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A Note About the Torsion of Null Curves in the 3-Dimensional Minkowski Spacetime and the Schwarzian Derivative

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Dedicated to the memory of Professor Włodzimierz Waliszewski

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Abstract. The main topic of this paper is to show that in the 3-dimensional Minkowski spacetime, the torsion of a null curve is equal to the Schwarzian derivative of a certain function appearing in a description of the curve. As applications, we obtain descriptions of the slant helices, and null curves for which the torsion is of the form $\tau = -2\lambda s$, s being the pseudo-arc parameter and $\lambda = \text{const} \neq 0$.

1. Introduction

There are very many papers about geometric properties of null curves in the Minkowski spacetimes. We refer the monographs [4, 5], and the survey articles [3, 11, 12], etc.

On the other hand, there is the classical notion of the Schwarzian derivative in mathematical analysis. This notion has many important applications in mathematical analysis (real and complex) and differential geometry; see [6, 7, 13–15], etc. The author is specially inspired by the paper [7], where it is shown a strict relation between the Schwarzian derivative and the curvature of worldlines in 2-dimenional Lorentzian manifolds of constant curvature.

In the presented short paper, we will show that the torsion of a null curve in the 3-dimensional Minkowski spacetime $\mathbb{E}^3_1$ is equal to the Schwarzian derivative of a certain function appearing in a description of the curve. Descriptions of the slant helices are obtained, and null curves for which the torsion is given by $\tau = -2\lambda s$, s being the pseudo-arc parameter and $\lambda = \text{const} \neq 0$.

2. Preliminaries

Let $\mathbb{E}^3_1$ be the 3-dimensional Minkowski spacetime, that is, the Cartesian $\mathbb{R}^3$ endowed with the standard Minkowski metric g given with respect to the Cartesian coordinates (x, y, z) by

$$g = dx \otimes dx + dy \otimes dy - dz \otimes dz, \tag{1}$$

or as the symmetric 2-form $g = dx^2 + dy^2 - dz^2$.

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Let $\alpha\colon I \to \mathbb{E}^3_1$ be a null (ligth-like) curve in $\mathbb{E}^3_1$, I being an open interval. Thus, $g(\alpha',\alpha') = 0$, that is, $g(\alpha'(t),\alpha'(t)) = 0$ for any $t \in I$. We also assume that the curve is non-degenerate, in the sense the three vector fields α', α'', α''' are linearly independent at every point of the curve.

Since $g(\alpha',\alpha') = 0$ and $g(\alpha',\alpha'') = 0$, it must be that $g(\alpha'',\alpha'') > 0$. A parametrization of the null curve is said to be pseudo-arc (or distinguished) if $g(\alpha'',\alpha'') = 1$. A null curve can always be parametrized by a pseudo-arc parameter. However, such a parameter is not uniquely defined. Precisely, for a null curve α, if s_1 is a pseudo-arc parameter, then s_2 is a pseudo-arc parameter if and only if there exists a constant c such that $s_2 = \pm s_1 + c$.

In the sequel, we assume that the parametrization of a null curve is pseudo-arc, and we denote such a parameter by s.

In the next section, we need the standard theorms concerning of null curves which can be formulated in the following manner (see e.g. [3, 5, 11, 12]):

Let α be a null curve in the 3-dimensional Minkowski spacetime $\mathbb{E}^3_1$. Then, there exists the only one Cartan moving frame $(\mathbf{L} = \alpha', \mathbf{N}, \mathbf{W})$ and the function τ defined along the curve α and such that

$$g(\mathbf{L},\mathbf{N}) = g(\mathbf{W},\mathbf{W}) = 1, \quad g(\mathbf{L},\mathbf{L}) = g(\mathbf{L},\mathbf{W}) = g(\mathbf{N},\mathbf{N}) = g(\mathbf{N},\mathbf{W}) = 0, \tag{2}$$

and the following system of differential equations

$$\mathbf{L}' = \mathbf{W}, \quad \mathbf{N}' = \tau\mathbf{W}, \quad \mathbf{W}' = -\tau\mathbf{L} - \mathbf{N} \tag{3}$$

