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Local Classification and Examples of an Important Class of Paracontact Metric Manifolds

Verónica Martín-Molina[a]

[a]*Departamento de Didáctica de las Matemáticas, Facultad de Ciencias de la Educación, Universidad de Sevilla, c/ Pirotecnia s/n, 41013, Sevilla, Spain*

Abstract. We study paracontact metric (κ,μ)-spaces with $\kappa = -1$, equivalent to $h^2 = 0$ but not $h = 0$. In particular, we will give an alternative proof of Theorem 3.2 of [11] and present examples of paracontact metric $(-1, 2)$-spaces and $(-1, 0)$-spaces of arbitrary dimension with tensor h of every possible constant rank. We will also show explicit examples of paracontact metric $(-1,\mu)$-spaces with tensor h of non-constant rank, which were not known to exist until now.

1. Introduction

Paracontact metric manifolds, the odd-dimensional analogue of paraHermitian manifolds, were first introduced in [10] and they have been the object of intense study recently, particularly since the publication of [14]. An important class among paracontact metric manifolds is that of the (κ,μ)-spaces, which satisfy the nullity condition [5]

$$R(X,Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY), \tag{1}$$

for all X, Y vector fields on M, where κ and μ are constants and $h = \frac{1}{2}L_\xi\varphi$.

This class includes the paraSasakian manifolds [10, 14], the paracontact metric manifolds satisfying $R(X,Y)\xi = 0$ for all X, Y [15], certain g-natural paracontact metric structures constructed on unit tangent sphere bundles [7], etc.

The definition of a paracontact metric (κ,μ)-space was motivated by the relationship between contact metric and paracontact geometry. More precisely, it was proved in [4] that any non-Sasakian contact metric (κ,μ)-space accepts two paracontact metric $(\widetilde{\kappa},\widetilde{\mu})$-structures with the same contact form. On the other hand, under certain natural conditions, every non-paraSasakian paracontact $(\widetilde{\kappa},\widetilde{\mu})$-space admits a contact metric (κ,μ)-structure compatible with the same contact form ([5]).

Paracontact metric (κ,μ)-spaces satisfy that $h^2 = (\kappa+1)\phi^2$ but this condition does not give any type of restriction over the value of κ, unlike in contact metric geometry, because the metric of a paracontact metric manifold is not positive definite. However, it is useful to distinguish the cases $\kappa > -1$, $\kappa < -1$ and $\kappa = -1$. In the first two, equation (1) determines the curvature completely and either the tensor h or φh are

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Email address: veronicamartin@us.es (Verónica Martín-Molina)

diagonalisable [5]. The case $\kappa = -1$ is equivalent to $h^2 = 0$ but not to $h = 0$. Indeed, there are examples of paracontact metric (κ, μ)-spaces with $h^2 = 0$ but $h \neq 0$, as was first shown in [2, 5, 8, 12].

However, only some particular examples were given of this type of space and no effort had been made to understand the general behaviour of the tensor h of a paracontact metric $(-1, \mu)$-space until the author published [11], where a local classification depending on the rank of h was given in Theorem 3.2. The author also provided explicit examples of all the possible constant values of the rank of h when $\mu = 2$. She explained why the values $\mu = 0$ and $\mu = 2$ are special and studying them is enough. Finally, she showed some paracontact metric $(-1, 0)$-spaces of any dimension with $\operatorname{rank}(h) = 1$ and of paracontact metric $(-1, 0)$-spaces of dimension 5 and 7 for any possible constant rank of h. These were the first examples of this type with $\mu \neq 2$ and dimension greater than 3.

In the present paper, after the preliminaries section, we will give an alternative proof of Theorem 3.2 of [11] that does not use [13] and we will complete the examples of all the possible cases of constant rank of h by presenting $(2n + 1)$-dimensional paracontact metric $(-1, 0)$-spaces with $\operatorname{rank}(h) = 2, \ldots, n$. Lastly, we will also show the first explicit examples ever known of paracontact metric $(-1, 2)$-spaces and $(-1, 0)$-spaces with h of non-constant rank.

2. Preliminaries

An *almost paracontact structure* on a $(2n + 1)$-dimensional smooth manifold M is given by a $(1, 1)$-tensor field φ, a vector field ξ and a 1-form η satisfying the following conditions [10]:

(i) $\eta(\xi) = 1,\ \varphi^2 = I - \eta \otimes \xi$,

(ii) the eigendistributions $\mathcal{D}^+$ and $\mathcal{D}^-$ of φ corresponding to the eigenvalues 1 and -1, respectively, have equal dimension n.

