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On Some Properties of Non-Symmetric Connections

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Abstract. On an N-dimensional manifold with non-symmetric connection L^i_{jk} four kinds of covariant derivative (1.1) are defined, and four curvature tensors are obtained. In the present paper specially the 3^{rd} and the 4^{th} kind of covariant derivative are studied, particularly their application on δ-symbols.

1. Introduction

Consider N-dimensional manifold $\mathcal{M}_N$ with local coordinates $x^i (i = 1, \dots, N)$. If we introduce non-symmetric affine connection L^i_{jk} on $\mathcal{M}_N$, one obtains a space L_N as a structure $L_N = (\mathcal{M}_N, L^i_{jk})$.

Because of non-symmetry, it is possible to define four kinds of covariant derivative. For example, for a tensor t^i_j we have

$$t^i_{j\underset{\substack{1\\2\\3\\4}}{|}m} = t^i_{j,m} + L^i_{\substack{pm\\mp\\pm\\mp}} t^p_j - L^p_{\substack{jm\\mj\\mj\\jm}} t^i_p, \tag{1.1}$$

where $t^i_{j,m} = \dfrac{\partial}{\partial x^m} t^i_j$. From here we obtain more Ricci-type identities. For example,

$$t^i_{j\underset{1}{|}mn} - t^i_{j\underset{1}{|}nm} = \underset{1}{R}{}^i_{pmn} t^p_j - \underset{1}{R}{}^p_{jmn} t^i_p - T^p_{mn} t^i_{j\underset{1}{|}p}, \tag{1.2}$$

$$t^i_{j\underset{2}{|}mn} - t^i_{j\underset{2}{|}nm} = \underset{2}{R}{}^i_{pmn} t^p_j - \underset{2}{R}{}^p_{jmn} t^i_p + T^p_{mn} t^i_{j\underset{2}{|}p}, \tag{1.3}$$

$$t^i_{j\underset{1}{|}m\underset{2}{|}n} - t^i_{j\underset{2}{|}n\underset{1}{|}m} = \underset{3}{R}{}^i_{pmn} t^p_j - \underset{3}{R}{}^p_{jmn} t^i_p, \tag{1.4}$$

$$t^i_{j\underset{3}{|}mn} - t^i_{j\underset{3}{|}nm} = \underset{1}{R}{}^i_{pmn} t^p_j - \underset{2}{R}{}^p_{jmn} t^i_p + T^p_{mn} t^i_{j\underset{3}{|}p}, \tag{1.5}$$

$$t^i_{j\underset{4}{|}mn} - t^i_{\underset{4}{jnm}} = \underset{2}{R}{}^i_{pmn} t^p_j - \underset{1}{R}{}^p_{jmn} t^i_p - T^p_{mn} t^i_{j\underset{4}{|}p}, \tag{1.6}$$

$$t^i_{j\underset{3}{|}m\underset{4}{|}n} - t^i_{j\underset{4}{|}n\underset{3}{|}m} = \underset{4}{R}{}^i_{pmn} t^p_j + \underset{3}{R}{}^p_{jnm} t^i_p, \tag{1.7}$$

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where

$$
\begin{aligned}
&T^i_{jk} = L^i_{jk} - L^i_{kj},\\
&\underset{1}{R}{}^i_{jmn} = L^i_{jm,n} - L^i_{jn,m} + L^p_{jm}L^i_{pn} - L^p_{jn}L^i_{pm},\\
&\underset{2}{R}{}^i_{jmn} = L^i_{mj,n} - L^i_{nj,m} + L^p_{mj}L^i_{np} - L^p_{nj}L^i_{mp},\\
&\underset{3}{R}{}^i_{jmn} = L^i_{jm,n} - L^i_{nj,m} + L^p_{jm}L^i_{np} - L^p_{nj}L^i_{pm} + L^p_{nm}T^i_{pj},\\
&\underset{4}{R}{}^i_{jmn} = L^i_{jm,n} - L^i_{nj,m} + L^p_{jm}L^i_{np} - L^p_{nj}L^i_{pm} + L^p_{mn}T^i_{pj},
\end{aligned}
\tag{1.8}
$$

2. The case when δ-symbols are not under covariant derivatives

2.1 In the case of the 1^{st} and the 2^{nd} kind of derivative, we have $\delta^i_{j\underset{\theta}{|}m} = 0$, based on which, in the case of total contraction with respect of contra- and covariant indices the partial derivative of invariant quantity is obtained. It is not so, if we use the 3^{rd} and the 4^{th} kind of derivative and we will study this matter in detail. Here we will engage specially in the 3^{rd} kind, because for the 4^{th} kind the matter is similar. The obtained structure will be denoted as $(L_N, \underset{3}{|})$.

First of all, for vectors u^i, v_i on the base of (1.1) is

$$u^i_{\underset{3}{|}m} = u^i_{\underset{1}{|}m}, \quad u^i_{\underset{4}{|}m} = u^i_{\underset{2}{|}m}, \quad v_{i\underset{3}{|}m} = v_{i\underset{2}{|}m}, \quad v_{i\underset{4}{|}m} = v_{i\underset{1}{|}m}. \tag{2.1}$$

For the 3^{rd} kind of derivative the Leibniz rule is valid, because, for example, for a vector u^i and a covector v_j we have:

$$
\begin{aligned}
(u^i v_j)_{\underset{3}{|}m} &\underset{(1.1)}{=} (u^i v_j)_{,m} + L^i_{pm}u^p v_j - L^p_{mj}u^i v_p = (u^i_{,m} + L^i_{pm}u^p)v_j + u^i(v_{j,m} - L^p_{mj}v_p)\\
&= u^i_{\underset{1}{|}m}v_j + u^i v_{j\underset{2}{|}m}
\end{aligned}
$$

and, based on (2.1):

$$(u^i v_j)_{\underset{3}{|}m} = u^i_{\underset{3}{|}m}v_j + u^i v_{j\underset{3}{|}m}. \tag{2.2}$$

Analogously it is generally, e.g.

$$(a^{ij}_k v_l)_{\underset{3}{|}m} = a^{ij}_{k\underset{3}{|}m}v_l + a^{ij}_k v_{l\underset{3}{|}m}, \tag{2.3}$$

where a is some tensor of the type (2,1), v-a covector.

