - 5\kappa_n\tau_g^2 + 3\kappa_g\tau_g'\right) - 4\kappa_n\tau_g\left(-\kappa_g\left(\kappa_n' + \kappa_g\tau_g\right) + \kappa_n\left(\kappa_g' + \kappa_n\tau_g\right)\right)\Big]_{s=l} \\ & -\left(\frac{2\tau(s)}{\left(\kappa_g^2-\kappa_n^2\right)^2}\Big[\left(\kappa_g^2-\kappa_n^2\right)\left(\kappa_n' + 4\kappa_g\tau_g\right) + 2\kappa_g\left(-\kappa_g\left(\kappa_n' + \kappa_g\tau_g\right) + \kappa_n\left(\kappa_g' + \kappa_n\tau_g\right)\right)\Big]\right)'_{s=l} \\ & -\left(\frac{2\kappa_n\tau(s)}{\kappa_g^2-\kappa_n^2}\right)''_{s=l}\Bigg\} + \mu'(l)\Bigg\{\Bigg(\frac{2\tau(s)}{\left(\kappa_g^2-\kappa_n^2\right)^2}\Big[\left(\kappa_g^2-\kappa_n^2\right)\left(\kappa_n' + 4\kappa_g\tau_g\right) \\ & +2\kappa_g\left(-\kappa_g\left(\kappa_n' + \kappa_g\tau_g\right) + \kappa_n\left(\kappa_g' + \kappa_n\tau_g\right)\right)\Big] + \left(\frac{2\kappa_n\tau(s)}{\kappa_g^2-\kappa_n^2}\right)'\Bigg\}_{s=l} - \mu''(l)\left(\frac{2\kappa_n\tau(s)}{\kappa_g^2-\kappa_n^2}\right)_{s=l}. \end{aligned}$$

In order that $H'(0) = 0$ for all choices of the function $\mu(s)$ satisfying (6), with arbitrary values of $\mu(l)$, $\mu'(l)$ and $\mu''(l)$, spacelike arc α must satisfy boundary conditions

$$\begin{aligned} & \frac{\tau(l)}{\left(\kappa_g^2(l)-\kappa_n^2(l)\right)^2}\Big[\left(\kappa_g^2-\kappa_n^2\right)\left(-\kappa_g^2\kappa_n + \kappa_n^3 - 2\tau_g\kappa_g' - 5\kappa_n\tau_g^2 + 3\kappa_g\tau_g'\right) - 4\kappa_n\tau_g\left(-\kappa_g\left(\kappa_n' + \kappa_g\tau_g\right) + \kappa_n\left(\kappa_g'\right.\right. \\ & \left.\left.+\kappa_n\tau_g\right)\right)\Big]_{s=l} - \left(\frac{\tau(s)}{\left(\kappa_g^2-\kappa_n^2\right)^2}\Big[\left(\kappa_g^2-\kappa_n^2\right)\left(\kappa_n' + 4\kappa_g\tau_g\right) + 2\kappa_g\left(-\kappa_g\left(\kappa_n' + \kappa_g\tau_g\right) + \kappa_n\left(\kappa_g' + \kappa_n\tau_g\right)\right)\Big]\right)'_{s=l} - \left(\frac{\kappa_n\tau(s)}{\kappa_g^2-\kappa_n^2}\right)''_{s=l} = 0, \end{aligned} \tag{20}$$

$$\left(\frac{\tau(s)}{\left(\kappa_g^2-\kappa_n^2\right)^2}\Big[\left(\kappa_g^2-\kappa_n^2\right)\left(\kappa_n' + 4\kappa_g\tau_g\right) + 2\kappa_g\left(-\kappa_g\left(\kappa_n' + \kappa_g\tau_g\right) + \kappa_n\left(\kappa_g' + \kappa_n\tau_g\right)\right)\Big]\right)_{s=l} + \left(\frac{\kappa_n\tau(s)}{\kappa_g^2-\kappa_n^2}\right)'_{s=l} = 0, \tag{21}$$