It follows that $\varphi\xi = 0$, $\eta \circ \varphi = 0$ and $\operatorname{rank}(\varphi) = 2n$. If an almost paracontact manifold admits a semi-Riemannian metric g such that

$$g(\varphi X, \varphi Y) = -g(X, Y) + \eta(X)\eta(Y),$$

for all X, Y on M, then $(M, \varphi, \xi, \eta, g)$ is called an *almost paracontact metric manifold*. Then g is necessarily of signature $(n + 1, n)$ and satisfies $\eta = g(\cdot, \xi)$ and $g(\cdot, \varphi\cdot) = -g(\varphi\cdot, \cdot)$.

We can now define the *fundamental* 2*-form* of the almost paracontact metric manifold by $\Phi(X, Y) = g(X, \varphi Y)$. If $d\eta = \Phi$, then η becomes a contact form (i.e. $\eta \wedge (d\eta)^n \neq 0$) and $(M, \varphi, \xi, \eta, g)$ is said to be a *paracontact metric manifold*.

We can also define on a paracontact metric manifold the tensor field $h := \frac{1}{2}L_\xi\varphi$, which is symmetric with respect to g (i.e. $g(hX, Y) = g(X, hY)$, for all X, Y), anti-commutes with φ and satisfies $h\xi = \operatorname{tr}h = 0$ and the identity $\nabla\xi = -\varphi + \varphi h$ ([14]). Moreover, it vanishes identically if and only if ξ is a Killing vector field, in which case $(M, \varphi, \xi, \eta, g)$ is called a *K-paracontact manifold*.

An almost paracontact structure is said to be *normal* if and only if the tensor $[\varphi, \varphi] - 2d\eta \otimes \xi = 0$, where $[\varphi, \varphi]$ is the Nijenhuis tensor of φ [14]:

$$[\varphi, \varphi](X, Y) = \varphi^2[X, Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y].$$

A normal paracontact metric manifold is said to be a *paraSasakian manifold* and is in particular *K*-paracontact. The converse holds in dimension 3 ([6]) but not in general in higher dimensions. However, it was proved in Theorem 3.1 of [11] that it also holds for $(-1, \mu)$-spaces. Every paraSasakian manifold satisfies

$$R(X, Y)\xi = -(\eta(Y)X - \eta(X)Y), \tag{2}$$

for every X, Y on M. The converse is not true, since Examples 3.8–3.11 of [11] and Examples 4.1 and 4.5 of the present one show that there are examples of paracontact metric manifolds satisfying equation (2) but with $h \neq 0$ (and therefore not K-paracontact or paraSasakian). Moreover, it is also clear in Example 4.5 that the rank of h does not need to be constant either, since h can be zero at some points and non-zero in others.

The main result of [11] is the following local classification of paracontact metric $(-1, \mu)$-spaces:

Theorem 2.1 ([11]). *Let M be a $(2n+1)$-dimensional paracontact metric $(-1,\mu)$-space. Then we have one of the following possibilities:*

1. *either $h = 0$ and M is paraSasakian,*
2. *or $h \neq 0$ and rank$(h_p) \in \{1,\dots,n\}$ at every $p \in M$ where $h_p \neq 0$. Moreover, there exists a basis $\{\xi_p, X_1, Y_1, \dots, X_n, Y_n\}$ of $T_p(M)$ such that the only non-vanishing components of g are*

$$g_p(\xi_p,\xi_p) = 1, \quad g_p(X_i,Y_i) = \pm 1,$$

and

$$h_{p|\langle X_i,Y_i\rangle} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad or \quad h_{p|\langle X_i,Y_i\rangle} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

where obviously there are exactly rank(h_p) submatrices of the first type.
If $n = 1$, such a basis $\{\xi_p, X_1, Y_1\}$ also satisfies that

$$\varphi_p X_1 = \pm X_1, \quad \varphi_p Y_1 = \mp Y_1,$$

and the tensor h can be written as

$$h_{p|\langle \xi_p, X_1, Y_1\rangle} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Many examples of paraSasakian manifolds are known. For instance, hyperboloids $\mathbb{H}^{2n+1}_{n+1}(1)$ and the hyperbolic Heisenberg group $\mathcal{H}^{2n+1} = \mathbb{R}^{2n} \times \mathbb{R}$, [9]. We can also obtain (η-Einstein) paraSasakian manifolds from contact (κ,μ)-spaces with $|1 - \frac{\mu}{2}| < \sqrt{1-\kappa}$. In particular, the tangent sphere bundle T_1N of any space form $N(c)$ with $c < 0$ admits a canonical η-Einstein paraSasakian structure, [3]. Finally, we can see how to construct explicitly a paraSasakian structure on a Lie group (see Example 3.4 of [11]) or on the unit tangent sphere bundle, [7].

