Psychology and Geometry
I. On the geometry of the human kind

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Abstract. In this note an attempt is made to describe *a personal look at some of the main steps in the history of geometry from a psychological point of view*, hereby basing on and sometimes merely formulating again parts of some previous papers, like [1–11]. For general references on elementary differential geometry, pseudo Riemannian geometry and geometry of submanifolds, see e.g. [12–22]. In reference [23], part II of some of the author's reflections on psychology and geometry, an attempt is made to describe *relativistic spacetimes* in a way as kind of a supplement to the contents of the present part I.

1. In his "*Philosophy of Mathematics and Natural Science*" Weyl wrote that "*science would perish without a supporting transcendental faith in truth and reality, and without the continuous interplay between its facts and constructions on the one hand and its imagery of ideas on the other*"; (the hereafter following statements of Weyl come from the same book). And, despite some actual decadent and perverse, or, put in a more positive way, despite some actual extra–terrestrial attitudes towards mathematics, mathematics definitely is included in the "science" of the foregoing citation. And, in any case, at least partially, the above could be rephrased as follows: *the geometry of submanifolds*, that is, named more romantically, *the geometry of the human kind*, is a basic and vital part of science.

From Bronowski's "*The Origins of Knowledge and Imagination*" come the following two quotes: "*The place of sight in human evolution is cardinal*" and "*The world of science is wholly dominated by the sense of sight*", and in "*The Ascent of Man*" the same author offers a discussion of *the theorem of Pythagoras* as a wonderful link between the two which presents a most direct view on *the fundamental role played by geometry in human cognition*. From this discussion, in Figure 1 is redrawn a *proof* of this principal theorem of Euclidean geometry based on "*the gravitational cross*".

Helmholtz stated in his "*Physiologische Optik*" that "*the sensations are signs to our consciousness and it is the task of our intelligence to learn to understand their meaning*". And Weyl gave the advice that "*in the ultimate description of the connection between appearance and reality one does better to ignore all intermediary levels of constitution*". Somewhat accordingly, a straightforward adaptation of the classical scale space in

the geometrical description of human visual sensation related to "the horizontal–vertical effect" (which too essentially results from our kind's experiences associated with the gravitational cross) and its consequential relevance for *the geometrical description of human visual perception* may be found in [7,8], and the light that this sheds on how in vision the functioning of various kinds of individual cells contributes to the overall formation of images, or, put in a more negative way, on the manifest failure to explain images by certain "theories" built upon the all in all not unwelcome information gotten by many experiments on this functioning, may be seen in [9]. In this context, "*it must be admitted that he who undertakes to deal with questions of natural sciences without the help of geometry is attempting the unfeasable*" as Galileo said in his "*Dialogo*". And, readily applicable to many busy people who have been busy in the recent times with what they do as "researches" in vision, physics, chemistry, etc., are Weyl's following comments in this respect: "*Enemies of the geometrical method are*, on the one hand, *the empiricists*, because any aprioristic construction is a thorn in their flesh; *they fondly imagine it to be possible to grasp reality as a thing of one stratum*, as it were, without aprioristic ingredients, *by a purely descriptive approach* (...). On the other hand, out of hatred for the freedom, the open field of geometrical construction, *those metaphysicians* oppose the method *who build up a rigid dialectic world of concepts as the true reality* (...)".

