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On the Existence of Isoperimetric Extremals of Rotation and the Fundamental Equations of Rotary Diffeomorphisms

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Abstract. In this paper we study the existence and the uniqueness of isoperimetric extremals of rotation on two-dimensional (pseudo-) Riemannian manifolds and on surfaces on Euclidean space. We find the new form of their equations which is easier than results by S. G. Leiko. He introduced the notion of rotary diffeomorphisms. In this paper we propose a new proof of the fundamental equations of rotary mappings.

1. Introduction

A rotary diffeomorphism of surfaces S_2 on a three-dimensional Euclidean space $\mathbb{E}_3$ and also of two-dimensional Riemannian manifolds V_2 is studied in papers of S. G. Leiko [10–18]. These results are local and are based on the known fact that a two-dimensional Riemannian manifold V_2 is implemented locally as a surface S_2 on $\mathbb{E}_3$. Therefore we will deal more with the study of V_2, i. e. the inner geometry of S_2. For recent studies of the deformation of surfaces from a different point, see [3, 7, 20–23, 29–31].

In [10, 11, 14] the following notion of special mapping is introduced .

Definition 1.1. A diffeomorphism $f\colon \overline{V}_2 \to V_2$ is called *rotary* if any geodesic $\overline{\gamma}$ is mapped onto isoperimetric extremal of rotation.

In our paper we have new proof of the fundamental equations of the rotary mappings (Section 5).

The *isoperimetric extremal of rotation* is a special curve on V_2 (resp. S_2) which is extremal of a certain variational problem of geodesic curvature (see [10–17] where the existence of these curves was shown for the case $V_2 \in C^4$, resp. on $S_2 \in C^5$).

The above curves have a physical meaning as can be interpreted as trajectories of particles with a spin, see [10, 12].

Our paper is devoted to the proof of the existence of isoperimetric extremal of rotation on $V_2 \in C^3$, resp. on $S_2 \in C^4$. Besides we find the fundamental equations of these curves in a more simple form of ordinary differential equation of Cauchy type. From the above the problem of a rotary diffeomorphism can be solved for the surfaces with the lower smoothness class.

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Remark. A two-dimensional Riemannian manifold V_2 belongs to the smoothness class C^r if its metric $g_{ij} \in C^r$. We suppose that the differentiability class r is equal to $0, 1, 2, \ldots, \infty, \omega$, where $0, \infty$ and ω denote continuous, infinitely differentiable, and real analytic functions respectively.

Surface S_2: $\mathbf{p} = \mathbf{p}(x^1, x^2)$ belongs to C^{r+1} if the vector function $\mathbf{p}(x^1, x^2) \in C^{r+1}$ and evidently inner two-dimensional Riemannian manifold V_2 belongs to C^r with induced metric $g_{ij}(x) = \mathbf{p}_i \cdot \mathbf{p}_j \in C^r$, where $\mathbf{p}_i = \partial_i \mathbf{p}$, $\partial_i = \partial/\partial x^i$. There $x = (x^1, x^2)$ are local coordinates of V_2, resp. S_2.

An immersion V_2 in Euclidean space is studied in detail, for example, in [24, 27]. In the study of surfaces S_2 we use the notation that is used in the books [1, 6, 9, 25, 26].

2. Isoperimetric extremals of rotation

Let us consider a two-dimensional Riemannian space $V_2 \in C^3$ with a metric tensor g. Let $g_{ij}(x^1, x^2) \in C^3$ $(i, j = 1, 2)$ be components of g in some local map.

For the curve ℓ: $(t_0, t_1) \to V_2$ with the parametric equation $x^h = x^h(t)$, we construct the tangent vector $\lambda^h = dx^h/dt$ and vectors
$$\lambda_1^h = \nabla_t \lambda^h \quad \text{and} \quad \lambda_2^h = \nabla_t \lambda_1^h.$$
Here ∇_t is an operator of covariant differentiation along ℓ with respect to the Levi-Civita connection ∇ of metric g, i.e.
$$\lambda_1^h = \nabla_t \lambda^h \equiv \frac{d\lambda^h}{dt} + \lambda^\alpha \Gamma^h_{\alpha\beta}(x(t))\, \frac{dx^b(t)}{dt}$$
and
$$\lambda_2^h = \nabla_t \lambda_1^h \equiv \frac{d\lambda_1^h}{dt} + \lambda_1^\alpha \Gamma^h_{\alpha\beta}(x(t))\, \frac{dx^b(t)}{dt},$$
where Γ^h_{ij} are the Christoffel symbols of V_2, i. e. components of ∇.