On the other hand, until [11] only some types of non-paraSasakian paracontact metric $(-1,\mu)$-spaces were known:

- $(2n+1)$-dimensional paracontact metric $(-1,2)$-space with rank$(h) = n$, [5].
- 3-dimensional paracontact metric $(-1,2)$-space with rank$(h) = n = 1$, [12].
- 3-dimensional paracontact metric $(-1,0)$-space with rank$(h) = n = 1$. This example is not paraSasakian but it satisfies (2), [8].

The answer to why there seems to be only examples of paracontact metric $(-1,\mu)$-spaces with $\mu = 2$ or $\mu = 0$ is a $\mathcal{D}_c$-homothetic deformation, i.e. the following change of a paracontact metric structure (M,φ,ξ,η,g) [14]:

$$\varphi' := \varphi, \quad \xi' := \frac{1}{c}\xi, \quad \eta' := c\eta, \quad g' := cg + c(c-1)\eta\otimes\eta,$$

for some $c \neq 0$.

It is known that (φ',ξ',η',g') is again a paracontact metric structure on M and that K-paracontact and paraSasakian structures are also preserved. However, curvature conditions like $R(X,Y)\xi = 0$ are destroyed, since paracontact metric (κ,μ)-spaces become other paracontact metric (κ',μ')-spaces with

$$\kappa' = \frac{\kappa + 1 - c^2}{c^2}, \quad \mu' = \frac{\mu - 2 + 2c}{c}.$$

In particular, if $(M, \varphi, \xi, \eta, g)$ is a paracontact metric $(-1, \mu)$-space, then the deformed manifold is another paracontact metric $(-1, \mu')$-space with $\mu' = \frac{\mu-2+2c}{c}$.

Therefore, given a $(-1, 2)$-space, a $\mathcal{D}_c$-homothetic deformation with arbitrary $c \neq 0$ will give us another paracontact metric $(-1, 2)$-space. Given a paracontact metric $(-1, 0)$-space, if we $\mathcal{D}_c$-homothetically deform it with $c = \frac{2}{2-\mu} \neq 0$ for some $\mu \neq 2$, we will obtain a paracontact metric $(-1, \mu)$-space with $\mu \neq 2$. A sort of converse is also possible: given a $(-1, \mu)$-space with $\mu \neq 2$, a $\mathcal{D}_c$-homothetic deformation with $c = 1 - \frac{\mu}{2} \neq 0$ will give us a paracontact metric $(-1, 0)$-space. The case $\mu = 0$, $h \neq 0$ is also special because the manifold satisfies (2) but it is not paraSasakian.

Examples of non-paraSasakian paracontact metric $(-1, 2)$-spaces of any possible dimension and constant rank of h were presented in [11]:

Example 2.2 ($(2n+1)$-dimensional paracontact metric $(-1, 2)$-space with rank$(h) = m \in \{1, \ldots, n\}$). *Let $\mathfrak{g}$ be the $(2n+1)$-dimensional Lie algebra with basis $\{\xi, X_1, Y_1, \ldots, X_n, Y_n\}$ such that the only non-zero Lie brackets are:*

$$[\xi, X_i] = Y_i, \quad i = 1, \ldots, m,$$

$$[X_i, Y_j] = \begin{cases} \delta_{ij}(2\xi + \sqrt{2}(1 + \delta_{im})Y_m) + (1 - \delta_{ij})\sqrt{2}(\delta_{im}Y_j + \delta_{jm}Y_i), & i, j = 1, \ldots, m, \\ \delta_{ij}(2\xi + \sqrt{2}Y_i), & i, j = m+1, \ldots, n, \\ \sqrt{2}Y_i, & i = 1, \ldots, m, \ j = m+1, \ldots, n. \end{cases}$$

If we denote by G the Lie group whose Lie algebra is $\mathfrak{g}$, we can define a left-invariant paracontact metric structure on G the following way:

$$\varphi\xi = 0, \quad \varphi X_i = X_i, \quad \varphi Y_i = -Y_i, \quad i = 1, \ldots, n,$$

$$\eta(\xi) = 1, \quad \eta(X_i) = \eta(Y_i) = 0, \quad i = 1, \ldots, n.$$

The only non-vanishing components of the metric are

$$g(\xi, \xi) = g(X_i, Y_i) = 1, \quad i = 1, \ldots, n.$$

A straightforward computation gives that $hX_i = Y_i$ if $i = 1, \ldots, m$, $hX_i = 0$ if $i = m+1, \ldots, n$ and $hY_j = 0$ if $j = 1, \ldots, n$, so $h^2 = 0$ and rank$(h) = m$. Furthermore, the manifold is a $(-1, 2)$-space.