2. Euclid's "*Elements*" (∼ -300) concerned *the state of geometry at his time as science of our environment as experienced by our visual and motoric senses*. The presentation was done in the axiomatic–deductive mode that had been developed in the Old Greek's schools aiming for security of the mathematical activities as protection against otherwise maybe too intuitive and loose proceedings; -the present note not being mathematical, it might very well allow for too loose and intuitive wordings and speculations, eventually, of course-. Descartes' "*Géométrie*" (1637) presented his programme to base the whole Euclidean geometry on *the determination of the distances of all pairs of points by means of the theorem of Pythagoras properly expressed in Cartesian co–ordinates*; cfr. Figure 2 for the planar case. In this analytic geometry the axiomatic foundation of the former synthetic geometry basically is replaced by a geometrical foundation of the real number system and this programme works equally well in all dimensions. In particular Thales' theorem on similarities and Stevin's "parallelogram rule" for the (de)composition of forces did play not unimportant roles in this evolution. And, in this co–ordinate setting of Euclidean geometry *the infinitesimal calculus could geometrically be developed*, and, in turn, this made possible to Newton (∼ 1670) to obtain *the general analytic formula's* for *the curvature of the curves in a Euclidean plane at any of their points* and to Euler (∼ 1760) to describe *the curvature behaviour of the surfaces M^2 in a Euclidean space E^3 at any of their points* in terms of the curvatures there of the planar normal sections in all tangent directions to M^2.

In his "*Disquisitiones generalis circa superficies curvas*" (1827), Gauss carried over Descartes' programme to "*the inner geometry*" of surfaces M^2 in E^3. It is plausible that his interests in cartography and in geodesy may hereby have been very inspirational indeed. The surfaces are described by *curvilinear co–ordinates*, say (u, v), and "*a geometrical structure*" is defined on these surfaces M^2 by their *line element ds* as expressed by the infinitesimal distance function which is naturally induced on these surfaces M^2 from the standard theorem of Pythagoras' Euclidean distance function of the ambient space E^3, i.e. via *a generalised theorem of Pythagoras* on M^2, namely, ds^2 is given by a general homogeneous quadratic polynomial in infinitesimal changes of the curvilinear co–ordinates: $ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2$; cfr. Figure 3. And, Gauss' main motivation in his studies of geometry likely was to finally settle the *Euclidean parallel postulate problem*, which could be reformulated as the problem to find out whether there exist valid geometrical spaces other than the Euclidean space, this latter one being "*the mathematical space corresponding to our direct visual experiences*". The main curvature invariant of the intrinsic geometry of surfaces (M^2, ds^2) is *the Gauss curvature K*, which in terms of the extremal external Euler curvatures k_1 and k_2 is given by their product, $K = k_1 k_2$, while from the same intrinsic point of view the most important curves on surfaces (M^2, ds^2) are their *geodesics*. And, locally, the 2*D Euclidean geometry* and the classical 2*D elliptical non–Euclidean geometry* and the classical 2*D hyperbolical non–Euclidean geometry* of Lobachevsky–Bolyai are realised on *surfaces M^2 in $\mathbb{E}^3$ with constant Gauss curvatures K = o* (i.e. on "developable surfaces"), $K > o$ (like on spheres) and $K < o$ (like on pseudo–spheres or tractroids), respectively, the geodesics of these surfaces taking the place of the straight lines in the Euclidean planes. The awareness of an intrinsic geometry on surfaces M^2 in Euclidean space $\mathbb{E}^3$ demonstrated Gauss' clear understanding of the eminent distinction of *local surface isometries* within the

class of, in his words, "*the maps which are similar in their smallest parts*", i.e. which are *locally conformal*.