It is known that the scalar product of vectors λ, ξ is defined by $\langle \lambda, \xi \rangle = g_{ij} \lambda^i \xi^j$. We denote
$$s[\ell] = \int_{t_0}^{t_1} \sqrt{\langle \lambda, \lambda \rangle}\, \mathrm{d}t \qquad \text{and} \qquad \theta[\ell] = \int_{t_0}^{t_1} k_g(s)\, \mathrm{d}s$$
functionals of length and rotation of the curve ℓ; k_g is the Frenet curvature[1)] and s is the arc length. In the case $S_2 \subset E_3$ the geodesic curvature of the curve is k_g.

Using these functionals we introduce the following definition

Definition 2.1 (Leiko [11]). A curve ℓ is called the *isoperimetric extremal of rotation* if ℓ is extremal of $\theta[\ell]$ and $s[\ell] = \text{const}$ with fixed ends.

It was shown in [11] that in a (not plain) space V_2 a curve is an isoperimetric extremal of rotation only if its Frenet curvature k_g and Gaussian curvature K are proportional:
$$k_g = c \cdot K, \tag{1}$$
where $c = \text{const}$.

In [11] it is proved that for a canonical parameter $t = a \cdot s + b$ $(a, b = \text{const})$ the condition (1) can be written in the following form
$$\lambda_2 = -\frac{\langle \lambda_1, \lambda_1 \rangle}{\langle \lambda, \lambda \rangle} \cdot \lambda + \frac{\nabla_\alpha K \cdot \lambda^\alpha}{K} \cdot \lambda_1, \tag{2}$$
where $\langle \lambda, \lambda_1 \rangle = 0$ and $\nabla_i K = \partial_i K$ is a gradient vector of the Gaussian curvature ($K \neq 0$).

Using these equations for the case of $V_2 \in C^4$ the uniqueness of the existence of isoperimetric extremals of rotation can be shown for the following initial conditions (see [14]):
$$x(0), \lambda(0), \lambda_1(0) \ \text{ such that } \ \langle \lambda(0), \lambda(0) \rangle = 1 \ \text{ and } \ \langle \lambda(0), \lambda_1(0) \rangle = 0.$$

---

1)In the original paper k_g is denoted as k. This fact can lead to confusion between k and the main curvature of the curve.

3. On new equations of isoperimetric extremals of rotation

First we recall the basic knowledge of theory of surfaces S_2 and (pseudo-) Riemannian manifolds, see [1, 5, 6, 9, 20, 25, 26].

For simplicity we will consider that a two-dimensional Riemannian manifold V_2 is a subspace of $S_2 \subset \mathbb{E}_3$ which is given by the equation $\mathbf{p} = \mathbf{p}(x^1, x^2)$. It is known that metric of S_2 is given by the following functions $g_{ij}(x) = \mathbf{p}_i \cdot \mathbf{p}_j \in C^r$, where $\mathbf{p}_i = \partial_i \mathbf{p}$.

The existence of the surface S_2 with metric g on V_2 results from the Bonnet Theorem; components g_{ij} of the first fundamental form belong to the smoothness class C^2 and components b_{ij} of the second fundamental form belong to the smoothness class C^1 both of them satisfy Gauss and Peterson-Codazzi equations.

For the Gaussian curvature K it holds that

$$K = \frac{b_{11}b_{22} - b_{12}^2}{g_{11}g_{22} - g_{12}^2},$$

where $b_{ij} = \partial_{ij}\mathbf{p} \cdot \mathbf{m}$ and $\mathbf{m} = \dfrac{\mathbf{p}_1 \times \mathbf{p}_2}{|\mathbf{p}_1 \times \mathbf{p}_2|}$ is a unit normal vector of the surface S_2. If $S_2 \in C^3$ then the curvature K is differentiable.

Now we recall the geometry of a (pseudo-) Riemannian manifold V_2 defined by the metric tensor g_{ij}. The Christoffel symbols of the first and the second kind are given by

$$\Gamma_{ijk} = \frac{1}{2}(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}) \quad \text{and} \quad \Gamma_{ij}^k = \Gamma_{ij\alpha} g^{\alpha k},$$

where g^{ij} are components of the matrix inverse to (g_{ij}).