$$\kappa_n(l)\,\tau(l) = 0, \tag{22}$$

and the differential equation

$$\kappa_g\tau^2(l)+\frac{2\tau(s)}{\left(\kappa_g^2-\kappa_n^2\right)^2}\left[\left(\kappa_g^2-\kappa_n^2\right)\left(-3\kappa_g^2\kappa_n'-2\kappa_g^3\tau_g+3\kappa_g\kappa_n\kappa_g'+\tau_g^2\kappa_n'+2\kappa_g\tau_g^3+2\kappa_g\kappa_n^2\tau_g-\kappa_g'\tau_g'-4\kappa_n\tau_g\tau_g'\right.\right. \quad (23)$$
$$\left.\left.+\kappa_g\tau_g''\right)+\left[-\kappa_g\left(\kappa_n'+\kappa_g\tau_g\right)+\kappa_n\left(\kappa_g'+\kappa_n\tau_g\right)\right]\left(-4\kappa_g^3+4\kappa_g\kappa_n^2+2\kappa_g\tau_g^2-2\kappa_n\tau_g'\right)\right]$$
$$-\left[\frac{2\tau(s)}{\left(\kappa_g^2-\kappa_n^2\right)^2}\left[\left(\kappa_g^2-\kappa_n^2\right)\left(-\kappa_g^2\kappa_n+\kappa_n^3-2\tau_g\kappa_g'-5\kappa_n\tau_g^2+3\kappa_g\tau_g'\right)-4\kappa_n\tau_g\left(-\kappa_g\left(\kappa_n'+\kappa_g\tau_g\right)+\kappa_n\left(\kappa_g'+\kappa_n\tau_g\right)\right)\right]\right]'$$
$$+\left[\frac{2\tau(s)}{\left(\kappa_g^2-\kappa_n^2\right)^2}\left[\left(\kappa_g^2-\kappa_n^2\right)\left(\kappa_n'+4\kappa_g\tau_g\right)+2\kappa_g\left(-\kappa_g\left(\kappa_n'+\kappa_g\tau_g\right)+\kappa_n\left(\kappa_g'+\kappa_n\tau_g\right)\right)\right]\right]''+\left(\frac{2\kappa_n\tau(s)}{\kappa_g^2-\kappa_n^2}\right)'''=0.$$

Theorem 2.1. *The intrinsic equations for a relaxed elastic line of second kind on a connected oriented spacelike surface in Minkowski 3-space are given by the differential equation* (23) *with the boundary conditions* (20) – (22) *at the free end, where κ_g, κ_n and τ_g are the geodesic curvature, the normal curvature and the geodesic torsion as functions of the arc length along the curve.*

2.2. Intrinsic equations for a relaxed elastic line of second kind on a timelike surface for timelike arc α

Since α is timelike T is timelike. So, Q and n are spacelike and $\varepsilon_1=\langle T,T\rangle=-1$, $\varepsilon_2=\langle Q,Q\rangle=1$, $\varepsilon_3=\langle n,n\rangle=1$. Hence $H'(0)$ can be written as