The intrinsic geometry of surfaces (M^2, ds^2) in E^3 was *abstrahated* -in the sense of being defined on "abstract" nD spaces, i.e. nD "manifolds" which are not a priori assumed to be situated in a surrounding standard space- *and generalised* to nD *Riemann–Finsler geometries* independently by Riemann and, respectively Helmholtz, in their *"Ueber die Hypothesen,* respectively *Tatsachen, welche der Geometrie zu Grunde liegen"* (1866 -Riemann's lecture of which this was the printed version having been given in 1854-, respectively 1868). Hereby both started from analytically n–fold extended "Mannigfaltigkeiten", i.e. both used systems of local co–ordinates $(x^1, x^2, \ldots, x^n)$, and, in the spirit of Descartes' programme for Euclidean geometry and as this had been further elaborated by Gauss for the intrinsic geometry of surfaces in $3D$ Euclidean spaces, then Riemann by hypothesis set off with a quite arbitrary "*Riemann–Finsler geometrical structure*" as line element ds and, to get more explicit and to go for most possible simplicity in his exposition, further on specified his hypothesis to *a positive definite metric tensor g* given in classical notation by $g = g_{hk}dx^h dx^k$, whereas Helmholtz right away came up with this same latter geometrical structure, i.e. at once set off with *a squared line element ds^2 given by a generalised theorem of Pythagoras* as a general quadratic homogeneous polynomial in infinitesimal changes of the co–ordinates as variables, because he found this to be the only factual possibility to allow for *measurements of distances invariant under congruences*; cfr. Figure 4. And the motivations of Riemann and Helmholtz for their introductions and first developments of Riemannian geometry came from their studies and reflections on problems in physics and human vision, respectively. For a general nD Riemannian space (M^n, g), the Gauss curvature K of surfaces (M^2, ds^2) in E^3 by *the theorema egregium formula* led to the notion of *the sectional or Riemann curvatures* $K(p, \pi)$ for any $2D$ tangent plane section π of M^n at any of its points p, as the Gauss curvature $K_{G^2}(p)$ at p of the $2D$ surface $(G^2, g|_{G^2})$ locally formed around p by the geodesics of (M^n, g) which are tangent to π at p. The abstract Riemannian spaces (M^n, g) which, in accordance with our natural expectation that the measurements of "beings" (à la Cartan) living in such spaces should not depend on the actual location in these spaces nor on the actual positioning of these beings at these places, or, put more academically, the Riemannian manifolds (M^n, g) which satisfy *the axiom of free mobility*, or, still, *the perfectly homogeneous and isotropic spaces*, (i.e. the spaces which behave in the same way at all points and at all points behave in the same way in all directions), are *the spaces* (M^n, g) *of constant sectional curvature K*, say $K = c$ (for possibly $c = o, c > o$ and $c < o$), these "*real space forms*" being denoted by $M^n(c)$, (the CC–spaces, *the spaces of constant curvature*). By the theorem of Beltrami, the real space forms $M^n(c)$ are the Riemannian spaces (M^n, g) which are *projectively equivalent with the locally Euclidean spaces*, i.e. with the spaces $M^n(o)$, of which *the Euclidean spaces* $\mathbb{E}^n$ are the prototypes, of course; *the nD classical elliptical and hyperbolical non–Euclidean spaces* $\mathbb{S}^n$ and $\mathbb{H}^n$ are the prototypes for the real space forms $M^n(c)$ for the cases $c > o$ and $c < o$, respectively. And, *these real space forms $M^n(c)$*, regardless $c = o, c > o$ or $c < o$, *all equally well do geometrically model "the ambient space of our direct visual sense experiences"*, since, as for instance discussed in Klein's "*Elementarmathematik vom höheren Standpunkte aus*", in view of the threshold of our sense perception and the fact that our space perception is adapted to a limited part of space only, our space perception can be described as closely as desired by Euclidean and non–Euclidean space forms alike. At this stage it could be worthwhile to look back at Gauss' point of departure in his studies of curved surfaces in order to see in how unforeseeable ways human knowledge and understanding may develop.

In the range of our kind's sense experiences (besides like feeling warm or thirsty etc.) likely next to our *notion of space* there is our *notion of time*, and, at least in our unsophisticated opinions of "where" and "when", we think of 3 space dimensions and we think of 1 time dimension. But, very naturally, whereas our appreciations of the space dimensions and of the time dimension certainly are so much of different natures indeed, already some of our kind's first serious studies in physics, namely in astronomy, were concerned with the very connection between these two fundamental notions, and this connection has remained of interest in various ways ever since. And, till still at present, as far as I know, the most straightforward and reasonable connection between physical space and time was described by Minkowski in his "*Raum und Zeit*" (1908), which started as follows: "M. H.! *Die Anschauugen über Raum und Zeit die ich Ihnen entwickeln möchte, sind auf experimentell–physikalischen Boden erwachsen. Darin liegt ihre Stärke. Ihre Tendenz ist eine Radikale. Von Stund an sollen Raum für sich und Zeit für sich völlig zu Schatten herabsinken, und nur noch eine Art Union der beiden soll Selbständigkeit bewahren*". In his physical spacetime $\mathbb{R}^4$ then, in local co–ordinates $(x, y, z; t)$ -whereby (x, y, z) denote the space co–ordinates and t the time co–ordinate-,