The Riemannian tensors of the first and the second type are given by

$$R_{hijk} = g_{h\alpha} R_{ijk}^{\alpha} \quad \text{and} \quad R_{ijk}^h = \partial_j \Gamma_{ik}^h - \partial_k \Gamma_{ij}^h + \Gamma_{ik}^{\alpha}\Gamma_{\alpha j}^h - \Gamma_{ij}\alpha \Gamma_{\alpha k}^h.$$

Then from Gauss's Theorema Egregium for surfaces $S_2 \in C^3$ it follows that ([5, § 22.2], [9, p. 145]):

$$K = \frac{R_{1212}}{g_{11}g_{22} - g_{12}^2}.$$

This formula defines the curvature K in a (pseudo-) Riemannian manifold V_2.

Finally, we recall the Gauss equations

$$\partial_{ij}\mathbf{p} = \Gamma_{ij}^k \cdot \mathbf{p}_k + b_{ij} \cdot \mathbf{m}. \tag{3}$$

Let a curve ℓ: $\mathbf{p} = \mathbf{p}(s)$ be an isoperimetric extremal of rotation on a surface S_2 parametrized by arclength s. On the other hand, because $\ell \subset S_2$: $\mathbf{p} = \mathbf{p}(x^1, x^2)$ there exist inner equations ℓ: $x^i = x^i(s)$ such that the following is valid

$$\mathbf{p}(s) = \mathbf{p}\,(x(s))$$

for all $s \in I$, where $\mathbf{p}$ on the left side is a vector function describing the curve ℓ and $\mathbf{p}$ on the right side is a vector function describing the surface S. Let us denote d/ds by a dot. Then $\dot{\mathbf{p}}(s)$ is a unit tangent vector of ℓ.

We compute the second order derivative for a vector $\mathbf{p}(s)$:

$$\begin{aligned} \dot{\mathbf{p}}(s) &= \mathbf{p}_i\,(x(s)) \cdot \dot{x}^i(s) \\ \ddot{\mathbf{p}}(s) &= \partial_{ij}\mathbf{p}\,(x(s))\,\dot{x}^i(s) \cdot \dot{x}^j(s) + \mathbf{p}_k \cdot \ddot{x}^k(s). \end{aligned}$$

Now we apply the Gauss equation (3) and we obtain

$$\ddot{\mathbf{p}}(s) = \left(\ddot{x}^k(s) + \Gamma_{ij}^k \cdot \dot{x}^i(s)\dot{x}^j(s)\right) \cdot \mathbf{p}_k + b_{ij} \cdot \mathbf{m}. \tag{4}$$

It is obvious that vector $\ddot{\mathbf{p}}(s)$ splits into two components: into a normal vector $\mathbf{m}$ and a unit vector $\mathbf{n}$ which is orthogonal to a vector $\mathbf{m}$ and $\dot{\mathbf{p}}(s)$. This vector is tangent to a surface S_2, therefore we can write $\mathbf{n} = n^k\, \mathbf{p}_k$, where n^k are components of the vector $\mathbf{n}$.

Therefore from (4) it follows that

$$(\ddot{x}^k(s) + \Gamma^k_{ij}(x(s)) \cdot \dot{x}^i(s)\, \dot{x}^j(s)) \cdot \mathbf{p}_k + b_{ij} \cdot \mathbf{m} = k_g \cdot n^k \cdot \mathbf{p}_k + k_n \cdot \mathbf{m},$$

where k_n is a normal curvature of S_2 in the direction of a tangent vector $\lambda = \dot{x}$.

Because vectors $\mathbf{p}_1, \mathbf{p}_2, \mathbf{m}$ are linearly independent, the following equation is true

$$\ddot{x}^k(s) + \Gamma^k_{ij}(x(s))\, \dot{x}^i(s)\, \dot{x}^j(s) = k_g \cdot n^k.$$

We can write this equation in the form:

$$\nabla_s \lambda = k_g \cdot n. \tag{5}$$

The formula above is an analogue of the Frenet formulas for the flat curves, see [5], [9, § 12], and for the curves with non-isotropic tangent vector λ ($|\lambda| \neq 0$) on (pseudo-) Riemannian manifolds V_2, see [19, pp. 22–26].

