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Some Properties of ET-Projective Tensors Obtained from Weyl Projective Tensor

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Abstract. Vanishing of linearly independent curvature tensors of a non-symmetric affine connection space as functions of vanished curvature tensor of the associated space of this one are analyzed in the first part of this paper. Projective curvature tensors of a non-symmetric affine connection space are expressed as functions of the affine connection coefficients and Weyl projective tensor of the corresponding associated affine connection space in the second part of this paper.

1. Introduction

Many authors have been interested in non-symmetric affine connection spaces theory research. Einstein was the first one who used non-symmetric affine connection spaces in his research area (see [2–4]). In the Unified Field Theory (UFT) instead of Riemannian space, he involves non-symmetric affine connection coefficients Γ^i_{jk} which are independent of metric tensor g_{ij}.

While at Riemannian space connection coefficients are expressed in terms of g_{ij}, functional connections of these quantities are determined with the following equations in Einstein's works about UFT:

$$g_{\underset{+-}{ij};m} \equiv g_{ij,m} - \Gamma^p_{im} g_{pj} - \Gamma^p_{mj} g_{ip} = 0. \tag{1}$$

After Einstein, a lot of authors have developed theory of non-symmetric affine connection spaces. Some of them are M. Prvanović [17], R. S. Mishra [15], S. M. Minčić [7–14], Lj. S. Velimirović [1, 13] and many others.

Definition 1.1. *An N-dimensional manifold $\mathcal{M}$ endowed with affine connection coefficients $L^i_{jk} = L^i_{jk}(x)$, $x \in \mathcal{M}$, is an N-dimensional* affine connection space.

Affine connection coefficients L^i_{jk} of this space satisfy the following equation

$$L^{i'}_{j'k'} = x^{i'}_i x^j_{j'} x^k_{k'} L^i_{jk} + x^{i'}_i x^i_{j'k'}, \tag{2}$$

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where e.g. $x^{i'}_i = \dfrac{\partial x^{i'}}{\partial x^i}$.

Based on eventually non-symmetry of affine connection coefficients L^i_{jk} by indices j and k, the symmetric part and the non-symmetric one (a half of a *torsion tensor* T^i_{jk}) of these connection coefficients respectively are defined as

$$L^i_{\underline{jk}} = \frac{1}{2}\left(L^i_{jk} + L^i_{kj}\right) \quad \text{and} \quad L^i_{\underset{\vee}{jk}} = \frac{1}{2}\left(L^i_{jk} - L^i_{kj}\right) = \frac{1}{2}T^i_{jk}. \tag{3}$$

It is evident it holds the equality

$$L^i_{jk} = L^i_{\underline{jk}} + L^i_{\underset{\vee}{jk}} = L^i_{\underline{jk}} + \frac{1}{2}T^i_{jk}. \tag{4}$$

Definition 1.2. *An N-dimensional space A_N with $T^i_{jk} = 0$ is* an affine connection space without torsion *and a space $\mathbb{G}\mathbb{A}_N$ where it exist indices i_0, j_0, k_0 such that $T^{i_0}_{j_0k_0} \neq 0$ is* an affine connection space with torsion.

Remark 1.3. *An affine connection space $\mathbb{A}_N$ without torsion is a symmetric affine connection space. An affine connection space $\mathbb{G}\mathbb{A}_N$ with torsion is a non-symmetric affine connection space.*

1.1. Affine connection space without torsion

An affine connection on an N-dimensional manifold M is a mapping ∇ which maps any pair (X, Y) of vector fields to a vector field $F = \nabla_X Y$ such that (see [6])

$$\begin{aligned} \nabla_X(Y + Z) &= \nabla_X Y + \nabla_X Z; \\ \nabla_X(fY) &= f\nabla_X Z + (Xf)Y; \\ \nabla_{fX+gY} Z &= f\nabla_X Z + g\nabla_Y Z, \end{aligned} \tag{5}$$

for any vector fields X, Y, Z and any differentiable functions f, g on M.

Remark 1.4. *In our case, operator ∇ is partial derivative operator.*

Let $\mathbb{A}_N$ be an N-dimensional space, and let $\ell : [a, b] \to \mathbb{A}_N$ be a curve on it. A **vector field along** ℓ is a function that assigns to each point of ℓ a tangent vector to the space at that point. That is, a vector field along ℓ is a smooth function $v : [a, b] \to \mathbb{R}^N$ with the property that $v(t) \in T_{\ell(t)}\mathbb{A}_N$ for any $t \in [a, b]$.

Let ℓ be a curve on an affine connection space $\mathbb{A}_N$. We say that a vector field X is parallel along a curve ℓ if X satisfies the condition

$$\nabla_t X \overset{def}{=} \nabla_\lambda X = 0, \tag{6}$$

for any t, where $\lambda = \lambda(t) = \dot{\ell}(t)$.

The definition below is the definition of parallel transport.

Definition 1.5. [6] *Let $x_0 = \ell(t_0)$ and $x_1 = \ell(t_1)$ be points on the given curve $\ell = \ell(t)$. A vector X_1 from the tangent space $T_{x_1}\mathbb{A}_N$ in x_1 is a result of the parallel transport along ℓ from the point x_0 to the point x_1 if along ℓ, there exists a parallel vector field $X(t)$ for which $X(t_0) = X_0$ and $X(t_1) = X_1$.*

The following definition is the one of a geodesic line.

Definition 1.6. [6] *A curve ℓ in space $\mathbb{A}_N$ is geodesic when its tangent vector field remains in tangent distribution of ℓ during parallel transport along the curve.*

Special diffeomorphisms are of interest for research in symmetric affine connection spaces and non-symmetric ones.

Definition 1.7. [6] *Let $\mathbb{A}_N$ and $\overline{\mathbb{A}}_N$ be symmetric affine connection spaces. A diffeomorphism $f : \mathbb{A}_N \to \overline{\mathbb{A}}_N$ is said to be a geodesic mapping of $\mathbb{A}_N$ onto $\overline{\mathbb{A}}_N$ if any geodesic curve in $\mathbb{A}_N$ it maps onto a geodesic curve in $\overline{\mathbb{A}}_N$.*

An invariant geometrical object under a geodesic mapping $f : \mathbb{A}_N \to \overline{\mathbb{A}}_N$ is a Weyl projective tensor

$$W^i_{jmn} = R^i_{jmn} + \tfrac{1}{N+1} R_{[mn]} + \tfrac{N}{N^2-1}\delta^i_{[m} R_{jn]} + \tfrac{1}{N^2-1}\delta^i_{[m} R_{n]j}, \tag{7}$$

where by $[\cdot,\cdot]$ it is denoted an anti-symmetrization without division. A symmetrization without division is going to be denoted by $(\cdot,\cdot)$. A magnitude

$$R^i_{jmn} = L^i_{jm,n} - L^i_{jn,m} + L^p_{jm} L^i_{pn} - L^p_{jn} L^i_{pm}, \tag{8}$$

is Riemann-Christoffel curvature tensor and $R_{mn} = R^p_{mnp}$ is Ricci tensor.