Minkowski made the extension of our natural measure of distances in space to a corresponding natural measure of distances in spacetime given by *an indefinite version of the classical theorem of Pythagoras* based on the physical "Weltpostulat" and on our psychologically distinct appreciations of time and of location, namely, to *the indefinite 4D Minkowski metric of index* 1: $ds^2 = dx^2 + dy^2 + dz^2 - dt^2$; (likely hereby the three letter combination "ict" showed up for the first and best time, and as Minkowski put it: "Man kann danach das Wesen dieses Postulates mathematisch sehr prägnant in die magische Formel kleiden: 300000 km = i sek", while, when looking at the time direction in conjunction with any basic space direction, or, still, at "the position of time with respect to space" -because, how else could we do than to mentally visualise time similar enough with our "every day" view of space though basically distinct from this following Minkowski's "Anschauungen über Raum und Zeit"-, one could hardly miss to see a kind of gravitational cross, once again). And, by generalisation of the 4*D* Minkowski space -which itself as briefly recalled before originated in the beginning understanding of our physical experiences of space and time-, one came to *the general pseudo Euclidean spaces E_k^n of arbitrary dimensions n and of arbitrary signatures $(n-k,k)$ or of arbitrary indices k*. And, then, still further, in analogy with the Riemannian spaces, *general pseudo Riemannian spaces* or *manifolds* too were introduced, by means of the geometrical structures defined by *generalised indefinite theorema of Pythagoras of arbitrary indices*. All in all, the development of *pseudo*, or also called *semi Riemannian geometry* could be done in the same way as Riemannain geometry proper, of course now devoting extra care to the special phenomena which present themselves for indefinite metrics (like: sectional curvatures only being determined for non–degenerate planes, the existence of real curves of zero length, etc.). The most well known application of pseudo Riemannian geometry may well be *the general theory of relativity* of Einstein, with, in particular, *the relativistic cosmology* of Friedmann–Lemaître.

So as ambient geometrical environment corresponding to our kind's experiences of space and time, from the Euclidean 3*D* spaces (or, for that matter, from the 3*D* real space forms) when only taking into account experiences of space, when moreover taking into account experiences of time arose the 4*D* spaces of Minkowski in which we could get accustomed soon enough to feel there too pretty well at ease. And, by formal extensions to arbitrary numbers of spacelike as well as of timelike dimensions, one arrived at the general pseudo Euclidean spaces of arbitrary dimensions and signatures. Since the direct observations of the manifold curves and surfaces in our most unsophisticated 3*D* Euclidean ambient world essentially constitute our most elementary visual experiences, in mathematical form, *the geometry of the curves and the surfaces in* $\mathbb{E}^3$, and in accordance with the above, in its most naturally generalised form, *the geometry of the submanifolds of pseudo Euclidean spaces* may well be considered as "*the geometry of the human kind*". On the one hand, the relevance of *the extrinsic geometry of submanifolds*, i.e. of the study of the shapes that these submanifolds assume in their ambient spaces, likely readily is pretty clear based on the obvious importance of our plain experiences of shape in so many fundamental situations during our life throughout. On the other hand, above was briefly recalled the emergence of *abstract Riemannian geometry* and of *abstract pseudo Riemannian geometry* as generalisation and extension of *the intrinsic geometry of surfaces M^2 in* $\mathbb{E}^3$; for more about these geometries, one could a.o. consult Chern's Preface to [12], Berger's Panorama [19] and references [16][17][20][21].