We show efficient construction of a unit vector n which is orthogonal to λ using a discriminant tensor ε and a structure tensor F on V_2 defined by relations

$$\varepsilon_{ij} = \sqrt{|g_{11}g_{22} - g_{12}^2|} \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \text{and} \qquad F_i^h = \varepsilon_{ij} \cdot g^{jh}.$$

The tensor ε is skew-symmetric and covariantly constant and tensor F defined on V_2 is a structure, for which it holds

$$F^2 = e\, Id \quad \text{and} \quad \nabla F = 0,$$

where $e = -1$ for a "properly" Riemannian V_2 and $e = +1$ for a pseudo-Riemannian V_2.

It can be easily proved that vector $F\lambda$ is also a unit vector orthogonal to a unit vector λ. Obviously, it holds that $n = \pm F\lambda$. Therefore from (1) and (5) follows the theorem.

Theorem 3.1. *The equation of isoperimetric extremal of rotation can be written in the form*

$$\nabla_s \lambda = c \cdot K \cdot F\lambda, \tag{6}$$

where $c = \text{const}$.

Remark. Further differentiation of the equation (6) gives the equation (2) by Leiko [10–17]. Note that the equation (6) has more simple form than the equation (2). If $c = 0$ is satisfied then the curve is geodesic.

4. On the uniqueness of the existence of isoperimetric extremals of rotation

Analysis of the equation (6) convinces of the validity of the following theorem which generalizes and refines the results of Leiko [10–17].

Theorem 4.1. *Let V_2 be a (non flat) Riemannian manifold of the smoothness class C^3. Then there is precisely one isoperimetric extremal of rotation going through a point $x_0 \in V_2$ in a given non-isotropic direction $\lambda_0 \in TV_2$ and constant c.*

Proof. Let x^h_0 be coordinates of a point x_0 at $V_2 \in C^3$ and λ^h_0 ($\neq 0$) be coordinates of a unit tangent vector λ_0 in a given point x_0.

We will find an isoperimetric extremal of rotation ℓ: $x^h = x^h(s)$, where s is the arc length, on a space V_2 such that $x^h(0) = x^h_0$ and $\dot{x}^h(0) = \lambda^h_0$, i. e. this curve goes through a point x_0 in the direction λ_0.

Let us write equation (6) as a system of ordinary differential equations:

$$\begin{aligned} \dot{x}^h(s) &= \lambda^h(s) \\ \dot{\lambda}^h(s) &= -\Gamma^h_{ij}(x(s)) \cdot \lambda^i(s) \cdot \lambda^j(s) + c \cdot K(x(s)) \cdot F^h_i(x(s)) \cdot \lambda^i(s). \end{aligned} \tag{7}$$

From the theory of differential equations it is known (see [4, 8]) that given the initial conditions $x^h(0) = x^h_0$ and $\lambda^h(0) = \dot{x}^h(0) = \lambda^h_0$ the system (7) has only one solution when

$$\Gamma^h_{ij} \in C^1, K \in C^1 \text{ and } F^h_i \in C^1. \tag{8}$$

These conditions (8) are met on a space $V_2 \in C^3$ (we consider that V_2 is a metric of some surface $S_2 \subset \mathbb{E}_3$ of the smoothness class C^4).

Correctness of the solution of (7) lies in the fact that the vector $\lambda(s)$ is unit for all s. Evidently, $\langle\lambda, \lambda\rangle$ is constant along ℓ, i.e. $\nabla_s\langle\lambda, \lambda\rangle = 2 \cdot \langle\lambda, \nabla_s\lambda\rangle = 0$, and from $\langle\lambda_0, \lambda_0\rangle = \pm 1$ it follows $\langle\lambda, \lambda\rangle = \pm 1$. □

Remark. It is possible to substitute the condition (8) by the Lipschitz's condition for these functions.

Continuity of these functions is guaranteed by the existence of a solution to (7). This is possible when $V_2 \in C^2$, resp. $S_2 \in C^3$.

5. On the fundamental equations of rotary diffeomorphisms of V_2

Assume to be given two-dimensional (pseudo-) Riemannian manifolds $V_2 = (M, g)$ and $\overline{V}_2 = (\overline{M}, \overline{g})$ with metrics g and $\overline{g}$, Levi-Civita connections ∇ and $\overline{\nabla}$, complex structures F and $\overline{F}$, respectively.