is satisfied. These vector fileds are given by

$$\mathbf{L} = \alpha', \quad \mathbf{W} = \alpha'', \quad \mathbf{N} = -\alpha''' - \frac{1}{2}g(\alpha''',\alpha''')\alpha', \tag{4}$$

and the function τ by

$$\tau = \frac{1}{2}g(\alpha''',\alpha'''). \tag{5}$$

From these results it can be deduced that a given function τ on an open interval I, there exists the only one null curve $\alpha\colon I \to \mathbb{E}^3_1$ realizing (2) and (3) up to the orientation of this curve and up to the isometries of the Minkowski space $\mathbb{E}^3_1$.

The triple $(\mathbf{L},\mathbf{N},\mathbf{W})$ defined in (4) is called the Frenet frame, the function τ defined in (5) is called the torsion, and the equations (3) are called the Frenet equations of the null curve α. Since

$$\det[\mathbf{L},\mathbf{N},\mathbf{W}] = \det[\alpha',\alpha'',\alpha'''], \tag{6}$$

the frames $(\mathbf{L},\mathbf{N},\mathbf{W})$ and $(\alpha',\alpha'',\alpha''')$ have the same orientations.

In the following section, we are going to expresse the torsion τ and the frame $(\mathbf{L},\mathbf{N},\mathbf{W})$ with the help of a special function related to a pseudo-arc parametrization of a null curve in $\mathbf{E}^3_1$.

3. A description of the torsion

Let $\alpha\colon I \to \mathbb{E}^3_1$ be a null curve. Simplifying denotations, we write $\alpha(s) = (x(s), y(s), z(s))$, $s \in I$, where s is a pseudo-arc parameter, and $x(s)$, $y(s)$, $z(s)$ are certain functions of s. Then, we have

$$\alpha' = x'\frac{\partial}{\partial x}\Big|_\alpha + y'\frac{\partial}{\partial y}\Big|_\alpha + z'\frac{\partial}{\partial z}\Big|_\alpha.$$

For simplicity, instead of that, we will write $\alpha' = (x', y', z')$. And in the similar manner, the next derivatives of α will be written, e.g., $\alpha'' = (x'', y'', z'')$.

Using (1), our two assumptions: $g(\alpha', \alpha') = 0$ (the nullity condition), and $g(\alpha'', \alpha'') = 1$ (the pseudo-arc parametrization) give the following two equalities

$$\begin{aligned} x'^2 + y'^2 - z'^2 &= 0, && (7)\\ x''^2 + y''^2 - z''^2 &= 1. && (8) \end{aligned}$$

One notes that the shapes of the equalities (7) and (8) exclude the situation when at least one of the functions x', y', z' vanishes on an open subinterval of I. In the sequel, restricting slightly the assumptions, we will consider only the case when $x' \neq 0$, $y' \neq 0$ and $z' \neq 0$ on I.

It is a standard and elementary idea that from (7), it follows that

$$x' = h, \quad y' = \frac{h}{2}\left(f - \frac{1}{f}\right), \quad z' = \frac{h}{2}\left(f + \frac{1}{f}\right), \tag{9}$$

f and h being certain non-zero functions on I. Hence,

$$\begin{aligned} x'' &= h', \\ y'' &= \frac{fh'(f^2 - 1) + hf'(f^2 + 1)}{2f^2}, \\ z'' &= \frac{fh'(f^2 + 1) + hf'(f^2 - 1)}{2f^2}. \end{aligned}$$

In view of the above relations, the equality (8) turns into $h^2 f'^2 = f^2$. Hence, f' is non-zero (and has constant sign) on I. Consequently,

$$h = \varepsilon \frac{f}{f'}, \ \varepsilon = \pm 1.$$

Thus, for the vector field $\mathbf{L}$ (cf. (4)), we have

$$\mathbf{L} = \alpha' = \frac{\varepsilon}{2f'}\left(2f, f^2 - 1, f^2 + 1\right). \tag{10}$$

Consequently, we get the following description of the curve α

$$\alpha(s) = \alpha(s_0) + \frac{\varepsilon}{2}\int_{s_0}^{s} \frac{1}{f'(t)}\left(2f(t), f^2(t) - 1, f^2(t) + 1\right) dt, \ s, s_0 \in I.$$

Conversely, if a curve α is given by the last formula, then (7) and (8) are fulfilled so that the curve is null and not geodesic, and the parameter s is distinguish.