Concerning Riemannian geometry, as was anticipated already right away after the abstract Riemannain manifolds (M^n, g) had been born, starting with Schläfli, every such abstract space (M^n, g) can be isometrically embedded in Eucliden ambient spaces $\mathbb{E}^{n+m}$ with appropriate co–dimensions m, i.e. every Riemannian manifold (M^n, g) can be geometrically identified with an *nD* submanifold M^n of $\mathbb{E}^{n+m}$, such that every abstract Riemannian geometry essentially is nothing else than the (more) concrete intrinsic geometry of some submanifolds in some Euclidean ambient spaces, by *the embedding theorem of Nash* (1956); and corresponding embedding theorems for the pseudo Riemannian situation later were obtained by Clarke and Greene. In conclusion: the general geometry of submanifolds in pseudo Euclidean spaces in its *extrinsic part* consists of the most natural mathematical study of what corresponds to our kind's most primitive visual experiences of the shapes of "things in space" and in its *intrinsic part*, i.e. with the inner geometry of submanifolds, or, still, with pseudo Riemannian geometry, consists of the most natural mathematical study of the geometrical properties of these submanifolds considered "on themselves", or, still, of the geometry which in a way was forced upon these submanifolds, now however considered as entities existing on their own, by their ambient geometrical worlds, in consequence of the shapes that they there assume, and, of course, the study

of the relations between intrinsic and extrinsic properties is of great value in the geometry of submanifolds as such. What in the course of time really has been considered to be geometry, certainly till not so long ago, has been *the part of mathematics that is essentially and very closely connected with our kind's vision, static and dynamic*, and in the first place hereby comes the geometry of submanifolds. And, in view of the formerly given references to Bronowski, at this stage certainly the following citation of Chern from [12] may well be at its place here: "*While algebra and analysis provide the foundations of mathematics, geometry is at the core*", and, further e.g. taking into consideration from Steinhaus' "*Kaleidoskop der Mathematik*" that "*der Gegenstand der Mathematik ist die Wirklichkeit, kein Hirngespinst*", the manifest importance of the geometry of submanifolds for science and technology and medicine may not come as a surprise.

3. But, as Descartes stated in his *"Géométrie"*: *"What is most important in the study of geometry is the way it cultivates the mind"*. And, at least at its very beginning, *psychology was the science with as object of study the workings of the mind*. From Dombrowski's "*150 years after Gauss' ≪ diquisitiones generales circa superficies curvas ≫*" comes the following: "Gauss saw clearly *the ambivalence of the use of analytical calculations in geometrical problems* (i.e. their effectiveness on the one hand, and on the other hand their inherent tendency to weaken the force of geometrical intuition), as is shown by the following excerpts from his *review of the "Géométrie descriptive" by G. Monge*: 'It is not to be denied that the advantages of an analytical treatment over a geometrical treatment, its conciseness, simplicity, uniformity, and especially its generality, usually become more and more decisive as the investigations become more difficult and more complicated. However, *it is always very important to continue to cultivate the geometrical method*. (...) In particular we must praise the work under consideration for its great clarity (...), and therefore recommend its study as *nourishing intellectual substance*, by which undoubtedly much can be contributed to *the revival and conservation of the genuine geometrical spirit*, sometimes missing in the mathematics of these times'. The latter recommendation is supplemented and rounded off in the review by the remark (which is also interesting didactically) that *the geometric method will 'remain indispensable in the early study of young people, to prevent one–sidedness (...) and to give to the understanding a lineliness and directness, which are much less developed and -occasionally- rather jeopardized by the analytical method'* ". Concerning "the revival and conservation of the geometrical spirit" hereby mentioned, the subsequent history of mathematics and of pure and applied sciences alike may show that these aspirations of Gauss have become well realised indeed, at least till some decades ago. Next follows the opening part of Struik's *Preface* to his 1953 MIT "*Lectures on Analytic and Projective Geometry*": "The extension of the mathematical curriculum in our colleges has not infrequently been at the expense of some of the most valuable fields of more 'old–fashioned' mathematics. *Among the victims we find elementary, projective and algebraic geometry, fields which used to stir the enthusiasm of an older generation. This decline into relative neglect not only means that mathematicians grow up poorly acquainted with one of the most attractive parts of their science, elegant in form and in results, but also means a loss of more fundamental values*. This 'modern geometry' of the nineteenth century was to a considerable extent responsible for the whole revolution in mathematical thinking typical of this period; out of it came the concepts of non–Euclidean geometry and geometry of more than three dimensions, and it contributed substantially to the formation of such topics as transformation, group, invariant, and oriented quantity. Moreover, it has profoundly influenced axiomatics. *Neglect of this part of mathematics therefore tends to stifle the understanding of some of the most important notions of modern mathematics, physics and engineering*. The task is set to find the legitimate place for this field inside our present mathematical curriculum, and to stress those fundamentals which are most vital for the understanding of our science as a whole".