Assume a rotary diffeomorphism $f\colon \overline{V}_2 \to V_2$. Since f is a diffeomorphism, we can impose local coordinate system on M and $\overline{M}$, respectively, such that locally $f\colon \overline{V}_2 \to V_2$ maps points onto points with the same coordinates x, and $M = \overline{M}$.

From Definition 1.1 it follows that any geodesic $\overline{\gamma}$ on V_2 is mapped onto an isoperimetric extremal of rotation on V_2.

Let $\overline{\gamma}: x^h = x^h(\overline{s})$ be a geodesic on $\overline{V}_2$ for which the following equation is valid

$$\frac{d^2x^h}{d\overline{s}^2} + \overline{\Gamma}^h_{ij}(x(s))\,\frac{dx^i}{d\overline{s}}\,\frac{dx^j}{d\overline{s}} = 0 \tag{9}$$

and let $\gamma: x^h = x^h(s)$ be an isoperimetric extremal of rotation on V_2 for which the following equation is valid

$$\lambda^h_1 \equiv d\lambda^h/ds + \Gamma^h_{ij}(x(s))\,\lambda^i\lambda^j = c \cdot K(x(s)) \cdot F^h_i(x(s)) \cdot \lambda^i, \tag{10}$$

where Γ^h_{ij} and $\overline{\Gamma}^h_{ij}$ are components of ∇ and $\overline{\nabla}$, parameters s and $\overline{s}$ are arc lengthes on γ and $\overline{\gamma}$, $\lambda^h = dx^h(s)/ds$.

Suppose that $\overline{s} = \overline{s}(s)$. In this case we modify equation (9):

$$d\lambda^h/ds + \overline{\Gamma}^h_{ij}(x(s))\,\lambda^i\lambda^j = \varrho(s) \cdot \lambda^h, \tag{11}$$

where $\varrho(s)$ is a certain function of parameter s, i.e. this equation is the equation of a geodesic with an arbitrary parameter.

We denote $P^h_{ij}(x) = \Gamma^h_{ij}(x) - \overline{\Gamma}^h_{ij}(x)$ the deformation tensor of connections ∇ and $\overline{\nabla}$ defined by the rotary diffeomorphism.

As a consequence of (11), we have

$$\lambda^h_1 = \varrho \cdot \lambda^h + P^h, \tag{12}$$

where $P^h = P^h_{ij} \cdot \lambda^i\lambda^j$, and from (12) it follows $\varrho = -\langle\lambda, P\rangle$.

After differentiating (12) along the curve ℓ and substituting the corresponding values in (10), we obtain

$$P_1^h - \langle \lambda, P_1 \rangle \cdot \lambda^h = (3\,\langle \lambda, P \rangle + \nabla_\alpha K \cdot \lambda^\alpha / K) \cdot (\langle \lambda, P \rangle \cdot \lambda^h - P^h), \tag{13}$$

where $P_1^h = (\nabla_k P_{ij}^h + 2P_{i\alpha}^h P_{jk}^\alpha)\lambda^i \lambda^j \lambda^k$.

Let us study the formula (13) in isothermal coordinates in the fixed point x' in which $g_{11} = g_{22} = 1$, $g_{12} = 0$. As $(\lambda^2)^2 = 1 - (\lambda^1)^2$ the formula (13) is the function of value λ^1. The coefficient of $(\lambda^1)^6$ is equal $A^2 + B^2$, where $A = P_{11}^1 - P_{22}^1 - 2P_{12}^2$ and $B = P_{22}^2 - P_{11}^2 - 2P_{12}^1$. From this it follows $A^2 + B^2 = 0$, and evidently $A = B = 0$, and hence we have in this coordinate system

$$P_{11}^1 - P_{22}^1 - 2P_{12}^2 = 0 \ \text{ and } \ P_{22}^2 - P_{11}^2 - 2P_{12}^1 = 0.$$

In this coordinate system we denote $\psi_1 = P_{12}^2$, $\psi_2 = P_{12}^1$, $\theta^1 = P_{22}^1$ and $\theta^2 = P_{11}^2$. We can rewrite the above formula equivalently to the following tensor equation

$$P_{ij}^h = \delta_i^h \psi_j + \delta_j^h \psi_i + \theta^h g_{ij}, \tag{14}$$

where ψ_i and θ^h are covector and vector fields.