From (10), we obtain for the vector field $\mathbf{W}$ (cf. (4)),

$$\mathbf{W} = \alpha'' = -\frac{\varepsilon f''}{2f'^2}\left(2f, f^2 - 1, f^2 + 1\right) + \varepsilon\,(1, f, f). \tag{11}$$

From (11), we find

$$\alpha''' = \varepsilon \frac{2f''^2 - f'f'''}{2f'^3}\left(2f, f^2 - 1, f^2 + 1\right) - \frac{\varepsilon f''}{f'}(1, f, f) + \varepsilon f'(0, 1, 1). \tag{12}$$

To compute $g(\alpha''', \alpha''')$, using (1), we find at first the following

$$\begin{aligned} &g\left(\left(2f, f^2 - 1, f^2 + 1\right), \left(2f, f^2 - 1, f^2 + 1\right)\right) = 0, \\ &g\left(\left(2f, f^2 - 1, f^2 + 1\right), (1, f, f)\right) = 0, \quad g\left(\left(2f, f^2 - 1, f^2 + 1\right), (0, 1, 1)\right) = -2, \\ &g\left((1, f, f), (1, f, f)\right) = 1, \quad g\left((1, f, f), (0, 1, 1)\right) = 0, \quad g\left((0, 1, 1), (0, 1, 1)\right) = 0. \end{aligned}$$

Then, having (12) and applying the above formulas, we get

$$g(\alpha''', \alpha''') = \frac{2f'f''' - 3f''^2}{f'^2}. \tag{13}$$

In view of (13) and (5), the torsion must be of the form

$$\tau = \frac{2f'f''' - 3f''^2}{2f'^2} = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2. \tag{14}$$

Now, it is important to note that the right hand side of the formula (14) is just the Schwarzian derivative of the function f, which is usually denoted by $S(f)$. Thus, $\tau = S(f)$.

Finally, applying (10), (12) and (13) into (4), we find the vector field

$$\mathbf{N} = -\frac{\varepsilon f''^2}{4f'^3}\left(2f, f^2 - 1, f^2 + 1\right) + \frac{\varepsilon f''}{f'}(1, f, f) - \varepsilon f'(0, 1, 1). \tag{15}$$

Summarizing the above considerations, we can formulate the following theorem.

Theorem 1. *Let $\mathbb{E}^3_1$ be the 3-dimensional Minkowski spacetime. Any (non-degenerate) null curve α in $\mathbb{E}^3_1$ can be parametrized in the following way*

$$\alpha(s) = \alpha(s_0) + \frac{\varepsilon}{2}\int_{s_0}^{s} \frac{1}{f'(t)}\left(2f(t), f^2(t) - 1, f^2(t) + 1\right) dt, \; s, s_0 \in I, \tag{16}$$

where s is a pseudo-arc parameter, I is a certain open interval, f is a non-zero function with non-zero derivative f' on I. The torsion τ of such a curve is equal to the Schwarzian derivative of the function f, that is,

$$\tau = S(f) = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2. \tag{17}$$

The vector fields forming the Frenet frame of the curve α are given by the formulas (10), (11) and (15).

Remark 1. *Applying formulas (10), (11), (15), it can be verified that*

$$\det[\mathbf{L}, \mathbf{N}, \mathbf{W}] = \varepsilon.$$

This together with (6) implies that the constant ε appering in (16) corresponds to the orientation of the curve α. Note that the torsion does not depend on the orientation of the curve. Moreover, the torsion and the orientation does not depend on the sign of the function f.