Examples of non-paraSasakian paracontact metric $(-1, 0)$-spaces of any possible dimension and rank$(h) = 1$ were also given in [11]:

Example 2.3 ($(2n+1)$-dimensional paracontact metric $(-1, 0)$-space with rank$(h) = 1$). *Let $\mathfrak{g}$ be the $(2n+1)$-dimensional Lie algebra with basis $\{\xi, X_1, Y_1, \ldots, X_n, Y_n\}$ such that the only non-zero Lie brackets are:*

$$[\xi, X_1] = X_1 + Y_1, \qquad [\xi, Y_1] = -Y_1, \qquad [X_1, Y_1] = 2\xi,$$
$$[X_i, Y_i] = 2(\xi + Y_i), \qquad [X_1, Y_i] = X_1 + Y_1, \qquad [Y_1, Y_i] = -Y_1, \quad i = 2, \ldots, n.$$

If we denote by G the Lie group whose Lie algebra is $\mathfrak{g}$, we can define a left-invariant paracontact metric structure on G the following way:

$$\varphi\xi = 0, \quad \varphi X_1 = X_1, \quad \varphi Y_1 = -Y_1, \quad \varphi X_i = -X_i, \quad \varphi Y_i = Y_i, \quad i = 2, \ldots, n,$$

$$\eta(\xi) = 1, \quad \eta(X_i) = \eta(Y_i) = 0, \quad i = 1, \ldots, n.$$

The only non-vanishing components of the metric are

$$g(\xi, \xi) = g(X_1, Y_1) = 1, \quad g(X_i, Y_i) = -1, \quad i = 2, \ldots, n.$$

A straightforward computation gives that $hX_1 = Y_1$, $hY_1 = 0$ and $hX_i = hY_i = 0$, $i = 2, \ldots, n$, so $h^2 = 0$ and rank$(h) = 1$.

Moreover, by basic paracontact metric properties and Koszul's formula we obtain that

$$\nabla_\xi X_1 = 0, \quad \nabla_\xi Y_1 = 0, \quad \nabla_\xi X_i = X_i, \quad \nabla_\xi Y_i = -Y_i, \quad i = 2, \dots, n,$$
$$\nabla_{X_i} Y_1 = \delta_{i1}\xi, \quad \nabla_{X_i} Y_j = \delta_{ij}(\xi + 2Y_i), \quad \nabla_{Y_1} X_1 = -\xi, \quad \nabla_{Y_i} X_j = -\delta_{ij}\xi, \quad i, j = 2, \dots, n,$$
$$\nabla_{X_1} X_j = 0, \quad \nabla_{Y_1} Y_1 = \nabla_{Y_1} Y_j = 0, \quad \nabla_{Y_j} Y_1 = Y_1, \quad i = 2, \dots, n,$$

and thus

$$R(X_i, \xi)\xi = -X_i, \quad i = 1, \dots, n,$$
$$R(Y_i, \xi)\xi = -Y_i, \quad i = 1, \dots, n,$$
$$R(X_i, X_j)\xi = R(X_i, Y_j)\xi = R(Y_i, Y_j)\xi = 0, \quad i, j = 1, \dots, n.$$

Therefore, the manifold is also a $(-1, 0)$*-space.*

To our knowledge, the previous example is the first paracontact metric $(-1, \mu)$-space with $h^2 = 0$, $h \neq 0$ and $\mu \neq 2$ that was constructed in dimensions greater than 3. For dimension 3, Example 4.6 of [8] was already known.

In dimension 5, there also exist examples of paracontact metric $(-1, 0)$-space with $\operatorname{rank}(h) = 2$ and in dimension 7 of $\operatorname{rank}(h) = 2, 3$, as shown in [11]. Higher-dimensional examples of paracontact metric $(-1, 0)$-spaces with $\operatorname{rank}(h) \geq 2$ were not included, which will be remedied in Example 4.1. We will also see how to construct a 3-dimensional paracontact metric $(-1, 0)$-space and $(-1, 2)$-space where the rank of h is not constant.

3. New proof of Theorem 2.1

We will now present a revised proof of Theorem 2.1 that does not use [13] when $h \neq 0$ but constructs the basis explicitly.