On the other hand, from (14) it follows $4\psi_i = 2P_{i\alpha}^\alpha - g_{ih} P_{\alpha\beta}^h g^{\alpha\beta}$ and $4\theta_i = 3g_{ih} P_{\alpha\beta}^h g^{\alpha\beta} - 2P_{i\alpha}^\alpha$.

As a consequence of (14), the formula (12) obtains following form

$$\lambda_1^h = \tilde{\varrho} \cdot \lambda^h + \theta^h, \tag{15}$$

and from (15) it follows $\tilde{\varrho} = -\langle \lambda, \theta \rangle$. After differentiating (15) along the curve ℓ and substituting the corresponding values in (10), we obtain

$$\nabla_\alpha \theta^h \lambda^\alpha - \theta^h \theta_\alpha \lambda^\alpha - \theta^h K_\alpha \lambda^\alpha / K = \lambda^h\,(\nabla_\beta \theta_\alpha \lambda^\alpha \lambda^\beta - \theta_\alpha \theta_\beta \lambda^\alpha \lambda^\beta - \theta_\alpha \lambda^\alpha\ K_\beta \lambda^\beta / K),$$

and by similar way we obtain the following formulas

$$\nabla_j \theta_i = \theta_i(\theta_j + K_j/K) + \nu\, g_{ij}, \tag{16}$$

where ν is a function on V_2.

The equations (14) and (16) are necessary and sufficient conditions of rotary diffeomorphism by V_2 onto $\bar{V}_2$. Our proof is straightforward and more comprehensive than the one proposed in [11]. We notice that the above considerations are possible when $V_2 \in C^3$ and $\overline{V}_2 \in C^3$.

The vector field θ_i is torse-forming, see [20, 28, 33]. Under furher conditions on differentiability of metrics it has been proved in [11] that θ_i is concircular. From this follows that V_2 is isometric to surfaces of revolution. Concircular vector fields were studied by many authors, such as [2, 20, 32].

References

[1] A. D. Aleksandrov, N. Yu. Necvetaev, *Geometry.* Moscow, Nauka, 1990.

[2] O. Chepurna, I. Hinterleitner, *On the mobility degree of (pseudo-) Riemannian spaces with respect to concircular mappings.* Miskolc Math. Notes **14** (2013), no. 2, 561–568.

[3] M. S. Ćirić, M. Lj. Zlatanović, M. S. Stanković, Lj. S. Velimirović, *On geodesic mappings of equidistant generalized Riemannian spaces.* Applied Mathematics and Computation **218:12** (2012), 6648–6655.

[4] J. Diblík, O. Přibyl, *Ordinary differential equations.* Brno, CERM, 2004.

[5] A. Gray, *Modern differential geometry of curves and surfaces with Mathematica.* 2nd ed. Boca Raton, FL: CRC Press, 1997.

[6] N. I. Guseva and all, *Geometry.* Moscow, Academia, 2013.

[7] I. Hinterleitner, *Geodesic mappings on compact riemannian manifolds with conditions on sectional curvature.* Publ. Inst. Math. New Ser. **94(108)** (2013), 125–130.

[8] E. Kamke, *Differentialgleichungen reeller Funktionen.* Leipzig, Akad. Verlag., 1956.

[9] E. Kreyzsig, *Differential geometry.* Toronto Univ. Press, 1959.

[10] S. G. Leiko, *Conservation laws for spin trajectories generated by isoperimetric extremals of rotation.* Gravitation and Theory of Relativity **26** (1988), 117–124.