Definition 1.8. [6] *An N-dimensional affine connection space $\mathbb{A}_N$ with vanished Riemann-Christoffel curvature tensor R^i_{jmn} is a* ***flat space****.*

Definition 1.9. [6] *An N-dimensional affine connection space $\mathbb{A}_N$ with vanished Ricci tensor R_{mn} is a* ***Ricci-flat space****.*

Definition 1.10. *An N-dimensional affine connection space $\mathbb{A}_N$ with vanished Weyl projective tensor is a* ***projectively flat space****.*

It is easy to conclude that a flat space M is a projectively flat one. On the contrary, Ricci-flat projective flat space M is a flat space.

1.2. Affine connection space with torsion

S. Minčić and M. Stanković tried to generalize Weyl projective tensor in non-symmetric case. They did not succeed in that. For this reason, they have started research into equitorsion mappings.

Definition 1.11. [8, 18] *A mapping $f : \mathbb{GA}_N \to \mathbb{G}\overline{\mathbb{A}}_N$ is* ***equitorsion*** (**ET**)-***mapping*** *if the torsion tensors of the spaces $\mathbb{GA}_N$ and $\mathbb{G}\overline{\mathbb{A}}_N$ are equal.*

Definition 1.12. *A space $\mathbb{A}_N$ is an* ***associated space*** *to a space $\mathbb{GA}_N$ if symmetric part $L^i_{\underline{jk}}$ of connection coefficients in space $\mathbb{GA}_N$ are used for connection coefficients in the space $\mathbb{A}_N$.*

Using non-symmetry of connection, S. Minčić (see [10]) involved four types of covariant differentiation of tensors. Let $X^{i_1 i_2 \dots i_A}_{j_1 j_2 \dots j_B}$ be a tensor in a non-symmetric affine connection space $\mathbb{GA}_N$ with affine connection coefficients L^i_{jk}. Covariant derivatives of this tensor are:

$$X^{i_1 i_2 \dots i_A}_{j_1 j_2 \dots j_B \underset{\substack{1\\2\\3\\4}}{|} k} = X^{i_1 i_2 \dots i_A}_{j_1 j_2 \dots j_B, k} + \sum_{\alpha=1}^{A} L^{i_\alpha}_{\substack{pm\\mp\\pm\\mp}} X^{i_1 \dots i_{\alpha-1} p i_{\alpha+1} \dots i_A}_{j_1 j_2 \dots j_B} - \sum_{\alpha=1}^{B} L^{p}_{\substack{j_\alpha m\\ m j_\alpha\\ m j_\alpha\\ j_\alpha m}} X^{i_1 \dots i_A}_{j_1 \dots j_{\alpha-1} p j_{\alpha+1} \dots j_B} \tag{9}$$

In the corresponding Ricci type identities, there are four curvature tensors [11]:

$$
\begin{aligned}
\underset{1}{R}{}^i_{jmn} &= L^i_{jm,n} - L^i_{jn,m} + L^p_{jm}L^i_{pn} - L^p_{jn}L^i_{pm},\\
\underset{2}{R}{}^i_{jmn} &= L^i_{mj,n} - L^i_{nj,m} + L^p_{mj}L^i_{np} - L^p_{nj}L^i_{mp},\\
\underset{3}{R}{}^i_{jmn} &= L^i_{jm,n} - L^i_{nj,m} + L^p_{jm}L^i_{np} - L^p_{nj}L^i_{pm} + L^p_{nm}\left(L^i_{pj} - L^i_{jp}\right),\\
\underset{4}{R}{}^i_{jmn} &= L^i_{jm,n} - L^i_{nj,m} + L^p_{jm}L^i_{np} - L^p_{nj}L^i_{pm} + L^p_{mn}(L^i_{pj} - L^i_{jp}).
\end{aligned}
\tag{10}
$$

In these Ricci type identities appear fifteen magnitudes $\underset{\theta}{A}{}^i_{jmn}$ which are not tensors:

$$
\begin{aligned}
\underset{1}{A}{}^i_{jmn} &= L^i_{jm,n} - L^i_{jn,m} + L^p_{jm}L^i_{np} - L^p_{jn}L^i_{mp}, & \underset{9}{A}{}^i_{jmn} &= L^i_{jm,n} - L^i_{nj,m} + L^p_{jm}L^i_{pn} - L^p_{nj}L^i_{pm},\\
\underset{2}{A}{}^i_{jmn} &= L^i_{jm,n} - L^i_{jn,m} + L^p_{mj}L^i_{pn} - L^p_{nj}L^i_{pm}, & \underset{10}{A}{}^i_{jmn} &= L^i_{jm,n} - L^i_{nj,m} + L^p_{jm}L^i_{np} - L^p_{jn}L^i_{pm},\\
\underset{3}{A}{}^i_{jmn} &= L^i_{mj,n} - L^i_{nj,m} + L^p_{mj}L^i_{pn} - L^p_{nj}L^i_{pm}, & \underset{11}{A}{}^i_{jmn} &= L^i_{mj,n} - L^i_{jn,m} + L^p_{mj}L^i_{np} - L^p_{nj}L^i_{mp},\\
\underset{4}{A}{}^i_{jmn} &= L^i_{mj,n} - L^i_{nj,m} + L^p_{jm}L^i_{np} - L^p_{jn}L^i_{mp}, & \underset{12}{A}{}^i_{jmn} &= L^i_{mj,n} - L^i_{jn,m} + L^p_{mj}L^i_{pn} - L^p_{nj}L^i_{mp},\\
\underset{5}{A}{}^i_{jmn} &= L^i_{jm,n} - L^i_{nj,m} + L^p_{jm}L^i_{pn} - L^p_{nj}L^i_{mp}, & \underset{13}{A}{}^i_{jmn} &= L^i_{mj,n} - L^i_{nj,m} + L^p_{mj}L^i_{np} - L^p_{nj}L^i_{pm},\\
\underset{6}{A}{}^i_{jmn} &= L^i_{jm,n} - L^i_{nj,m} + L^p_{mj}L^i_{np} - L^p_{jn}L^i_{pm}, & \underset{14}{A}{}^i_{jmn} &= L^i_{mj,n} - L^i_{nj,m} + L^p_{jm}L^i_{np} - L^p_{nj}L^i_{mp},\\
\underset{7}{A}{}^i_{jmn} &= L^i_{jm,n} - L^i_{jn,m} + L^p_{jm}L^i_{pn} - L^p_{jn}L^i_{mp}, & \underset{15}{A}{}^i_{jmn} &= L^i_{jm,n} - L^i_{nj,m} + L^p_{jm}L^i_{np} - L^p_{nj}L^i_{pm},\\
\underset{8}{A}{}^i_{jmn} &= L^i_{jm,n} - L^i_{jn,m} + L^p_{mj}L^i_{pn} - L^p_{jn}L^i_{pm}. & &
\end{aligned}
\tag{11}
$$