Proof. Since $\kappa = -1$, we know from [5] that $h^2 = 0$. If $h = 0$, then $R(X, Y)\xi = -(\eta(Y)X - \eta(X)Y)$, for all X, Y on M and ξ is a Killing vector field, so Theorem 3.1 of [11] gives us that the manifold is paraSasakian.

If $h \neq 0$, then let us take a point $p \in M$ such that $h_p \neq 0$. We know that ξ is a global vector field such that $g(\xi, \xi) = 1$, that $h\xi = 0$ and that h is self-adjoint, so $Ker\eta_p$ is h-invariant and $h_p : Ker\eta_p \mapsto Ker\eta_p$ is a non-zero linear map such that $h_p^2 = 0$. We will now construct a basis $\{X_1, Y_1, \dots, X_n, Y_n\}$ of $Ker\eta_p$ that satisfies all of our requirements.

Take a non-zero vector $v \in Ker\eta_p$ such that $h_p v \neq 0$, which we know exists because $h_p \neq 0$. Then we write $Ker\eta_p = L_1 \oplus L_1^\perp$, where $L_1 = \langle v, h_p v \rangle$. Both L_1 and $L_1^\perp$ are h_p-invariant because h_p is self-adjoint. Moreover, $g_p(v, h_p v) \neq 0$ because $g_p(h_p v, h_p v) = 0 = g_p(h_p v, w)$ for all $w \in L_1^\perp$, $h_p v \neq 0$ and g is a non-degenerate metric.We now distinguish two cases:

1. If $g_p(v, v) = 0$, then we can take $X_i = \frac{1}{\sqrt{|g_p(v, h_p v)|}} v$ and $Y_i = \frac{1}{\sqrt{|g_p(v, h_p v)|}} h_p v$, which satisfy that $g_p(X_i, X_i) = 0 = g_p(Y_i, Y_i)$, $g_p(X_i, Y_i) = \pm 1$ and $h_p X_i = Y_i$.
2. If $g_p(v, v) \neq 0$, then $v' = v - \frac{g_p(v,v)}{g_p(v, h_p v)} h_p v$ satisfies that $g_p(v', v') = 0$, so we can take $X_i = \frac{1}{\sqrt{|g_p(v', h_p v)'|}} v'$, $Y_i = \frac{1}{\sqrt{|g_p(v', h_p v')|}} hv'$. We have again that $g_p(X_i, X_i) = 0 = g_p(Y_i, Y_i)$, $g_p(X_i, Y_i) = \pm 1$ and $h_p X_i = Y_i$.

In both cases, $L_1 = \langle X_i, Y_i \rangle$, so we now take a non-zero vector $v \in L_1^\perp$ and check if $h_p v \neq 0$. We know that we can take v such that $h_p v \neq 0$ in this step as many times as the rank of h_p, which is at minimum 1 (since $h_p \neq 0$) and at most n because $\dim Ker\eta_p = 2n$ and the space L_1 has dimension 2.

If we denote by m the rank of h_p, then we can write $Ker\eta_p$ as the following direct sum of mutually orthogonal subspaces:

$$Ker\eta_p = L_1 \oplus L_2 \oplus \cdots \oplus L_m \oplus V = \langle X_1, Y_1, \dots, X_m, Y_m \rangle \oplus V,$$

where $h_p v = 0$ for all $v \in V$. Each L_i is of signature $(1,1)$ because $\{\tilde{X}_i = \frac{1}{\sqrt{2}}(X_i + Y_i), \tilde{Y}_i = \frac{1}{\sqrt{2}}(X_i - Y_i)\}$ is a pseudo-orthonormal basis such that $g_p(\tilde{X}_i, \tilde{X}_i) = -g_p(\tilde{Y}_i, \tilde{Y}_i) = g_p(X_i, Y_i) = \pm 1$, $g_p(\tilde{X}_i, \tilde{Y}_i) = 0$. Therefore, $\langle X_1, Y_1, \ldots, X_m, Y_m \rangle$ is of signature (m,m) and, since $Ker\eta_p$ is of signature (n,n), we can take a pseudo-orthonormal basis $\{v_1, \ldots, v_{n-m}, w_1, \ldots, w_{n-m}\}$ of V such that $g_p(v_i, v_j) = \delta_{ij}$ and $g_p(w_i, w_j) = -\delta_{ij}$. Therefore, it suffices to define $X_{m+j} = \frac{1}{\sqrt{2}}(v_j + w_j), Y_{m+j} = \frac{1}{\sqrt{2}}(v_j - w_j)$, $j = 1, \ldots, n-m$, to have $g_p(X_i, X_i) = 0 = g_p(Y_i, Y_i)$, $g_p(X_i, Y_i) = 1$ and $h_p X_i = h_p Y_i = 0$, $i = m+1, \ldots, n$.