Remark 2. *The Schwarzian derivative S is an invariant of a fractional-linear transformation T of the 1-dimensional real projective space $\mathbb{R}P^1 = \mathbb{R} \cup \infty$ (cf. e.g. [13]). That is, $S(T \circ f) = S(f)$ if f is a function on $\mathbb{R}P^1$ and $T\colon \mathbb{R}P^1 \to \mathbb{R}P^1$ is given by*

$$T(r) = \frac{ar + b}{cr + d}, \; r \in \mathbb{R}P^1, \; a, b, c, d \in \mathbb{R}, ad - bc \neq 0. \tag{18}$$

We can apply the above fact seeking for null curves with given torsion τ. However, we should be careful since the domains of our functions f and $T \circ f$ may be defined only on some open subintervals lying on the real line $\mathbb{R}$.

4. Null Cartan helices

It is well-known that there are exactly three types of null curves with constant torsion in the Minkowski spacetime $\mathbb{E}^3_1$ (cf e.g., [9]) up to the orientation of the curve and up to the isometries of the space. They are often called the null Cartan helices.

As a first application of the results from the previous section, we demonstrate how these classes of curves can be recovered from their torsions.

(a) For $f(s) = s$, it holds $S(f) = 0$. In (16), we put $f(s) = s,\ s_0 = 0,\ \alpha(s_0) = (0, 0, 0),\ \varepsilon = 1$. Then, we obtain the curve

$$\alpha(s) = \frac{1}{6}\left(3s^2, s^3 - 3s, s^3 + 3s\right),$$

for which by (17) we have $\tau = 0$. Thus, α is a positively oriented null Cartan helix of zero torsion.

(b) For $f(s) = -\cot(cs/2)$, it holds $S(f) = c^2/2$. In (16), we put

$$f(s) = -\cot\frac{cs}{2},\ \alpha(0) = \left(\frac{1}{c^2}, 0, 0\right),\ \varepsilon = 1,\ c = \mathrm{const.} > 0.$$

Then, we obtain the curve

$$\alpha(s) = \frac{1}{c^2}\left(\cos(cs), \sin(cs), cs\right),$$

for which by (17) it holds $\tau = c^2/2$. Thus, α is a positively oriented null Cartan helix of constant positive torsion.

(c) For $f(s) = e^{cs}$, it holds $S(f) = -c^2/2$. In (16), we put

$$f(s) = e^{cs},\ \alpha(0) = \left(0, \frac{1}{c^2}, 0\right),\ \varepsilon = 1,\ c = \mathrm{const.} > 0.$$

Then, we obtain the curve

$$\alpha(s) = \frac{1}{c^2}\left(cs, \cosh(cs), \sinh(cs)\right),$$

for which by (17) we have $\tau = -c^2/2$. Thus, α is a positively oriented null Cartan helix of constant negative torsion.

Thus, we have seen the following:

Corollary 1. *Null helices in $\mathbb{E}^3_1$ form the three classes described in (a) – (c) in the above. The description is valid up to the pseudo-arc parameter changies, up to the orientation of the curve, and up to the isometries of the space.*

A curve $\alpha\colon I \to \mathbb{E}^3_1$ is called a general (or generalized) helix if there exists a non-zero vector V in $\mathbb{E}^3_1$ such that $g(\alpha', V) = \mathrm{const.}$; cf. [8, 9, 17], etc. This means that tangent indicatrix is laid in a plane or, equivalently, there exists a non-zero constant vector V in $\mathbb{E}^3_1$ for which $g(\alpha'', V) = 0$, that is, V is orthogonal to the acceleration vector field α''.

For null curves, it is already proved that null general helices in $\mathbb{E}^3_1$ are precisely the null Cartan helices; cf. ibidem.

5. Null slant helices

Following the ideas of [1, 2, 10], a slant helix is defined to be the curve (null as well as non-null) in $\mathbb{E}^3_1$ which satisfies the condition

$$g(\alpha'', V) = c = \mathrm{const.} \tag{19}$$

along the curve α, where V is a constant vector. Thus, a general helix is a slant helix with $c = 0$. Conversely, a slant helix with $c = 0$ becomes a general helix. In [1, Theorem 1.4], it is proved that a null curve in $\mathbb{E}^3_1$ is a slant helix if and only if its torsion is given by

$$\tau = \frac{a}{(cs+b)^2},\ a, b, c = \text{const}., \tag{20}$$

where c is just the constant realizing (19).