$$H'(0) = \int_0^l \mu\left\{-\kappa_g\tau^2(l)+\frac{2\tau(s)}{\left(\kappa_g^2+\kappa_n^2\right)^2}\left(\left(\kappa_g^2+\kappa_n^2\right)\left(3\kappa_g^2\kappa_n'+2\kappa_g^3\tau_g-3\kappa_g\kappa_n\kappa_g'-\tau_g^2\kappa_n'-2\kappa_g\tau_g^3+2\kappa_g\kappa_n^2\tau_g\right.\right.\right.$$
$$\left.\left.-\kappa_g'\tau_g'+4\kappa_n\tau_g\tau_g'+\kappa_g\tau_g''\right)+\left[\kappa_g\left(\kappa_n'+\kappa_g\tau_g\right)-\kappa_n\left(\kappa_g'-\kappa_n\tau_g\right)\right]\left(-4\kappa_g^3-4\kappa_g\kappa_n^2+2\kappa_g\tau_g^2-2\kappa_n\tau_g'\right)\right)$$
$$-\left[\frac{2\tau(s)}{\left(\kappa_g^2+\kappa_n^2\right)^2}\left[\left(\kappa_g^2+\kappa_n^2\right)\left(\kappa_g^2\kappa_n+\kappa_n^3-2\tau_g\kappa_g'+5\kappa_n\tau_g^2+3\kappa_g\tau_g'\right)-4\kappa_n\tau_g\left(\kappa_g\left(\kappa_n'+\kappa_g\tau_g\right)\right.\right.\right.$$
$$\left.\left.\left.-\kappa_n\left(\kappa_g'-\kappa_n\tau_g\right)\right)\right]\right]'+\left[\frac{2\tau(s)}{\left(\kappa_g^2+\kappa_n^2\right)^2}\left[\left(\kappa_g^2+\kappa_n^2\right)\left(\kappa_n'+4\kappa_g\tau_g\right)-2\kappa_g\left(\kappa_g\left(\kappa_n'+\kappa_g\tau_g\right)-\kappa_n\left(\kappa_g'-\kappa_n\tau_g\right)\right)\right]\right]''$$
$$\left.+\left(\frac{2\kappa_n\tau(s)}{\kappa_g^2+\kappa_n^2}\right)'''\right\}ds+\mu(l)\left\{\frac{2\tau(l)}{\left(\kappa_g^2(l)+\kappa_n^2(l)\right)^2}\left[\left(\kappa_g^2+\kappa_n^2\right)\left(\kappa_g^2\kappa_n+\kappa_n^3-2\tau_g\kappa_g'+5\kappa_n\tau_g^2+3\kappa_g\tau_g'\right)\right.\right.$$
$$\left.-4\kappa_n\tau_g\left(\kappa_g\left(\kappa_n'+\kappa_g\tau_g\right)-\kappa_n\left(\kappa_g'-\kappa_n\tau_g\right)\right)\right]_{s=l}-\left(\frac{2\tau(s)}{\left(\kappa_g^2+\kappa_n^2\right)^2}\left[\left(\kappa_g^2+\kappa_n^2\right)\left(\kappa_n'+4\kappa_g\tau_g\right)\right.\right.$$
$$\left.\left.\left.-2\kappa_g\left(\kappa_g\left(\kappa_n'+\kappa_g\tau_g\right)-\kappa_n\left(\kappa_g'-\kappa_n\tau_g\right)\right)\right]\right)'_{s=l}-\left(\frac{2\kappa_n\tau(s)}{\kappa_g^2+\kappa_n^2}\right)''_{s=l}\right\}$$
$$+\mu'(l)\left\{\left(\frac{2\tau(s)}{\left(\kappa_g^2+\kappa_n^2\right)^2}\left[\left(\kappa_g^2+\kappa_n^2\right)\left(\kappa_n'+4\kappa_g\tau_g\right)-2\kappa_g\left(\kappa_g\left(\kappa_n'+\kappa_g\tau_g\right)-\kappa_n\left(\kappa_g'-\kappa_n\tau_g\right)\right)\right]\right.\right.$$
$$\left.\left.+\left(\frac{2\kappa_n\tau(s)}{\kappa_g^2+\kappa_n^2}\right)'\right\}_{s=l}-\mu''(l)\left(\frac{2\kappa_n\tau(s)}{\kappa_g^2+\kappa_n^2}\right)_{s=l}.$$

In order that $H'(0) = 0$ for all choices of the function $\mu(s)$ satisfying (6), with arbitrary values of $\mu(l)$, $\mu'(l)$ and $\mu''(l)$, spacelike arc α must satisfy boundary conditions

$$\frac{\tau(l)}{\left(\kappa_g^2(l)+\kappa_n^2(l)\right)^2}\left[\left(\kappa_g^2+\kappa_n^2\right)\left(\kappa_g^2\kappa_n+\kappa_n^3-2\tau_g\kappa_g'+5\kappa_n\tau_g^2+3\kappa_g\tau_g'\right)-4\kappa_n\tau_g\left(\kappa_g\left(\kappa_n'+\kappa_g\tau_g\right)\right.\right.$$
$$\left.\left.-\kappa_n\left(\kappa_g'-\kappa_n\tau_g\right)\right)\right]_{s=l}-\left(\frac{\tau(s)}{\left(\kappa_g^2+\kappa_n^2\right)^2}\left[\left(\kappa_g^2+\kappa_n^2\right)\left(\kappa_n'+4\kappa_g\tau_g\right)\right.\right.$$
$$\left.\left.-2\kappa_g\left(\kappa_g\left(\kappa_n'+\kappa_g\tau_g\right)-\kappa_n\left(\kappa_g'-\kappa_n\tau_g\right)\right)\right]\right)'_{s=l}-\left(\frac{\kappa_n\tau(s)}{\kappa_g^2+\kappa_n^2}\right)''_{s=l}=0, \tag{24}$$