If $n = 1$, then $\varphi_p X_1 = \pm X_1$ and $\varphi_p Y_1 = \mp Y_1$ follow directly from paracontact metric properties and the definition of the basis $\{X_1, Y_1, \ldots, X_n, Y_n\}$. □

It is worth mentioning that Theorem 2.1 is true only pointwise, i.e. rank(h_p) does not need to be the same for every $p \in M$. Indeed, we will see in Examples 4.3 and 4.5 that we can construct paracontact metric $(-1,\mu)$-spaces such that h is zero in some points and non-zero in others.

4. New examples

We will first present an example of $(2n+1)$-dimensional paracontact metric $(-1,0)$-space with rank of h greater than 1. This means that, together with Examples 2.2 and 2.3, we have examples of paracontact metric $(-1,\mu)$-spaces of every possible dimension and constant rank of h when $\mu = 0$ and $\mu = 2$.

Example 4.1 ($(2n+1)$-dimensional paracontact metric $(-1,0)$-space with rank$(h) = m \in \{2,\ldots,n\}$). *Let $\mathfrak{g}$ be the $(2n+1)$-dimensional Lie algebra with basis $\{\xi, X_1, Y_1, \ldots, X_n, Y_n\}$ such that the only non-zero Lie brackets are:*

$$[\xi, X_1] = X_1 + X_2 + Y_1, \qquad [\xi, Y_1] = -Y_1 + Y_2,$$
$$[\xi, X_2] = X_1 + X_2 + Y_2, \qquad [\xi, Y_2] = Y_1 - Y_2,$$
$$[\xi, X_i] = X_i + Y_i, \quad i = 3, \ldots, m, \qquad [\xi, Y_i] = -Y_i, \quad i = 3, \ldots, m,$$

$$[X_i, X_j] = \begin{cases} \sqrt{2}X_1, & \text{if } i = 1,\ j = 2, \\ -\sqrt{2}X_j & \text{if } i = 2,\ j = 3\ldots, m, \\ \sqrt{2}[\xi, X_i], & \text{if } i = 1, \ldots, m,\ j = m+1, \ldots, n, \end{cases}$$

$$[Y_i, Y_j] = \begin{cases} \sqrt{2}(-Y_1 + Y_2), & \text{if } i = 1,\ j = 2, \\ \sqrt{2}Y_j, & \text{if } i = 1,2,\ j = 3, \ldots, m, \end{cases}$$

$$[X_i, Y_i] = \begin{cases} 2\xi + \sqrt{2}(X_2 + Y_2) & \text{if } i = 1, \\ -2\xi + \sqrt{2}X_1, & \text{if } i = 2, \\ -2\xi + \sqrt{2}(X_1 - X_2 - Y_2), & \text{if } i = 3, \ldots, m, \\ -2\xi - \sqrt{2}X_i, & \text{if } i = m+1, \ldots, n, \end{cases}$$

$$\underset{i \neq j}{[X_i, Y_j]} = \begin{cases} \sqrt{2}(Y_1 + X_2) & \text{if } i = 1,\ j = 2, \\ \sqrt{2}X_1, & \text{if } i = 2,\ j = 1, \\ \sqrt{2}X_j, & \text{if } i = 1,2,\ j = 3, \ldots, m, \\ \sqrt{2}Y_i, & \text{if } i = 3, \ldots, m,\ j = 2, \\ -\sqrt{2}[\xi, Y_j], & \text{if } i = m+1, \ldots, n,\ j = 1, \ldots, m. \end{cases}$$

If we denote by G the Lie group whose Lie algebra is $\mathfrak{g}$, we can define a left-invariant paracontact metric structure on G the following way:

$$\varphi\xi = 0, \quad \varphi X_i = X_i, \quad \varphi Y_i = -Y_i, \quad i = 1, \ldots, n,$$

$$\eta(\xi) = 1, \quad \eta(X_i) = \eta(Y_i) = 0, \quad i = 1, \ldots, n.$$

The only non-vanishing components of the metric are

$$g(\xi,\xi) = g(X_1, Y_1) = 1, \quad g(X_i, Y_i) = -1, \quad i = 2, \dots, n.$$

A straightforward computation gives that $hX_i = Y_i$, $i = 1, \dots, m$, $hX_i = 0$, $i = m+1, \dots, n$ *and* $hY_i = 0$, $i = 1, \dots, n$, *so* $h^2 = 0$ *and* $rank(h) = m$.