Derived curvature tensors of this space are:

$$
\begin{aligned}
\underset{1}{\widetilde{R}}{}^i_{jmn} &= \tfrac{1}{2}(\underset{1}{A} + \underset{3}{A})^i_{jmn} = \tfrac{1}{2}(\underset{2}{A} + \underset{4}{A})^i_{jmn},\\
\underset{2}{\widetilde{R}}{}^i_{jmn} &= \tfrac{1}{2}(\underset{7}{A} + \underset{13}{A})^i_{jmn} = \tfrac{1}{2}(\underset{9}{A} + \underset{11}{A})^i_{jmn},\\
\underset{3}{\widetilde{R}}{}^i_{jmn} &= \tfrac{1}{2}(\underset{8}{A} + \underset{14}{A})^i_{jmn} = \tfrac{1}{2}(\underset{10}{A} + \underset{12}{A})^i_{jmn},\\
\underset{4}{\widetilde{R}}{}^i_{jmn} &= \tfrac{1}{6}(\underset{3}{R} + \underset{11}{A} + \underset{13}{A})^i_{j[mn]} = \tfrac{1}{6}(\underset{3}{R} + \underset{12}{A} + \underset{14}{A})^i_{j[mn]},\\
\underset{5}{\widetilde{R}}{}^i_{jmn} &= (\underset{1}{A} - \underset{7}{A})^i_{jmn} - \underset{13}{A}{}^i_{jnm} = -\underset{7}{A}{}^i_{jmn} - (\underset{11}{A} + \underset{15}{A})^i_{jnm},\\
\underset{6}{\widetilde{R}}{}^i_{jmn} &= (\underset{2}{A} - \underset{8}{A})^i_{jmn} - \underset{14}{A}{}^i_{jnm} = -\underset{8}{A}{}^i_{jmn} - (\underset{12}{A} + \underset{15}{A})^i_{jnm},\\
\underset{7}{\widetilde{R}}{}^i_{jmn} &= (\underset{3}{A} + \underset{7}{A})^i_{jmn} + \underset{13}{A}{}^i_{jnm} = \underset{9}{A}{}^i_{jmn} + (\underset{13}{A} - \underset{15}{A})^i_{jnm},\\
\underset{8}{\widetilde{R}}{}^i_{jmn} &= (\underset{4}{A} + \underset{8}{A})^i_{jmn} + \underset{14}{A}{}^i_{jnm} = \underset{10}{A}{}^i_{jmn} + (\underset{14}{A} - \underset{15}{A})^i_{jnm}.
\end{aligned}
\tag{12}
$$

Curvature tensors (10, 12) are generalizations of a curvature tensor R^i_{jmn} (8) of a symmetric affine connection space $\mathbb{A}_N$. After non-symmetric affine connection coefficients L^i_{jk} symmetrization, all of these curvature tensors become equal to R^i_{jmn} (curvature tensor of the associated space).

S. Minčić proved in the set $\{\underset{1}{R}{}^i_{jmn}, \ldots, \underset{4}{R}{}^i_{jmn}, \underset{1}{\widetilde{R}}{}^i_{jmn}, \ldots, \underset{8}{\widetilde{R}}{}^i_{jmn}\}$ of four curvature tensors and eight derived ones of a non-symmetric affine connection space $\mathbb{GA}_N$ it exist five linearly independent tensors in [12]. We are going to use the following ones (see [18]) in further research:

$$
\underset{1}{K}{}^i_{jmn} = \underset{1}{R}{}^i_{jmn}, \quad \underset{2}{K}{}^i_{jmn} = \underset{1}{\widetilde{R}}{}^i_{jmn}, \quad \underset{3}{K}{}^i_{jmn} = \underset{3}{R}{}^i_{jmn}, \quad \underset{4}{K}{}^i_{jmn} = \underset{3}{\widetilde{R}}{}^i_{jmn}, \quad \underset{5}{K}{}^i_{jmn} = \frac{1}{4}(3\underset{4}{\widetilde{R}}{}^i_{jmn} + \underset{1}{R}{}^i_{jmn}).
\tag{13}
$$

Definition 1.13. *Space with* $\underset{r}{K}{}^i_{jmn} = 0, r \in \{1, 2, \ldots, 5\}$, *is the* ***r-th flat space****. A space which is the r-th flat one, for all* $r = 1, \ldots, 5$, *is* ***flat space****.*

1.3. Invariants of ET-mappings

In a non-symmetric affine connection space $\mathbb{GA}_N$ exist five linearly independent invariants under ET-mappings caused from linearly independent curvature tensors and the corresponding torsion ones. These invariants are

$$
\begin{aligned}
\underset{1}{\mathcal{E}}{}^i_{jmn} &= \underset{1}{K}{}^i_{jmn} + \tfrac{1}{N+1}\delta^i_j \underset{1}{K}_{[mn]} + \tfrac{N}{N^2-1}\delta^i_{[m}\underset{1}{K}_{jn]} + \tfrac{1}{N^2-1}\delta^i_{[m}\underset{1}{K}_{n]j}\\
&+ \frac{1}{(N+1)^2(N-1)} L^p_{\underline{mp}}\left((N-1)\,\delta^i_j T^q_{qn} + \delta^i_n T^q_{jq}\right)\\
&+ \frac{1}{(N+1)^2(N-1)} L^p_{\underline{np}}\left(-(N-1)\,\delta^i_j T^q_{qm} - \delta^i_m T^q_{jq}\right)\\
&+ \frac{1}{(N+1)^2(N-1)} L^p_{\underline{jp}}\left(N\delta^i_{[m} T^q_{qn]} + \left(N^2-1\right) T^i_{nm}\right)\\
&+ \frac{2}{(N+1)^2} L^q_{\underline{pq}}\left(-(N-1)\delta^i_j T^p_{mn} + \delta^i_m T^p_{jn} - \delta^i_n T^p_{jm}\right);\\[1ex]
&\boxed{\underset{2}{\mathcal{E}}{}^i_{jmn} = \underset{2}{K}{}^i_{jmn} + \tfrac{1}{N+1}\delta^i_j \underset{2}{K}_{[mn]} + \tfrac{N}{N^2-1}\delta^i_{[m}\underset{2}{K}_{jn]} + \tfrac{1}{N^2-1}\delta^i_{[m}\underset{2}{K}_{n]j}};\\[1ex]
\underset{3}{\mathcal{E}}{}^i_{jmn} &= \underset{3}{K}{}^i_{jmn} + \tfrac{1}{N+1}\delta^i_j \underset{3}{K}_{[mn]} + \tfrac{N}{N^2-1}\delta^i_{[m}\underset{3}{K}_{jn]} + \tfrac{1}{N^2-1}\delta^i_{[m}\underset{3}{K}_{n]j}\\
&+ \frac{1}{(N+1)^2(N-1)} L^p_{\underline{mp}}\left((N-1)\,\delta^i_j T^q_{qn} - \delta^i_n T^q_{jq} + (N^2-1)T^i_{jn}\right)\\
&+ \frac{1}{(N+1)^2(N-1)} L^p_{\underline{np}}\left(-(N-1)\,\delta^i_j T^q_{qm} + \delta^i_m T^q_{jq} + (N^2-1)T^i_{jm}\right)\\
&- \frac{1}{(N+1)^2(N-1)} L^p_{\underline{jp}}\delta^i_{[m} T^q_{qn]}\\
&+ \frac{1}{(N+1)^2} L^q_{\underline{pq}}\left(-(N-1)\delta^i_j T^p_{mn} - N\delta^i_m T^p_{jn} - \delta^i_n T^p_{jm}\right);\\[1ex]
&\boxed{\underset{4}{\mathcal{E}}{}^i_{jmn} = \underset{4}{K}{}^i_{jmn} + \tfrac{1}{N+1}\delta^i_j \underset{4}{K}_{[mn]} + \tfrac{N}{N^2-1}\delta^i_{[m}\underset{4}{K}_{jn]} + \tfrac{1}{N^2-1}\delta^i_{[m}\underset{4}{K}_{n]j}};\\[1ex]
&\boxed{\underset{5}{\mathcal{E}}{}^i_{jmn} = \underset{5}{K}{}^i_{jmn} + \tfrac{1}{N+1}\delta^i_j \underset{5}{K}_{[mn]} + \tfrac{N}{N^2-1}\delta^i_{[m}\underset{5}{K}_{jn]} + \tfrac{1}{N^2-1}\delta^i_{[m}\underset{5}{K}_{n]j}}.
\end{aligned}
\tag{14}
$$