As the second applications of the results from Section 3, we will describe the null slant helices in $\mathbb{E}^3_1$ which are different from the usual helices ($a \neq 0$ and $c \neq 0$ in (20)).

Note that moving the pseudo-arc parameter s into $s - b/c$ and next modifying slightly the constant a, we can write the condition (20) as

$$\tau = \frac{a}{2s^2},\ a = \text{const} \neq 0. \tag{21}$$

We can also assume that $s > 0$. Using (2) and (3), it can be checked that when the relation (21) is fulfilled, then for the vector

$$V = -\frac{a}{2s}\mathbf{L} + s\mathbf{N} + \mathbf{W}$$

it holds $V' = 0$ and $g(\alpha'', V) = g(\mathbf{W}, V) = 1$ (cf. ibidem).

(a) In (16), we put

$$f(s) = \ln s,\ s_0 = 1,\ \alpha(s_0) = \frac{1}{8}(-2, -1, 3),\ \varepsilon = 1.$$

Then, we obtain the curve

$$\alpha(s) = \frac{s^2}{8}\left(2(2\ln s - 1), 2\ln^2 s - 2\ln s - 1, 2\ln^2 s - 2\ln s + 3\right),$$

for which by (17) it holds

$$\tau = S(f) = \frac{1}{2s^2}.$$

Thus, α is a slant helix realizing (21) with $a = 1$.

(b) Let $a > 1$ and $b = \sqrt{a-1} > 0$. In (16), we put

$$f(s) = \tan\left(\frac{1}{2}\ln s^b\right),\ s_0 = 1,\ \alpha(s_0) = \frac{1}{b}\left(-\frac{b}{b^2+4}, -\frac{2}{b^2+4}, \frac{1}{2}\right),\ \varepsilon = 1.$$

Then, we obtain the curve

$$\alpha(s) = \frac{s^2}{b}\left(\frac{2\sin(\ln s^b) - b\cos(\ln s^b)}{b^2+4}, -\frac{2\cos(\ln s^b) + b\sin(\ln s^b)}{b^2+4}, \frac{1}{2}\right),$$

for which by (17) it holds

$$\tau = S(f) = \frac{1+b^2}{2s^2} = \frac{a}{2s^2}.$$

Thus, α is a slant helix realizing (21) with $a > 1$.

(c) Let $0 \neq a < 1$. Then for $b = \sqrt{1-a}$, we have $b > 0$ and $b \neq 1$. Consider the case $a \neq -3$, that is, $b \neq 2$. In (16), we put

$$f(s) = s^{-b},\ s_0 = 1,\ \alpha(s_0) = \frac{1}{2b}\left(-1, \frac{2b}{b^2-4}, \frac{4}{b^2-4}\right),\ \varepsilon = 1.$$

Then, we obtain the curve

$$\alpha(s) = \frac{s^2}{2b}\left(-1, \frac{s^{-b}}{b-2} + \frac{s^b}{b+2}, \frac{s^{-b}}{b-2} - \frac{s^b}{b+2}\right),$$

for which by (17) it holds

$$\tau = S(f) = \frac{1-b^2}{2s^2} = \frac{a}{2s^2}.$$

Thus, α is a slant helix realizing (21) with $-3 \neq a < 1$.

(d) In (16), we put

$$f(s) = \frac{1}{s^2},\ s_0 = 1,\ \alpha(s_0) = \frac{1}{16}(-4, 1, -1),\ \varepsilon = 1.$$

Then, we obtain the curve

$$\alpha(s) = \frac{1}{16}\left(-4s^2, s^4 - 4\ln s, -s^4 - 4\ln s\right),$$

for which by (17) it holds

$$\tau = S(f) = -\frac{3}{2s^2}.$$

Thus, α is a slant helix realizing (21) with $a = -3$.