Moreover, very long but direct computations give that

$$R(X_i, \xi)\xi = -X_i, \quad i = 1, \dots, n,$$
$$R(Y_i, \xi)\xi = -Y_i, \quad i = 1, \dots, n,$$
$$R(X_i, X_j)\xi = R(X_i, Y_j)\xi = R(Y_i, Y_j)\xi = 0, \quad i, j = 1, \dots, n.$$

Therefore, the manifold is also a $(-1, 0)$*-space.*

Remark 4.2. *Note that the previous example is only possible when* $n \geq 2$. *If* $n = 1$, *then we can only construct examples of* $rank(h) = 1$, *as in Example 2.3.*

In the definition of the Lie algebra of the previous example, some values of i *and* j *are not possible for* $m = 2$ *or* $m = n$. *In that case, removing the affected Lie brackets from the definition will give us valid examples nonetheless.*

We will present now an example of 3-dimensional paracontact metric $(-1, 2)$-space and one of 3-dimensional paracontact metric $(-1, 0)$-space, such that $\operatorname{rank}(h_p) = 0$ or 1 depending on the point p of the manifold. These are the first examples of paracontact metric (κ, μ)-spaces with h of non-constant rank that are known.

Example 4.3 (3-dimensional paracontact metric $(-1, 2)$-space with rank(h_p) not constant). *We consider the manifold* $M = \mathbb{R}^3$ *with the usual cartesian coordinates* (x, y, z). *The vector fields*

$$e_1 = \frac{\partial}{\partial x} + xz\frac{\partial}{\partial y} - 2y\frac{\partial}{\partial z}, \quad e_2 = \frac{\partial}{\partial y}, \quad \xi = \frac{\partial}{\partial z}$$

are linearly independent at each point of M. *We can compute*

$$[e_1, e_2] = 2\,\xi, \quad [e_1, \xi] = -x\,e_2, \quad [e_2, \xi] = 0.$$

We define the semi-Riemannian metric g *as the non-degenerate one whose only non-vanishing components are* $g(e_1, e_2) = g(\xi, \xi) = 1$, *and the* 1*-form* η *as* $\eta = 2ydx + dz$, *which satisfies* $\eta(e_1) = \eta(e_2) = 0$, $\eta(\xi) = 1$. *Let* φ *be the* $(1, 1)$*-tensor field defined by* $\varphi e_1 = e_1, \varphi e_2 = -e_2, \varphi\xi = 0$. *Then*

$$d\eta(e_1, e_2) = \frac{1}{2}(e_1(\eta(e_2)) - e_2(\eta(e_1)) - \eta([e_1, e_2])) = -1 = -g(e_1, e_2) = g(e_1, \varphi e_2),$$
$$d\eta(e_1, \xi) = \frac{1}{2}(e_1(\eta(\xi)) - \xi(\eta(e_1)) - \eta([e_1, \xi]) = 0 = g(e_1, \varphi\xi),$$
$$d\eta(e_2, \xi) = \frac{1}{2}(e_2(\eta(\xi)) - \xi(\eta(e_2)) - \eta([e_2, \xi]) = 0 = g(e_2, \varphi\xi).$$

Therefore, (φ, ξ, η, g) *is a paracontact metric structure on* M.

Moreover, $h\xi = 0$, $he_1 = xe_2$, $he_2 = 0$. *Hence,* $h^2 = 0$ *and, given* $p = (x, y, z) \in \mathbb{R}^3$, $rank(h_p) = 0$ *if* $x = 0$ *and* $rank(h_p) = 1$ *if* $x \neq 0$.

Let ∇ *be the Levi-Civita connection. Using the properties of a paracontact metric structure and Koszul's formula*

$$2g(\nabla_X Y, Z) = X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)) - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]), \tag{3}$$

we can compute

$$\nabla_\xi \xi = 0, \quad \nabla_{e_1} \xi = -e_1 - xe_2, \quad \nabla_{e_2} \xi = e_2, \quad \nabla_\xi e_1 = -e_1, \quad \nabla_\xi e_2 = e_2,$$

$$\nabla_{e_1}e_1 = x\xi, \quad \nabla_{e_2}e_2 = 0, \quad \nabla_{e_1}e_2 = \xi, \quad \nabla_{e_2}e_1 = -\xi.$$

Using the following definition of the Riemannian curvature

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, \tag{4}$$

we obtain

$$R(e_1,\xi)\xi = -e_1 + 2he_1, \quad R(e_2,\xi)\xi = -e_2 + 2he_2, \quad R(e_1,e_2)\xi = 0,$$

so the paracontact metric manifold M is also a $(-1,2)$-space.