Theorem 1.14. [18] *The magnitudes* $\underset{2}{\mathcal{E}}{}^i_{jmn}, \underset{4}{\mathcal{E}}{}^i_{jmn}, \underset{5}{\mathcal{E}}{}^i_{jmn}$ *are ET-projective tensors, and the other ones* $\underset{1}{\mathcal{E}}{}^i_{jmn}, \underset{3}{\mathcal{E}}{}^i_{jmn}$ *are ET-projective parameters.*

Definition 1.15. *A non-symmetric affine connection space* $\mathbb{GA}_N$ *with* $\underset{r}{\mathcal{E}}{}^i_{jmn} = 0, r \in \{1, 2, \ldots, 5\}$, *is the* ***r-th projectively flat space****. A space which is the r-th projectively flat one, for all* $r = 1, \ldots, 5$, *is* ***projectively flat space****.*

2. Flatness and projective flatness of affine connection spaces with torsion

Moffat's results [16] are equations of motion of test particles and linear weak field approximation in Ricci and Minkowski flat spaces. Corollary of these results are galaxy rotational velocity curves and effective gravitational constant at infinity.

In this research, Moffat used flat and Ricci-flat spaces. We are going to connect flatness and projective flatness of non-symmetric affine connection spaces.

Firstly, we consider whether the r-th flat space is the r-th projectively flat one, $r = 1, 2, \ldots, 5$, or not. If it is not the case, we are going to analyze what is a necessary and sufficient condition which the r-th flat space have to satisfy to be the r-th projectively flat one.

Next what we analyze in this paper is an expression of ET- projective tensors $\underset{r}{\mathcal{E}}{}^i_{jmn}$, $r = 1, \ldots, 5$, (14), of a space $\mathbb{GA}_N$ as linear functions of the Weyl projective tensor W^i_{jmn} of the associated space $\mathbb{A}_N$.

The first question is whether flat non-symmetric affine connection spaces are the projectively flat ones or not. In the following theorems the r-th flat and the r-th projective flat non-symmetric spaces, $r = 1, 2, \ldots, 5$, are connected.

Before we prove theorems which present connection of the r-th flatness and the r-th projective flatness of a space $\mathbb{GA}_N$, we have to prove the following propositions.

Proposition 2.1. *Connection coefficients L^i_{jk} satisfy the equation*

$$L^p_{jm}L^i_{np} + L^p_{mj}L^i_{pn} = 2L^p_{\underline{jm}}L^i_{\underline{np}} + \frac{1}{2}T^p_{jm}T^i_{np}. \tag{15}$$

Proof. Based on the equation (4), the equalities

$$\begin{aligned} L^p_{jm}L^i_{np} + L^p_{mj}L^i_{pn} &= (L^p_{\underline{jm}} + \frac{1}{2}T^p_{jm})(L^i_{\underline{np}} + \frac{1}{2}T^i_{np}) + (L^p_{\underline{mj}} + \frac{1}{2}T^p_{mj})(L^i_{\underline{pn}} + \frac{1}{2}T^i_{pn}) \\ &= L^p_{\underline{jm}}L^i_{\underline{np}} + \frac{1}{2}L^p_{\underline{jm}}T^i_{np} + \frac{1}{2}T^p_{jm}L^i_{\underline{np}} + \frac{1}{4}T^p_{jm}L^i_{np} + L^p_{\underline{mj}}L^i_{\underline{pn}} + \frac{1}{2}L^p_{\underline{mj}}T^i_{pn} + \frac{1}{2}T^p_{mj}L^i_{\underline{np}} + \frac{1}{4}T^p_{mj}T^i_{pn} \\ &= 2L^p_{\underline{jm}}L^i_{\underline{np}} + \frac{1}{2}T^p_{mj}T^i_{pn}, \end{aligned}$$

are valid. □

Proposition 2.2. *The equalities*

$$\underset{1}{\tilde{R}}{}^i_{jmn} = R^i_{jmn} + \frac{1}{4}T^p_{jm}T^i_{np} - \frac{1}{4}T^p_{jn}T^i_{mp} \quad \text{and} \quad \underset{3}{\tilde{R}}{}^i_{jmn} = R^i_{jmn} + \frac{1}{4}T^p_{jm}T^i_{np} - \frac{1}{4}T^p_{jn}T^i_{pm} \tag{16}$$

hold, where $R^i_{jmn} = L^i_{\underline{jm},n} - L^i_{\underline{jn},m} + L^p_{\underline{jm}}L^i_{\underline{pn}} - L^p_{\underline{jn}}L^i_{\underline{pm}}$.

Proof. From the equation (11), we obtain

$$\begin{aligned} \underset{1}{\tilde{R}} = \frac{1}{2}(\underset{1}{A} + \underset{3}{A})^i_{jmn} &= \frac{1}{2}(L^i_{jm,n} - L^i_{jn,m} + L^p_{jm}L^i_{np} - L^p_{jn}L^i_{mp}) + \frac{1}{2}(L^i_{mj,n} - L^i_{nj,m} + L^p_{mj}L^i_{pn} - L^p_{nj}L^i_{pm}) \\ &\overset{(3,15)}{=} L^i_{\underline{jm},n} - L^i_{\underline{jn},m} + \left(L^p_{\underline{jm}}L^i_{\underline{np}} + \frac{1}{4}T^p_{jm}T^i_{np}\right) - \left(L^p_{\underline{jn}}L^i_{\underline{mp}} + \frac{1}{4}T^p_{jn}T^i_{mp}\right) \\ &= R^i_{jmn} + \frac{1}{4}T^p_{jm}T^i_{np} - \frac{1}{4}T^p_{jn}T^i_{mp}, \end{aligned}$$

$$\begin{aligned} \underset{3}{\tilde{R}} = \frac{1}{2}(\underset{10}{A} + \underset{12}{A})^i_{jmn} &= L^i_{\underline{jm},n} - L^i_{\underline{nj},m} + \left(L^p_{\underline{jm}}L^i_{\underline{np}} + \frac{1}{4}T^p_{jm}T^i_{np}\right) - \left(L^p_{\underline{jn}}L^i_{\underline{pm}} + \frac{1}{4}T^p_{jn}T^i_{pm}\right) \\ &= R^i_{jmn} + \frac{1}{4}T^p_{jm}T^i_{np} - \frac{1}{4}T^p_{jn}T^i_{pm}, \end{aligned}$$

which proves this proposition. □

Curvature pseudotensors $\underset{12}{A}$ and $\underset{14}{A}$ satisfy the equation

$$\frac{1}{2}(\underset{12}{A} + \underset{14}{A})^i_{jmn} = L^i_{mj,n} - L^i_{\underline{jn},m} + L^p_{\underline{mj}}L^i_{\underline{pn}} + \frac{1}{4}T^p_{mj}T^i_{pn} - L^p_{nj}L^i_{mp}. \tag{17}$$

Let us consider the mentioned problems.