Yet as far as I can see, at present even at most universities around the globe, (universities which form -as far as teaching is concerned- a dense global network of pretty identical expensive strange sort of youth camps), no more teaching of geometry as such is done anymore. To make matters not better: before ever starting with a chance of success any type of "higher studies" worthy for members of our kind, *a delicate initiation of the young children in intuitive geometry*, including manifold observations of fascinating natural and other forms, extremely simple ones as well as beautiful more complicated ones, and *a constructive development in them of a serious dose of common sense*, both starting from in the *kindergartens*, are necessary processes they should have enjoyed going through in order to attain the mental capability to really understand basic geometry, and *a proper education in geometry* should then be offered to these pupils, continuously and

patiently, during the last ten years or so of *the primary and the secondary schools*, and one could look around to see how very very few young people do have the luck of being "brought up" more or less in this way while all are entitled to this, and, apart from exceptions, are able to do well in this all along the way, and this way moreover should be open for free.
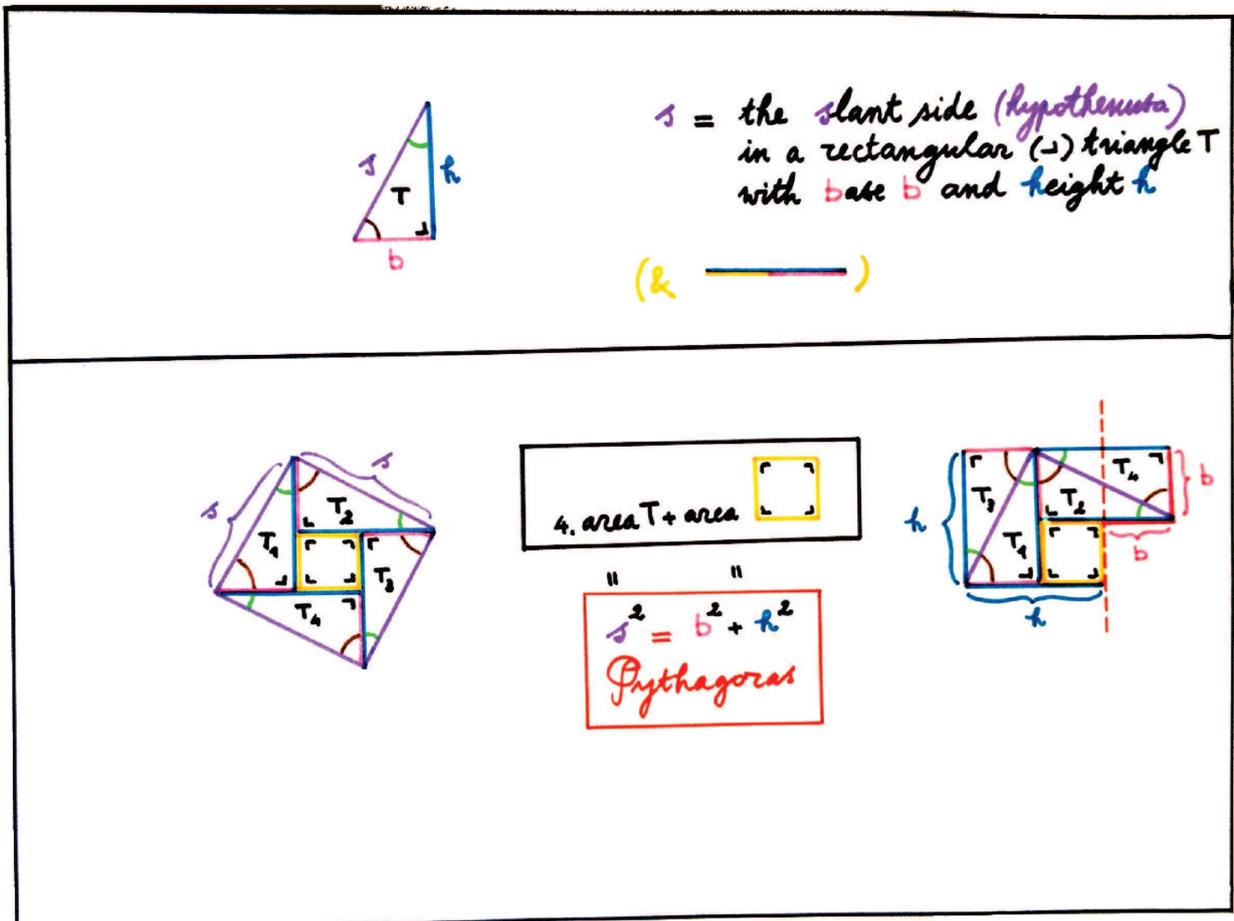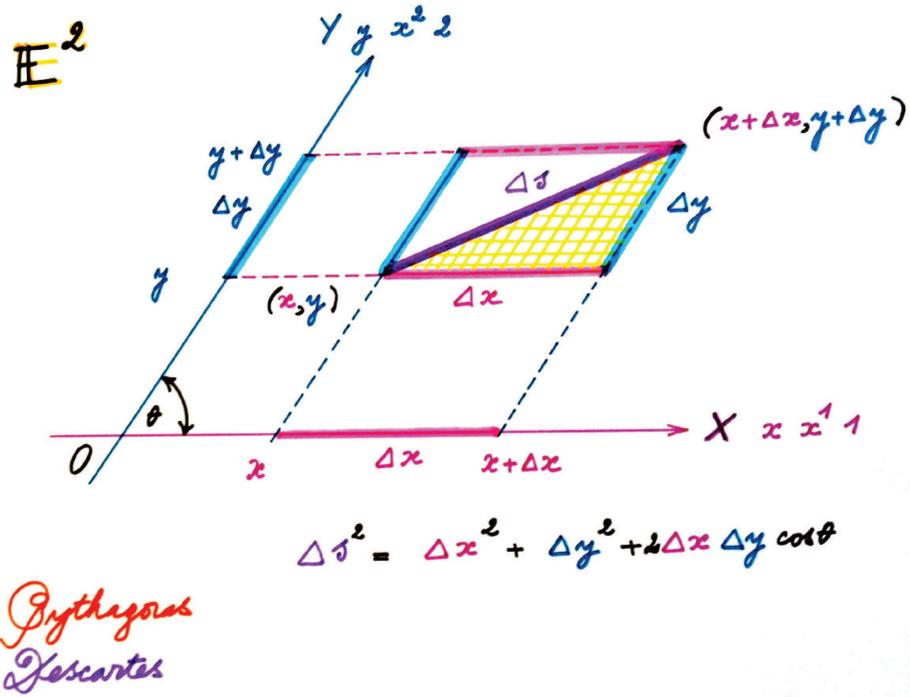

Figure 1

$\mathbb{E}^2$

$Y \quad y \quad x^2 \quad 2$

$y+\Delta y$

Δy

$(x+\Delta x, y+\Delta y)$

Δs

Δy

y

(x,y)