$$\left(\frac{\tau(s)}{\left(\kappa_g^2+\kappa_n^2\right)^2}\left[\left(\kappa_g^2+\kappa_n^2\right)\left(\kappa_n'+4\kappa_g\tau_g\right)-2\kappa_g\left(\kappa_g\left(\kappa_n'+\kappa_g\tau_g\right)-\kappa_n\left(\kappa_g'-\kappa_n\tau_g\right)\right)\right]\right)_{s=l}+\left(\frac{\kappa_n\tau(s)}{\kappa_g^2+\kappa_n^2}\right)'_{s=l}=0, \tag{25}$$

$$\kappa_n(l)\,\tau(l)=0, \tag{26}$$

and the differential equation

$$-\kappa_g\tau^2(l)+\frac{2\tau(s)}{\left(\kappa_g^2+\kappa_n^2\right)^2}\left(\left(\kappa_g^2+\kappa_n^2\right)\left(3\kappa_g^2\kappa_n'+2\kappa_g^3\tau_g-3\kappa_g\kappa_n\kappa_g'-\tau_g^2\kappa_n'-2\kappa_g\tau_g^3+2\kappa_g\kappa_n^2\tau_g-\kappa_g'\tau_g'+4\kappa_n\tau_g\tau_g'\right.\right.$$
$$\left.\left.+\kappa_g\tau_g''\right)+\left[\kappa_g\left(\kappa_n'+\kappa_g\tau_g\right)-\kappa_n\left(\kappa_g'-\kappa_n\tau_g\right)\right]\left(-4\kappa_g^3-4\kappa_g\kappa_n^2+2\kappa_g\tau_g^2-2\kappa_n\tau_g'\right)\right) \tag{27}$$
$$-\left[\frac{2\tau(s)}{\left(\kappa_g^2+\kappa_n^2\right)^2}\left[\left(\kappa_g^2+\kappa_n^2\right)\left(\kappa_g^2\kappa_n+\kappa_n^3-2\tau_g\kappa_g'+5\kappa_n\tau_g^2+3\kappa_g\tau_g'\right)-4\kappa_n\tau_g\left(\kappa_g\left(\kappa_n'+\kappa_g\tau_g\right)-\kappa_n\left(\kappa_g'-\kappa_n\tau_g\right)\right)\right]\right]'$$
$$+\left[\frac{2\tau(s)}{\left(\kappa_g^2+\kappa_n^2\right)^2}\left[\left(\kappa_g^2+\kappa_n^2\right)\left(\kappa_n'+4\kappa_g\tau_g\right)-2\kappa_g\left(\kappa_g\left(\kappa_n'+\kappa_g\tau_g\right)-\kappa_n\left(\kappa_g'-\kappa_n\tau_g\right)\right)\right]\right]''+\left(\frac{2\kappa_n\tau(s)}{\kappa_g^2+\kappa_n^2}\right)'''=0.$$

Theorem 2.2. *Let α be a timelike arc. The intrinsic equations for α to be a relaxed elastic line of second kind on a connected oriented timelike surface in Minkowski 3-space are given by the differential equation* (27) *with the boundary conditions* (24) – (26) *at the free end, where κ_g, κ_n and τ_g are the geodesic curvature, the normal curvature and the geodesic torsion as functions of the arc length along the curve.*

2.3. Intrinsic equations for a relaxed elastic line of second kind on a timelike surface for spacelike arc α

Now Q is timelike, T and n are spacelike. So, $\varepsilon_1 = \langle T, T\rangle = 1$, $\varepsilon_2 = \langle Q, Q\rangle = -1$, $\varepsilon_3 = \langle n, n\rangle = 1$. Hence $H'(0)$ can be written as