Theorem 2.3. *A first-flat space $\mathbb{GA}_N$ is the first-projectively flat one if and only if it satisfies the condition*

$$\begin{aligned} &L^p_{\underline{mp}}\left((N-1)\delta^i_j T^q_{qn} + \delta^i_n T^q_{jq}\right) + L^p_{\underline{jp}}\left(N\delta^i_{[m} T^q_{qn]} + (N^2-1)T^i_{nm}\right) + (N-1)\delta^i_m L^q_{\underline{pq}} T^p_{jn} \\ &= L^p_{\underline{np}}\left((N-1)\delta^i_j T^q_{qm} + \delta^i_m T^q_{jq}\right) + (N-1)L^q_{\underline{pq}}\left((N-1)\delta^i_j T^p_{mn} + \delta^i_n T^p_{jm}\right). \end{aligned} \tag{18}$$

Proof. For a first-flat space the equation $\underset{1}{K}{}^i_{jmn} = 0$ is valid, which implies $\underset{1}{K}_{mn} = \underset{1}{K}{}^p_{mnp} = 0$. Then, it holds

$$\underset{1}{K}{}^i_{jmn} + \frac{1}{N+1}\delta^i_j K_{[mn]} + \frac{N}{N^2-1}\delta^i_{[m}\underset{r}{K}_{jn]} + \frac{1}{N^2-1}\delta^i_{[m}\underset{r}{K}_{n]j} = 0. \tag{19}$$

From the equation (19) and the definition of $\underset{1}{\mathcal{E}}{}^i_{jmn}$ (14), we get

$$\begin{aligned} \underset{1}{\mathcal{E}}{}^i_{jmn} = &\frac{1}{(N+1)^2(N-1)}L^p_{\underline{mp}}\left((N-1)\,\delta^i_j T^q_{qn} + \delta^i_n T^q_{jq}\right) \\ &+ \frac{1}{(N+1)^2(N-1)}L^p_{\underline{np}}\left(-(N-1)\,\delta^i_j T^q_{qm} - \delta^i_m T^q_{jq}\right) \\ &+ \frac{1}{(N+1)^2(N-1)}L^p_{\underline{jp}}\left(N\delta^i_{[m} T^q_{qn]} + \left(N^2-1\right)T^i_{nm}\right) \\ &+ \frac{N-1}{(N+1)^2(N-1)}L^q_{\underline{pq}}\left(-(N-1)\delta^i_j T^p_{mn} + \delta^i_m T^p_{jn} - \delta^i_n T^p_{jm}\right) \end{aligned}$$

which is equal zero if and only if the equality (18) is satisfied. □

Theorem 2.4. *A second-flat space $\mathbb{GA}_N$ is the second-projectively flat one.*

Proof. For a second-flat space we have it holds $\underset{2}{K}{}^i_{jmn} = 0$, i.e.

$$\underset{2}{K}_{mn} = 0. \tag{20}$$

The second equation in (14), immediately causes

$$\underset{2}{\mathcal{E}}{}^i_{jmn} = \underset{2}{K}{}^i_{jmn} + \frac{1}{N+1}\delta^i_j \underset{2}{K}_{[mn]} + \frac{N}{N^2-1}\delta^i_{[m}\underset{2}{K}_{jn]} + \frac{1}{N^2-1}\delta^i_{[m}\underset{2}{K}_{n]j} \overset{(20)}{=} 0,$$

which proves this theorem. □

Theorem 2.5. *A third-flat space $\mathbb{GA}_N$ is the third-projectively flat one if and only if it satisfies the condition*

$$\begin{aligned} &L^p_{\underline{mp}}\left((N-1)\delta^i_j T^q_{qn} + (N^2-1)T^i_{jn}\right) + L^p_{\underline{np}}\left(\delta^i_m T^q_{jq} + (N^2-1)T^i_{jm}\right) \\ &\quad = \delta^i_n L^p_{\underline{mp}} T^q_{jq} + (N-1)\delta^i_j L^p_{\underline{np}} T^q_{qm} + \delta^i_{[m} T^q_{qn]} L^p_{\underline{jp}} \\ &\quad + (N-1)L^q_{\underline{pq}}\left((N-1)\delta^i_j T^p_{mn} + N\delta^i_m T^p_{jn} + \delta^i_n T^p_{jm}\right). \end{aligned} \tag{21}$$

Proof. Analogously as in the proof of the Theorem 2.1, we have it holds

$$\begin{aligned} \underset{3}{\mathcal{E}}{}^i_{jmn} = &\frac{1}{(N+1)^2(N-1)}L^p_{\underline{mp}}\left((N-1)\,\delta^i_j T^q_{qn} - \delta^i_n T^q_{jq} + (N^2-1)T^i_{jn}\right) \\ &+ \frac{1}{(N+1)^2(N-1)}L^p_{\underline{np}}\left(-(N-1)\,\delta^i_j T^q_{qm} + \delta^i_m T^q_{jq} + (N^2-1)T^i_{jm}\right) \\ &- \frac{1}{(N+1)^2(N-1)}L^p_{\underline{jp}}\delta^i_{[m} T^q_{qn]} \\ &+ \frac{N-1}{(N+1)^2(N-1)}L^q_{\underline{pq}}\left(-(N-1)\delta^i_j T^p_{mn} - N\delta^i_m T^p_{jn} - \delta^i_n T^p_{jm}\right) \end{aligned}$$

which is equal zero if and only if it holds the equality (21). □

Following the procedures used into the last three proofs, it is easy to prove next two theorems.

Theorem 2.6. *A fourth-flat space* $\mathbb{GA}_N$ *is the fourth-projectively flat one.*

Theorem 2.7. *A fifth-flat space* $\mathbb{GA}_N$ *is the fifth-projectively flat one.*

Based on the theorems presented above, we directly conclude the following corollary is valid.