[11] S. G. Leiko, *Rotary diffeomorphisms on Euclidean spaces.* Mat. Zametki **47:3** (1990), 52–57.
[12] S. G. Leiko, *Variational problems for rotation functionals, and spin-mappings of pseudo-Riemannian spaces.* Sov. Math. **34:10** (1990), 9–18; transl. from Izv. Vyssh. Uchebn. Zaved. Mat. **2** (1990), 9–17.
[13] S. G. Leiko, *Extremals of rotation functionals of curves in a pseudo-Riemannian space, and trajectories of spinning particles in gravitational fields.* Russ. Acad. Sci., Dokl., Math. **46:1** (1993), 84–87; transl. from Dokl. Akad. Nauk, Ross. Akad. Nauk **325:4** (1992), 659–663.
[14] S. G. Leiko, *Isoperimetric extremals of a turn on surfaces in the Euclidean space* $\mathbb{E}^3$. Izv. Vyshh. Uchebn. Zaved. Mat. **6** (1996), 25–32.
[15] S. G. Leiko, *On the conformal, concircular, and spin mappings of gravitational fields.* J. Math. Sci., New York **90:2** (1998), 1941–1944; transl. from Mat. Metody Fiz.-Mekh. Polya **40:2** (1997), 44–47.
[16] S. G. Leiko, *Isoperimetric problems for rotation functionals of the first and second orders in (pseudo) Riemannian manifolds.* Russ. Math. **49:5** (2005), 45–51; transl. from Izv. Vyssh. Uchebn. Zaved., Mat. **5** (2005), 49–55.
[17] S. G. Leiko, Yu. S. Fedchenko, Infinitesimal rotary deformations of surfaces and their application to the theory of elastic shells. Ukr. Math. J. 55, No. 12, 2031-2040 (2003); translation from Ukr. Mat. Zh. 55 (2003), No. 12, 1697–1703.
[18] S. G. Leiko, S. al Hussin *Rotary quasiconcircular diffeomorphisms on the (pseudo) Riemannian spaces.* (Ukrainian) Mat. Metody Fiz.-Mekh. Polya **44:1** (2001), 22–27.
[19] S. G. Leiko, J. Mikeš, *Methodological guidelines on the course "Riemannian geometry".* (Russian) Odessa Univ. Press, 1985.
[20] J. Mikeš, A. Vanžurová, I. Hinterleitner, *Geodesic mappings and some generalizations.* Olomouc, Palacky Univ. Press, 2009.
[21] S. M. Minčić, Lj. S. Velimirović, M. S. Stanković., *New Integrability conditions of derivational equations of a submanifold in a generalized Riemannian space.* FILOMAT **24:4** (2010), 137–146.
[22] S. M. Minčić, Lj. S. Velimirović, M. S. Stanković., *On spaces with non-symmetric affine connection, containing subspaces without torsion.* Applied Mathematics and Computation, **219:9** (2013), 4346–4353.
[23] S. M. Minčić, Lj. S. Velimirović, M. S. Stanković., *Integrability conditions of derivational equations of a submanifold of a generalized Riemannian space.* Applied Mathematics and Computation **226:1** (2014), 3–9.
[24] J. Nash, *The imbedding problem for Riemannian manifolds.* Ann. of Math (2) **63** (1956), 20–63.
[25] A. G. Popov, *Lobachevsky geometry and mathematical physics.* (Russian) Phys. faculty of Moscow univ., 2012. Transl. in Springer, 2014.
[26] E. G. Poznyak, E. V. Shikin, *Differential geometry. The first acquaintance.* (Russian) Moscow Univ. Press, 1990.
[27] E. G. Poznyak, A. G. Popov, *The sine-Gordon equation: geometry and physics.* (Russian) Moscow, Znanie, 1991.
[28] L. Rachůnek, J. Mikeš, *On tensor fields semiconjugated with torse-forming vector fields.* Acta Univ. Palack. Olomuc. Fac. Rerum Natur. Math. **44** (2005), 151–160.
[29] Ž. Radulović, J. Mikeš, M. L. Gavrilchenko, *Geodesic mappings and deformations of Riemannian spaces.* (Russian) Podgorica, CID, Odessa Univ. Press, 1997.
[30] M. S. Stanković, M. Lj. Zlatanović, Lj. S. Velimirović, *Equitorsion holomorphically projective mappings of generalized Kahlerian space of the second kind.* International Electronic Journal of Geometry, **3:2** (2010), 26–39.
[31] L. S. Velimirović, M. D. Cvetković, M. S. Ćirić, N. Velimirović, *Analysis of Gaudi surfaces at small deformations.* Appl. Math. Comput. **218:13** (2012), 6999–7004.
[32] K. Yano, *Concircular Geometry, I-IV.* Proc. Imp. Acad. Tokyo **16** (1940).
[33] K. Yano, *On the torse-forming directions in Riemannian spaces.* Proc. Imp. Acad. Tokyo **20** (1944), 340–345.