Δx

θ

$X \quad x \quad x^1 \quad 1$

0

$x \qquad \Delta x \qquad x+\Delta x$

$$\Delta s^2 = \Delta x^2 + \Delta y^2 + 2\Delta x\,\Delta y \cos\theta$$

Pythagoras
Descartes

Figure 2

$M^2 \subset \mathbb{E}^3$

ds

(u,v)

du

$(u+du, v+dv)$
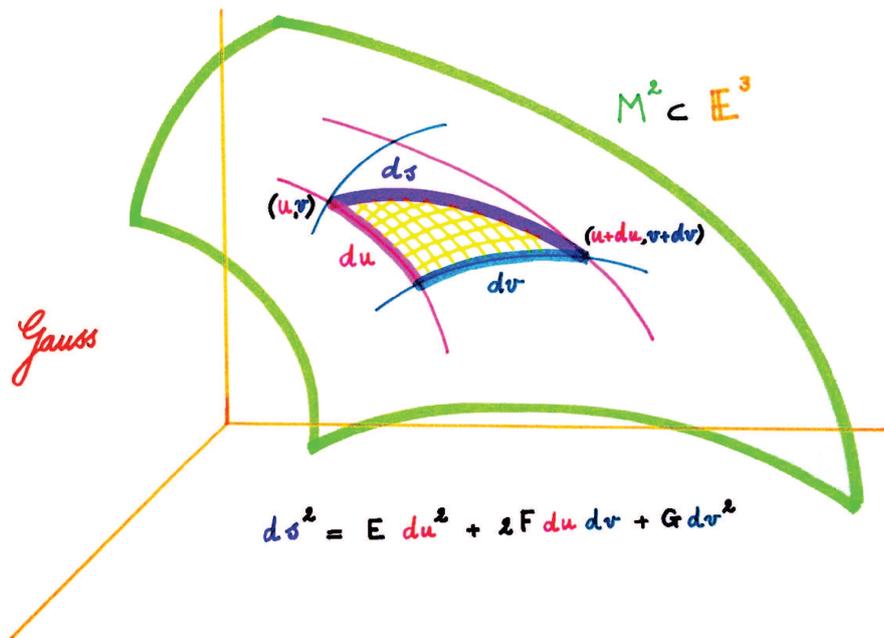
dv

Gauss

$$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2$$

Figure 3
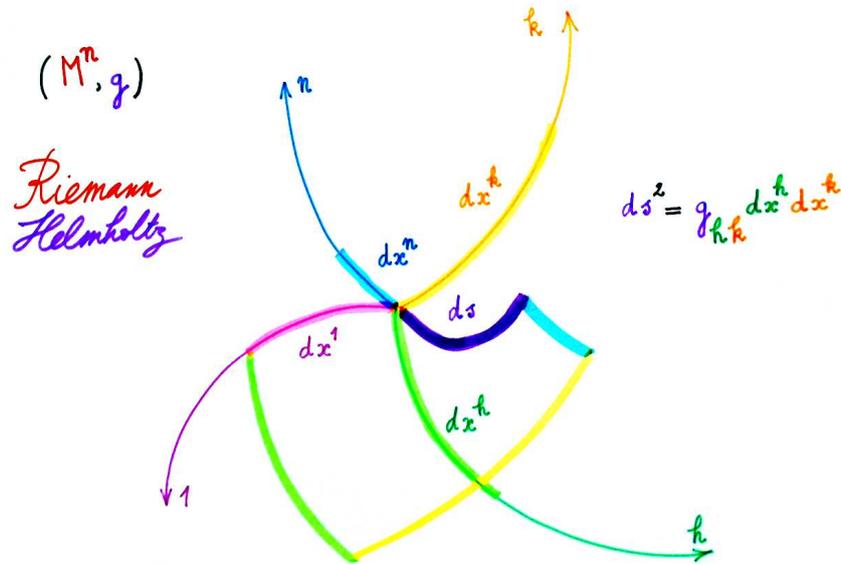
Figure 4

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