$$\begin{aligned}
H'(0) \;=\; & \int_0^l \mu \Bigg\{ \kappa_g \tau^2(l) + \frac{2\tau(s)}{\left(\kappa_n^2-\kappa_g^2\right)^2}\Big(\left(\kappa_n^2-\kappa_g^2\right)\left(-3\kappa_g^2\kappa_n' + 2\kappa_g^3\tau_g + 3\kappa_g\kappa_n\kappa_g' - \tau_g^2\kappa_n' + 2\kappa_g\tau_g^3 - 2\kappa_g\kappa_n^2\tau_g\right. \\
& \left. -\kappa_g'\tau_g' + 4\kappa_n\tau_g\tau_g' + \kappa_g\tau_g''\right) + \left[-\kappa_g\left(\kappa_n' - \kappa_g\tau_g\right) - \kappa_n\left(-\kappa_g' + \kappa_n\tau_g\right)\right]\left(4\kappa_g^3 - 4\kappa_g\kappa_n^2 + 2\kappa_g\tau_g^2 + 2\kappa_n\tau_g'\right)\Big) \\
& - \Bigg[\frac{2\tau(s)}{\left(\kappa_n^2-\kappa_g^2\right)^2}\Big[\left(\kappa_n^2-\kappa_g^2\right)\left(-\kappa_g^2\kappa_n + \kappa_n^3 - 2\tau_g\kappa_g' + 5\kappa_n\tau_g^2 + 3\kappa_g\tau_g'\right) + 4\kappa_n\tau_g\Big(-\kappa_g\left(\kappa_n' - \kappa_g\tau_g\right) \\
& -\kappa_n\left(-\kappa_g' + \kappa_n\tau_g\right)\Big)\Big]\Bigg]' + \Bigg[\frac{2\tau(s)}{\left(\kappa_n^2-\kappa_g^2\right)^2}\Big[\left(\kappa_n^2-\kappa_g^2\right)\left(-\kappa_n' + 4\kappa_g\tau_g\right) \\
& +2\kappa_g\left(-\kappa_g\left(\kappa_n' - \kappa_g\tau_g\right) - \kappa_n\left(-\kappa_g' + \kappa_n\tau_g\right)\right)\Big]\Bigg]'' - \left(\frac{2\kappa_n\tau(s)}{\kappa_n^2-\kappa_g^2}\right)''' \Bigg\} ds \\
& +\mu(l)\Bigg\{\frac{2\tau(l)}{\left(\kappa_n^2(l)-\kappa_g^2(l)\right)^2}\Big[\left(\kappa_n^2-\kappa_g^2\right)\left(-\kappa_g^2\kappa_n + \kappa_n^3 - 2\tau_g\kappa_g' + 5\kappa_n\tau_g^2 + 3\kappa_g\tau_g'\right) + 4\kappa_n\tau_g\Big(-\kappa_g\left(\kappa_n' - \kappa_g\tau_g\right) \\
& -\kappa_n\left(-\kappa_g' + \kappa_n\tau_g\right)\Big)\Big]_{s=l} - \Bigg(\frac{2\tau(s)}{\left(\kappa_n^2-\kappa_g^2\right)^2}\Big[\left(\kappa_n^2-\kappa_g^2\right)\left(-\kappa_n' + 4\kappa_g\tau_g\right) + 2\kappa_g\Big(-\kappa_g\left(\kappa_n' - \kappa_g\tau_g\right) \\
& -\kappa_n\left(-\kappa_g' + \kappa_n\tau_g\right)\Big)\Big]\Bigg)'_{s=l} + \left(\frac{2\kappa_n\tau(s)}{\kappa_n^2-\kappa_g^2}\right)''_{s=l}\Bigg\} + \mu'(l)\Bigg\{\Bigg(\frac{2\tau(s)}{\left(\kappa_n^2-\kappa_g^2\right)^2}\Big[\left(\kappa_n^2-\kappa_g^2\right)\left(-\kappa_n' + 4\kappa_g\tau_g\right) \\
& +2\kappa_g\left(-\kappa_g\left(\kappa_n' - \kappa_g\tau_g\right) - \kappa_n\left(-\kappa_g' + \kappa_n\tau_g\right)\right)\Big] - \left(\frac{2\kappa_n\tau(s)}{\kappa_n^2-\kappa_g^2}\right)'\Bigg\}_{s=l} + \mu''(l)\left(\frac{2\kappa_n\tau(s)}{\kappa_n^2-\kappa_g^2}\right)_{s=l}.
\end{aligned}$$

In order that $H'(0) = 0$ for all choices of the function $\mu(s)$ satisfying (6), with arbitrary values of $\mu(l)$, $\mu'(l)$ and $\mu''(l)$, spacelike arc α must satisfy boundary conditions