Thus, we have shown the following:

Corollary 2. *Null slant helices in $\mathbb{E}^3_1$ form the four classes described in (a) – (d) in the above. The description is valid up to the pseudo-arc parameter changies, up to the orientation of the curve, and up to the isometries of the space.*

6. Null curves with the torsion proportional to the pseudo-arc parameter

In this section, we determine the null curves in $\mathbb{E}^3_1$ for which $\tau = -2\lambda s$, $\lambda = \mathrm{const.} \neq 0$. We will use the formula (17).

According to our Theorem, we need at first to find a solution of the differential equation

$$\left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2 = -2\lambda s. \tag{22}$$

We seek for solutions of this equation in the form

$$f(s) = \int \frac{ds}{\phi^2(s)}, \tag{23}$$

ϕ being an unknown fucntion. Then the equation (22) becomes the following differential equation

$$\phi'' - \lambda s\phi = 0. \tag{24}$$

The general solution of the above equation is

$$\phi(s) = c_1 \operatorname{Ai}(\mu s) + c_2 \operatorname{Bi}(\mu s),\ \mu = \sqrt[3]{\lambda},\ c_1, c_2 = \mathrm{const.}$$

where Ai and Bi are the Airy functions of the first and second kind, respectively. For the solutions of (24) and for the special Airy functions, we refer [16], [18], etc. In the below calculations, we use the basic properties of these functions.

For our purpose, we take the only one solution of (24), say $\phi(s) = \mathrm{Ai}\,(\mu s)$. Then, from (23) we get

$$f(s) = \frac{\pi}{\mu} \cdot \frac{\mathrm{Bi}(\mu s)}{\mathrm{Ai}(\mu s)}. \tag{25}$$

Next,

$$f'(s) = \frac{1}{\mathrm{Ai}^2(\mu s)}. \tag{26}$$

Having (10) with $\varepsilon = 1$, and using (25) and (26), we can write α' as

$$\alpha'(s) = \left(\frac{\pi}{\mu}\,\mathrm{Ai}(\mu s)\,\mathrm{Bi}(\mu s), \frac{1}{2\mu^2}\left(\pi^2\,\mathrm{Bi}^2(\mu s) - \mu^2\,\mathrm{Ai}^2(\mu s)\right), \frac{1}{2\mu^2}\left(\pi^2\,\mathrm{Bi}^2(\mu s) + \mu^2\,\mathrm{Ai}^2(\mu s)\right)\right).$$

The integration of the last equality gives the following curve

$$\begin{aligned}\alpha(s) \;=\; &\left(\frac{\pi}{\mu^2}\left(\mu s\,\mathrm{Ai}(\mu s)\,\mathrm{Bi}(\mu s) - \mathrm{Ai}'(\mu s)\,\mathrm{Bi}'(\mu s)\right),\right.\\ &\frac{1}{2\mu^3}\left(\pi^2\left(\mu s\,\mathrm{Bi}^2(\mu s) - \mathrm{Bi}'^2(\mu s)\right) - \mu^3 s\,\mathrm{Ai}^2(\mu s) + \mu^2\,\mathrm{Ai}'^2(\mu s)\right),\\ &\left.\frac{1}{2\mu^3}\left(\pi^2\left(\mu s\,\mathrm{Bi}^2(\mu s) - \mathrm{Bi}'^2(\mu s)\right) + \mu^3 s\,\mathrm{Ai}^2(\mu s) - \mu^2\,\mathrm{Ai}'^2(\mu s)\right)\right),\end{aligned} \tag{27}$$

if the the initial condition at $s_0 = 0$ is

$$\alpha(0) = \frac{1}{2\sqrt[3]{9}\,\mu^3\,\Gamma^2(\frac{1}{3})}\left(2\sqrt{3}\,\mu\,\pi, \mu^2 - 3\pi^2, -\mu^2 - 3\pi^2\right).$$

Thus, we can formulate the following:

Corollary 3. *Null curves in $\mathbb{E}^3_1$ for which $\tau = -2\lambda s$, $\lambda = \mathrm{const}. \neq 0$, are given by the formula (27) with $\mu = \sqrt[3]{\lambda}$ up to the pseudo-arc parameter changies, up to the orientation of the curve, and up to the isometries of the space.*

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