Corollary 2.8. *A flat space* $\mathbb{GA}_N$ *which satisfies the conditions* (18) *and* (21) *is a projectively flat one.*

3. Functionally connected projective curvature tensors and parameters

Invariants of different geometrical mappings and their exact presentations are some of research subjects which results are used in different applications (for example, see[5]). Expressions of projective curvature tensors $\underset{k}{\mathcal{E}}{}^i_{jmn}$, Eq. (14), of a non-symmetric affine connection space $\mathbb{GA}_N$ as functions of Weyl projective tensor W^i_{jmn}, Eq. (7), of the corresponding associated space $\mathbb{A}_N$ are the main purposes of this part of our research.

Theorem 3.1. *A Weyl projective tensor* W^i_{jmn} *of an affine connection space* $\mathbb{A}_N$ *and parameter* $\underset{1}{\mathcal{E}}{}^i_{jmn}$ *of the generalized one* $\mathbb{GA}_N$ *satisfy the equation*

$$\underset{1}{\mathcal{E}}{}^i_{jmn} = W^i_{jmn} + \frac{1}{4}\underset{1}{\widetilde{\mathcal{V}}}{}^i_{jmn}, \tag{22}$$

where

$$\begin{aligned}
\underset{1}{\widetilde{\mathcal{V}}}{}^i_{jmn} &= 2T^i_{j[m;n]} + T^p_{j[m}T^i_{pn]} - \frac{4}{N+1}L^p_{\underline{jp}}T^i_{mn}\\
&+ \frac{2}{N+1}\delta^i_j\left(2T^p_{mn;p} - T^p_{[mp;n]} + T^p_{mn}T^q_{pq}\right) + \frac{4}{(N+1)^2}\delta^i_jL^p_{[\underline{mp}}T^q_{qn]}\\
&+ \frac{1}{N+1}\left(2\delta^i_{[m}T^p_{jn];p} + \delta^i_{[m}T^p_{jn]}T^q_{pq} - \delta^i_{[m}T^p_{qn]}T^q_{jp}\right) - \frac{2}{N^2-1}\left(N\delta^i_{[m}T^p_{jp;n]} + \delta^i_{[m}T^p_{qn]}T^q_{jp}\right)\\
&+ \frac{4}{(N+1)^2(N-1)}\left(N\delta^i_{[m}L^p_{\underline{jp}}T^q_{qn]} - \delta^i_{[m}L^p_{\underline{n]p}}T^q_{jq}\right).
\end{aligned}$$

Proof. The equality

$$\begin{aligned}
\underset{1}{\mathcal{E}}{}^i_{jmn} &= \underset{1}{K}{}^i_{jmn} + \frac{1}{N+1}\delta^i_j\underset{1}{K}_{[mn]} + \frac{N}{N^2-1}\delta^i_{[m}\underset{1}{K}_{jn]} + \frac{1}{N^2-1}\delta^i_{[m}\underset{1}{K}_{n]j}\\
&+ \frac{1}{(N+1)^2(N-1)}\left(L^p_{\underline{mp}}\left((N-1)\,\delta^i_jT^q_{qn} + \delta^i_nT^q_{jq}\right) + L^p_{\underline{np}}\left(-(N-1)\,\delta^i_jT^q_{qm} - \delta^i_mT^q_{jq}\right)\right)\\
&+ \frac{1}{(N+1)^2(N-1)}L^p_{\underline{jp}}\left(N\delta^i_{[m}T^q_{qn]} + \left(N^2-1\right)T^i_{nm}\right)\\
&+ \frac{1}{(N+1)^2(N-1)}L^q_{\underline{pq}}\left(-(N-1)^2\delta^i_jT^p_{mn} + (N-1)\delta^i_mT^p_{jn} - (N-1)\delta^i_nT^p_{jm}\right)
\end{aligned}$$

holds.

After the transformation of the definition of $\underset{1}{K}{}^i_{jmn}$, we obtain the following equation holds:

$$\underset{1}{K}{}^i_{jmn} = R^i_{jmn} + \frac{1}{2}T^i_{jm;n} - \frac{1}{2}T^i_{jn;m} + \frac{1}{4}T^p_{jm}T^i_{jn} - \frac{1}{4}T^p_{jn}T^i_{pm}$$

Based on the previous equation, we conclude Ricci tensor of the first type becomes

$$\underset{1}{K}{}_{mn} = R_{mn} + \frac{1}{2}T^p_{mn;p} - \frac{1}{2}T^p_{mp;n} + \frac{1}{4}T^p_{mn}T^q_{pq} - \frac{1}{4}T^p_{mq}T^q_{qn}$$

which induces the succeeding equation:

$$\underset{1}{K}{}_{[mn]} = R_{[mn]} + T^p_{mn;p} - \frac{1}{2}T^p_{mp;n} + \frac{1}{2}T^p_{np;m} + \frac{1}{2}T^p_{mn}T^q_{pq}.$$

Further,

$$\begin{aligned}
\frac{N}{N^2-1}\delta^i_{[m}\underset{1}{K}{}_{jn]} + \frac{1}{N^2-1}\delta^i_{[m}\underset{1}{K}{}_{n]j} &= \frac{N}{N^2-1}\delta^i_{[m}R_{jn]} + \frac{1}{N^2-1}\delta^i_{[m}R_{n]j}\\
&+ \frac{N}{N^2-1}\delta^i_m\left(\frac{1}{2}T^p_{jn;p} - \frac{1}{2}T^p_{jp;n} + \frac{1}{4}T^p_{jn}T^q_{Tpq} - \frac{1}{4}T^p_{jq}T^q_{pn}\right)\\
&+ \frac{1}{N^2-1}\delta^i_m\left(\frac{1}{2}T^p_{nj;p} - \frac{1}{2}T^p_{np;j} + \frac{1}{4}T^p_{nj}T^q_{pq} - \frac{1}{4}T^p_{nq}T^q_{pj}\right)\\
&- \frac{N}{N^2-1}\delta^i_n\left(\frac{1}{2}T^p_{jm;p} - \frac{1}{2}T^p_{jp;m} + \frac{1}{4}T^p_{jm}T^q_{pq} - \frac{1}{4}T^p_{jq}T^q_{pm}\right)\\
&- \frac{1}{N^2-1}\delta^i_n\left(\frac{1}{2}T^p_{mj;p} - \frac{1}{2}T^p_{mp;j} + \frac{1}{4}T^p_{mj}T^q_{pq} - \frac{1}{4}T^p_{mq}T^q_{pj}\right)\\
&= \frac{N}{N^2-1}\delta^i_{[m}R_{jn]} + \frac{1}{N^2-1}\delta^i_{[m}R_{n]j} + \frac{1}{N+1}\delta^i_{[m}\delta^r_{n]}\left(\frac{1}{2}T^p_{jr;p} + \frac{1}{4}T^p_{jr}T^q_{pq}\right)\\
&- \frac{1}{4(N+1)}\delta^i_{[m}T^p_{jq}T^q_{pn]} - \frac{N}{2(N^2-1)}\delta^i_{[m}T^p_{jp;n]} - \frac{1}{2(N^2-1)}\delta^i_{[m}T^p_{n]p;j}.
\end{aligned}$$