$$\begin{aligned}
& \frac{\tau(l)}{\left(\kappa_n^2(l)-\kappa_g^2(l)\right)^2}\Big[\left(\kappa_n^2-\kappa_g^2\right)\left(-\kappa_g^2\kappa_n + \kappa_n^3 - 2\tau_g\kappa_g' + 5\kappa_n\tau_g^2 + 3\kappa_g\tau_g'\right) + 4\kappa_n\tau_g\Big(-\kappa_g\left(\kappa_n' - \kappa_g\tau_g\right) \\
& -\kappa_n\left(-\kappa_g' + \kappa_n\tau_g\right)\Big)\Big]_{s=l} - \Bigg(\frac{\tau(s)}{\left(\kappa_n^2-\kappa_g^2\right)^2}\Big[\left(\kappa_n^2-\kappa_g^2\right)\left(-\kappa_n' + 4\kappa_g\tau_g\right) + 2\kappa_g\Big(-\kappa_g\left(\kappa_n' - \kappa_g\tau_g\right) \\
& -\kappa_n\left(-\kappa_g' + \kappa_n\tau_g\right)\Big)\Big]\Bigg)'_{s=l} + \left(\frac{\kappa_n\tau(s)}{\kappa_n^2-\kappa_g^2}\right)''_{s=l} = 0,
\end{aligned} \tag{28}$$

$$\frac{\tau(l)}{\left(\kappa_n^2(l)-\kappa_g^2(l)\right)^2}\Big[\left(\kappa_n^2-\kappa_g^2\right)\left(-\kappa_n' + 4\kappa_g\tau_g\right) + 2\kappa_g\left(-\kappa_g\left(\kappa_n' - \kappa_g\tau_g\right) - \kappa_n\left(-\kappa_g' + \kappa_n\tau_g\right)\right)\Big]_{s=l} - \left(\frac{\kappa_n\tau(s)}{\kappa_n^2-\kappa_g^2}\right)'_{s=l} = 0, \tag{29}$$

$$\kappa_n(l)\,\tau(l) = 0, \tag{30}$$

and the differential equation

$$\begin{aligned}
&\kappa_g \tau^2(l) + \frac{2\tau(s)}{\left(\kappa_n^2-\kappa_g^2\right)^2}\Big(\left(\kappa_n^2-\kappa_g^2\right)\left(-3\kappa_g^2\kappa_n' + 2\kappa_g^3\tau_g + 3\kappa_g\kappa_n\kappa_g' - \tau_g^2\kappa_n' + 2\kappa_g\tau_g^3 - 2\kappa_g\kappa_n^2\tau_g - \kappa_g'\tau_g' + 4\kappa_n\tau_g\tau_g'\right.\\
&\left.+\kappa_g\tau_g''\right) + \left[-\kappa_g\left(\kappa_n' - \kappa_g\tau_g\right) - \kappa_n\left(-\kappa_g' + \kappa_n\tau_g\right)\right]\left(4\kappa_g^3 - 4\kappa_g\kappa_n^2 + 2\kappa_g\tau_g^2 + 2\kappa_n\tau_g'\right)\Big) \qquad (31)\\
&-\left[\frac{2\tau(s)}{\left(\kappa_n^2-\kappa_g^2\right)^2}\left[\left(\kappa_n^2-\kappa_g^2\right)\left(-\kappa_g^2\kappa_n + \kappa_n^3 - 2\tau_g\kappa_g' + 5\kappa_n\tau_g^2 + 3\kappa_g\tau_g'\right) + 4\kappa_n\tau_g\left(-\kappa_g\left(\kappa_n' - \kappa_g\tau_g\right)\right.\right.\right.\\
&\left.\left.\left.-\kappa_n\left(-\kappa_g' + \kappa_n\tau_g\right)\right)\right]\right]' + \left[\frac{2\tau(s)}{\left(\kappa_n^2-\kappa_g^2\right)^2}\left[\left(\kappa_n^2-\kappa_g^2\right)\left(-\kappa_n' + 4\kappa_g\tau_g\right)\right.\right.\\
&\left.\left.+2\kappa_g\left(-\kappa_g\left(\kappa_n' - \kappa_g\tau_g\right) - \kappa_n\left(-\kappa_g' + \kappa_n\tau_g\right)\right)\right]\right]'' - \left(\frac{2\kappa_n\tau(s)}{\kappa_n^2-\kappa_g^2}\right)''' = 0.
\end{aligned}$$

Theorem 2.3. *Let α be a spacelike arc. The intrinsic equations for α to be a relaxed elastic line of second kind on a connected oriented timelike surface in Minkowski 3-space are given by the differential equation* (31) *with the boundary conditions* (28) – (30) *at the free end, where κ_g, κ_n and τ_g are the geodesic curvature, the normal curvature and the geodesic torsion as functions of the arc length along the curve.*

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