Remark 4.4. *The previous example does not contradict Theorem 2.1, as we will see by constructing explicitly the basis of the theorem on each point p where $h_p \neq 0$, i.e., on every point $p = (x,y,z)$ such that $x \neq 0$.*

Indeed, let us take a point $p = (x,y,z) \in \mathbb{R}^3$. If $x \neq 0$, then we define $X_1 = \frac{e_{1p}}{\sqrt{|x|}}$, $Y_1 = \frac{h_p e_{1p}}{\sqrt{|x|}}$. We obtain that $\{\xi_p, X_1, Y_1\}$ is a basis of $T_p(\mathbb{R}^3)$ that satisfies that:

- *the only non-vanishing components of g are $g_p(\xi_p,\xi_p) = 1$, $\quad g_p(X_1,Y_1) = sign(x)$,*
- *the tensor h can be written as $h_{p|\langle\xi_p,X_1,Y_1\rangle} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$,*
- $\varphi_p\xi = 0, \quad \varphi_p X_1 = X_1, \quad \varphi_p Y_1 = -Y_1.$

Example 4.5 (3-dimensional paracontact metric $(-1,0)$-space with rank(h_p) not constant). *We consider the manifold $M = \mathbb{R}^3$ with the usual cartesian coordinates (x,y,z). The vector fields*

$$e_1 = \frac{\partial}{\partial x} + xe^{-2z}\frac{\partial}{\partial y} - 2y\frac{\partial}{\partial z}, \quad e_2 = \frac{\partial}{\partial y}, \quad \xi = \frac{\partial}{\partial z}$$

are linearly independent at each point of M. We can compute

$$[e_1,e_2] = 2\,\xi, \quad [e_1,\xi] = 2xe^{-2z}\,e_2, \quad [e_2,\xi] = 0.$$

We define the semi-Riemannian metric g as the non-degenerate one whose only non-vanishing components are $g(e_1,e_2) = g(\xi,\xi) = 1$, and the 1-form η as $\eta = 2ydx + dz$, which satisfies $\eta(e_1) = \eta(e_2) = 0$, $\eta(\xi) = 1$. Let φ be the $(1,1)$-tensor field defined by $\varphi e_1 = e_1, \varphi e_2 = -e_2, \varphi\xi = 0$. Then

$$d\eta(e_1,e_2) = \frac{1}{2}(e_1(\eta(e_2)) - e_2(\eta(e_1)) - \eta([e_1,e_2])) = -1 = -g(e_1,e_2) = g(e_1,\varphi e_2),$$
$$d\eta(e_1,\xi) = \frac{1}{2}(e_1(\eta(\xi)) - \xi(\eta(e_1)) - \eta([e_1,\xi]) = 0 = g(e_1,\varphi\xi),$$
$$d\eta(e_2,\xi) = \frac{1}{2}(e_2(\eta(\xi)) - \xi(\eta(e_2)) - \eta([e_2,\xi]) = 0 = g(e_2,\varphi\xi).$$

Therefore, (φ,ξ,η,g) is a paracontact metric structure on M.

Moreover, $h\xi = 0$, $he_1 = -2xe^{-2z}e_2$, $he_2 = 0$. Hence, $h^2 = 0$ and, given $p = (x,y,z) \in \mathbb{R}^3$, $rank(h_p) = 0$ if $x = 0$ and $rank(h_p) = 1$ if $x \neq 0$.

Let ∇ be the Levi-Civita connection. Using the properties of a paracontact metric structure and Koszul's formula (3)*, we can compute*

$$\nabla_\xi\xi = 0, \quad \nabla_{e_1}\xi = -e_1 + 2xe^{-2z}e_2, \quad \nabla_{e_2}\xi = e_2, \quad \nabla_\xi e_1 = -e_1, \quad \nabla_\xi e_2 = e_2,$$
$$\nabla_{e_1}e_1 = -2xe^{-2z}\xi, \quad \nabla_{e_2}e_2 = 0, \quad \nabla_{e_1}e_2 = \xi, \quad \nabla_{e_2}e_1 = -\xi.$$

Using now (4)*, we obtain*

$$R(e_1,\xi)\xi = -e_1, \quad R(e_2,\xi)\xi = -e_2, \quad R(e_1,e_2)\xi = 0,$$

so the paracontact metric manifold M is also a $(-1,0)$-space.

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