Because of that, it holds

$$\underset{1}{\mathcal{E}}{}^i_{jmn} = W^i_{jmn} + \frac{1}{4}\underset{1}{\widetilde{\mathcal{V}}}{}^i_{jmn},$$

which proves this theorem. □

Theorem 3.2. *A Weyl projective tensor W^i_{jmn} of an affine connection space $\mathbb{A}_N$ and parameter $\underset{2}{\mathcal{E}}{}^i_{jmn}$ of the generalized one $\mathbb{GA}_N$ satisfy the equation*

$$\underset{2}{\mathcal{E}}{}^i_{jmn} = W^i_{jmn} + \frac{1}{4}\underset{2}{\widetilde{\mathcal{V}}}{}^i_{jmn}, \tag{23}$$

where

$$\underset{2}{\widetilde{\mathcal{V}}}{}^i_{jmn} = -T^p_{j[m}T^i_{pn]} - \frac{2}{N+1}\delta^i_jT^p_{mn}T^q_{pq} - \frac{1}{N+1}\delta^i_{[m}T^p_{jn]}T^q_{pq} + \frac{1}{N-1}\delta^i_{[m}T^q_{pn]}T^p_{jq}.$$

Proof. Let we start with $\underset{2}{\mathcal{E}}{}^i_{jmn}$. It holds

$$\underset{2}{\mathcal{E}}{}^i_{jmn} = \underset{2}{K}{}^i_{jmn} + \frac{1}{N+1}\delta^i_j \underset{2}{K}_{[mn]} + \frac{N}{N^2-1}\delta^i_{[m}\underset{2}{K}_{jn]} + \frac{1}{N+1}\delta^i_{[m}\underset{2}{K}_{n]j}. \tag{24}$$

Directly from the equations (10, 13), we have it is valid the following equation:

$$\underset{2}{K}{}^i_{jmn} = R^i_{jmn} - \frac{1}{4}T^p_{jm}T^i_{pn} + \frac{1}{4}T^p_{jn}T^i_{pm}.$$

This equation gives us

$$\underset{2}{K}_{mn} = R_{mn} - \frac{1}{4}T^p_{mn}T^q_{pq} + \frac{1}{4}T^p_{mq}T^q_{pn} \quad \text{and} \quad \underset{2}{K}_{[mn]} = R_{[mn]} - \frac{1}{2}T^p_{mn}T^q_{pq}.$$

After submission of these results in (24), we obtain it is valid

$$\begin{aligned}\underset{2}{\mathcal{E}}{}^i_{jmn} = R^i_{jmn} &+ \frac{1}{N+1}\delta^i_j R_{[mn]} + \frac{N}{N^2-1}\delta^i_{[m}R_{jn]} + \frac{1}{N^2-1}\delta^i_{[m}R_{n]j}\\ &- \frac{1}{4}T^p_{jm}T^i_{pn} + \frac{1}{4}T^p_{jn}T^i_{pm} - \frac{1}{2(N+1)}\delta^i_j T^p_{mn}T^q_{pq} - \frac{1}{4(N+1)}\delta^i_m T^p_{jn}T^q_{pq}\\ &+ \frac{1}{4(N+1)}\delta^i_n T^p_{jm}T^q_{pq} + \frac{1}{4(N-1)}\delta^i_m T^p_{jq}T^q_{pn} - \frac{1}{4(N-1)}\delta^i_n T^p_{jq}T^q_{pm},\end{aligned}$$

which proves this theorem. □

Theorem 3.3. *A Weyl projective tensor W^i_{jmn} of an affine connection space $\mathbb{A}_N$ and parameter $\underset{3}{\mathcal{E}}{}^i_{jmn}$ of the generalized one $\mathbb{GA}_N$ satisfy the equation*

$$\underset{3}{\mathcal{E}}{}^i_{jmn} = W^i_{jmn} + \frac{1}{4}\underset{3}{\widetilde{\mathcal{V}}}{}^i_{jmn}, \tag{25}$$

where

$$\begin{aligned}\underset{3}{\widetilde{\mathcal{V}}}{}^i_{jmn} = 2T^i_{j(m;n)} &- T^p_{j[m}T^i_{pn]} - 2T^p_{mn}T^i_{pj} + \frac{4}{N+1}L^p_{(\underline{mp}}T^i_{jn)}\\ &+ \frac{2}{N+1}\delta^i_j\left(2T^p_{mn;p} + T^p_{[mp;n]} - T^p_{mn}T^q_{pq}\right) + \frac{4}{(N+1)^2}\delta^i_j\left(L^p_{[\underline{mp}}T^q_{qn]} - L^q_{\underline{pq}}T^p_{mn}\right)\\ &+ \frac{1}{N+1}\left(2\delta^i_{[m}T^p_{jn];p} - \delta^i_{[m}T^p_{jn]}T^q_{pq} - \delta^i_{[m}T^p_{n]q}T^q_{pj}\right) + \frac{2}{N^2-1}\left(N\delta^i_{[m}T^p_{jp;n]} + \delta^i_{[m}T^p_{n]p;j}\right)\\ &+ \frac{4}{(N+1)^2(N-1)}\left(\delta^i_{[m}L^p_{\underline{n]p}}T^q_{jq} + \delta^i_{[m}L^p_{\underline{jp}}T^q_{n]q}\right) - \frac{4}{(N+1)^2}L^q_{\underline{pq}}\left(N\delta^i_m T^p_{jn} + \delta^i_n T^p_{jm}\right).\end{aligned}$$

Proof. It holds

$$\underset{3}{K}{}^i_{jmn} = R^i_{jmn} + \frac{1}{2}T^i_{jm;n} + \frac{1}{2}T^i_{jn;m} - \frac{1}{4}T^p_{jm}T^i_{pn} + \frac{1}{4}T^p_{jn}T^i_{pm} - \frac{1}{2}T^p_{mn}T^i_{pj}$$

Now, we get it is valid

$$\underset{3}{K}_{[mn]} = R_{[mn]} + T^p_{mn;p} + \frac{1}{2}T^p_{mp;n} - \frac{1}{2}T^p_{np;m} - \frac{1}{2}T^p_{mn}T^q_{pq}.$$

Based on the previous equations, we obtain it holds

$$\begin{aligned}\frac{N}{N^2-1}\delta^i_{[m}\underset{3}{K}_{jn]} + \frac{1}{N^2-1}\delta^i_{[m}\underset{3}{K}_{n]j} = {} & \frac{N}{N^2-1}\delta^i_{[m}R_{jn]} + \frac{1}{N^2-1}\delta^i_{[m}R_{n]j} \\ & + \frac{1}{N+1}\delta^i_m\left(\frac{1}{2}T^p_{jn;p} - \frac{1}{4}T^p_{jn}T^q_{pq} + \frac{1}{4}T^p_{jq}T^q_{pn} - \frac{1}{2}T^p_{nq}T^q_{pj}\right) \\ & - \frac{1}{N+1}\delta^i_n\left(\frac{1}{2}T^p_{jm;p} - \frac{1}{4}T^p_{jm}T^q_{pq} + \frac{1}{4}T^p_{jq}T^q_{pm} - \frac{1}{2}T^p_{mq}T^q_{pj}\right) \\ & + \frac{1}{2}\left(\frac{N}{N^2-1}\delta^i_m T^p_{jp;n} + \frac{1}{N^2-1}\delta^i_m T^p_{np;j} - \frac{N}{N^2-1}\delta^i_n T^p_{jp;m} - \frac{1}{N^2-1}\delta^i_n T^p_{mp;j}\right),\end{aligned}$$

which combined with the definition of $\underset{3}{\mathcal{E}}{}^i_{jmn}$, proves this theorem. □

We are proving the following two theorems with using of procedures similarly to the ones applied in the last three proves.

Theorem 3.4. *A Weyl projective tensor W^i_{jmn} of an affine connection space $\mathbb{A}_N$ and parameter $\underset{4}{\mathcal{E}}{}^i_{jmn}$ of the generalized one $\mathbb{GA}_N$ satisfy the equation*

$$\underset{4}{\mathcal{E}}{}^i_{jmn} = W^i_{jmn} + \frac{1}{4}\underset{4}{\widetilde{\mathcal{V}}}{}^i_{jmn}, \tag{26}$$

where

$$\underset{4}{\widetilde{\mathcal{V}}}{}^i_{jmn} = -T^p_{j(m}T^i_{pn)} - \frac{2}{N+1}\delta^i_j T^p_{mn}T^q_{pq} - \frac{1}{N+1}\delta^i_{[m}T^p_{jn]}T^q_{pq} + \frac{1}{N-1}\delta^i_{[m}T^p_{qn]}T^q_{jp}.$$

□

Theorem 3.5. *A Weyl projective tensor W^i_{jmn} of an affine connection space $\mathbb{A}_N$ and parameter $\underset{5}{\mathcal{E}}{}^i_{jmn}$ of the generalized one $\mathbb{GA}_N$ satisfy the equation*

$$\underset{5}{\mathcal{E}}{}^i_{jmn} = W^i_{jmn} + \frac{1}{8}\underset{5}{\widetilde{\mathcal{V}}}{}^i_{jmn}, \tag{27}$$

where

$$\underset{5}{\widetilde{\mathcal{V}}}{}^i_{jmn} = -T^p_{mn}T^i_{pj} - \frac{1}{N-1}\delta^i_{[m}T^p_{n]q}T^q_{pj}.$$

□

Corollary 3.6. *Magnitudes $\underset{r}{\mathcal{V}}{}^i_{jmn}, r = 1, 3$ defined in Theorems 3.1 and 3.3 are parameters. Other ones $\underset{r}{\mathcal{V}}{}^i_{jmn}, r = 2, 4, 5$ are tensors.*

Proof. Based on the equations (22-27), we have

$$\underset{r}{\mathcal{E}}{}^i_{jmn} = W^i_{jmn} + \frac{1}{4}\underset{r}{\mathcal{V}}{}^i_{jmn}, r = 1, \dots, 4, \quad \text{and} \quad \underset{5}{\mathcal{E}}{}^i_{jmn} = W^i_{jmn} + \frac{1}{8}\underset{5}{\mathcal{V}}{}^i_{jmn}.$$

The proof of this corollary holds directly from the Theorem 1.14 proved in [18], i.e. it holds directly from the tensor characters of $\underset{r}{\mathcal{E}}{}^i_{jmn}$ and the facts that difference of tensor and parameter is parameter and difference of two tensors is a tensor. □

4. Conclusion

Flatness and projective flatness of spaces $\mathbb{A}_N$ and $\mathbb{GA}_N$ were analyzed in this paper. In the second section it is proved that non-symmetric flat spaces $\mathbb{GA}_N$ are projectively flat if they satisfy conditions (18) and (21).

In the third section, ET-projective tensors in space $\mathbb{GA}_N$ are presented as linear functions of Weyl projective tensor in an associated space $\mathbb{A}_N$.

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References

[1] M. S. Ćirić, M. Lj. Zlatanović, M. S. Stanković, Lj. S. Velimirović, On geodesic mappings of equidistant generalized Riemannian spaces, Applied Mathematics and Computation 218, 12 (2012), 6648–6655.
[2] A. Einstein, A Generalization of the Relativistic Theory of Gravitation, Annals of Mathematics, Princeton, 46, (1945), 576-584.
[3] A. Einstein, Bianchi Identities in the Generalized Theory of Gravitation, Canadian Journal of Mathematics, 2, (1950), 120–128.
[4] A. Einstein, Relativistic Theory of the Non-symmetric field, Appendix II in the Book: The Meaning of Relativity, (5th edition), Princeton, 49, 1955.
[5] F. Fu, X. Yang, P. Zhao, Geometrical and physical characteristics of a class of conformal mappings, Journal of Geometry and Physics, 62 (2012), 1467–1479.
[6] J. Mikeš, A. Vanžurová, I. Hinterleitner, Geodesic Mappings and Some Generalizations, 2009.
[7] S. M. Minčić, M. S. Stanković, On geodesic mapping of general affine connection spaces and of generalized Riemannian spaces, Mat.vesnik, 49 (1997), 27–33.
[8] S. M. Minčić, M. S. Stanković, Equitorsion geodesic mappings of generalized Rimannian spaces, Publications de L'Institut Mathématique, 61 (75) (1997) 97–104.
[9] S. M. Minčić, Independent curvature tensors and pseudotensors of spaces with non-summetric affine connexion, Colloquia Mathematica Societatis János Bolayai, 31, Differential Geometry, Budapest (Hungary), (1979), 445–460.
[10] S. M. Minčić, Ricci identities in the space of non-symmetric affine connection, Matematički Vesinik, 10(25), Vol. 2, (1973), 161–172.
[11] S. M. Minčić, Curvature tensors of the space of non-symmetric affine connexion, obtained from the curvature pseudotensors, Matematički Vesnik, 13, (28), (1976), 421–435.
[12] S. M. Minčić, Independent curvature tensors and pseudotensors of spaces with non-symmetric affine connexion, Colloquia Mathematica Societatis Janos Bolayai, 31, Differential Geometry, Budapest (Hungary), (1979), 445–460.
[13] S. M. Minčić, Lj. S. Velimirović, M. S. Stanković, On spaces with non-symmetric affine connection, containing subspaces without torsion, Applied Mathematics and Computation 219, 9 (2013), 4346–4353.
[14] S. M. Minčić, M. S. Stanković, Lj. S. Velimirović, Generalized Riemannian Spaces and Spaces of Non-symmetric Affine Connection, Faculty of Science and Mathematics Niš, 2013.
[15] R. S. Mishra, Subspaces of a generalized Riemannian space, Bull. Acad. Roy. Belgique, (1954), 1058–1071.
[16] J. W. Moffat, Gravitational Theory, Galaxy Rotation Curves and Cosmology without Dark Matter, arXiv: astro-ph/0412195v3 5 May 2005.
[17] M. Prvanović, Four curvature tensors of non-symmetric affine connexion (in Russian), Proceedings of the conference 150 years of Lobachevski geometry, Kazan 1976, Moscow 1997, 199–205.
[18] M. Lj. Zlatanović, New projective tensors for equitorsion geodesic mappings, Applied Mathematics Letters, 25, (2012), 890–897.