2.2 As we will see in the following exposition, δ-symbol can not be introduced under the sign of derivative $\underset{3}{|}$ without of influence to the result. So, we have e. g.

$$
\begin{aligned}
\delta^i_j(u^j v_i)_{\underset{3}{|}m} &= \delta^i_j(u^j_{\underset{3}{|}m}v_i + u^j v_{i\underset{3}{|}m}) = \delta^i_j\left[(u^j_{,m} + L^j_{pm}u^p)v_i + u^j(v_{i,m} - L^p_{mi}v_p)\right]\\
&= (u^i v_i)_{,m} + L^i_{pm}u^p v_j - u^i L^p_{mj}v^p,
\end{aligned}
$$

from where, after exchanging some mute indices,

$$\delta^j_i(u^i v_j)_{\underset{3}{|}m} = (u^i v_i)_{,m} + T^i_{jm}u^j v_i. \tag{2.4}$$

On the other hand, starting from (1.1) and puting $j = i$, we obtain

$$\begin{aligned}(u^i v_i)_{\underset{3}{|}m} &= (u^i v_i)_{,m} + L^i_{pm} u^p v_i - L^p_{mi} u^i v_p \\ &= (u^i v_i)_{,m} + T^i_{pm} u^p v_i\end{aligned}$$

and we have

$$\delta^i_j (u^j v_i)_{\underset{3}{|}m} = (u^i v_i)_{\underset{3}{|}m} = (u^i v_i)_{,m} + T^i_{jm} u^j v_i. \tag{2.5}$$

Consider now the next example. With respect of (2.3) is

$$\begin{aligned}\delta^k_i \delta^l_j (a^{ij}_k v_l)_{\underset{3}{|}m} &= \delta^k_i \delta^l_j (a^{ij}_{k\underset{3}{|}m} v_l + a^{ij}_k v_{l\underset{3}{|}m}) = (a^{ij}_{i,m} + L^i_{pm} a^{pj}_i + L^j_{pm} a^{ip}_i - L^p_{mi} a^{ij}_p) v_j + a^{ij}_i (v_{j,m} - L^p_{mj} v_p) \\ &= (a^{ij}_i v_j)_{,m} + T^i_{pm} a^{pj}_i v_j + T^j_{pm} a^{ip}_i v_j,\end{aligned}$$

i. e.

$$\delta^k_i \delta^l_j (a^{ij}_k v_l)_{\underset{3}{|}m} = (a^{ij}_i v_j)_{\underset{3}{|}m} = (a^{ij}_i v_j)_{,m} + T^i_{jm} (a^{jp}_i v_p + a^{pj}_p v_i) \tag{2.6}$$

For a tensor t^i_j, from (1.1) is

$$\delta^i_j (t^j_{i\underset{3}{|}m}) = \delta^i_j (t^j_{i,m} + L^j_{pm} t^p_i - L^p_{mi} t^j_p) = t^i_{i,m} + L^i_{pm} t^p_i - L^p_{mi} t^i_p = (t^i_i)_{\underset{3}{|}m},$$

from where, by corresponding exchanging of indices:

$$\delta^i_j (t^j_{i\underset{3}{|}m}) = t^i_{i\underset{3}{|}m} = t^i_{i,m} + T^i_{jm} t^j_i. \tag{2.7}$$

Further, we have

$$\begin{aligned}\delta^i_j \delta^k_l (a^j_i b^l_k)_{\underset{3}{|}m} &= \delta^i_j \delta^k_l (a^j_i b^l_k)_{\underset{3}{|}m} \underset{(2.3)}{=} \delta^i_j \delta^k_l (a^j_{i\underset{3}{|}m} b^l_k + a^j_i b^l_{k\underset{3}{|}m}) \\ &= \delta^i_j \delta^k_l (a^j_{i,m} b^l_k + L^j_{pm} a^p_i b^l_k - L^p_{mi} a^j_p b^l_k + b^l_{k,m} a^j_i + L^l_{pm} b^p_k a^j_i - L^p_{mk} b^l_p a^j_i) \\ &= a^i_{i,m} b^k_k + L^i_{pm} a^p_i b^k_k - L^p_{mi} a^i_p b^k_k + b^k_{k,m} a^i_i + L^k_{pm} b^p_k a^i_i - L^p_{mk} b^k_p a^i_i \\ &= (a^i_i b^k_k)_{,m} + T^i_{pm} a^p_i b^k_k + T^k_{pm} b^p_k a^i_i,\end{aligned}$$

i.e.

$$\delta^i_j \delta^k_l (a^j_i b^l_k)_{\underset{3}{|}m} = (a^i_i b^k_k)_{\underset{3}{|}m} = (a^i_i b^k_k)_{,m} + T^i_{jm} (a^j_i b^k_k + b^j_i a^k_k) \quad . \tag{2.8}$$

Generally, the following theorem, that we will prove by total induction, is valid.

Theorem 2.1. *If $\delta^{i_1}_{j_1}, \dots, \delta^{i_n}_{j_n}$ are Kronecker δ-sumbols and $\underset{1}{a}{}^{j_1}_{i_1}, \underset{2}{a}{}^{j_2}_{i_2}, \dots, \underset{n}{a}{}^{j_n}_{i_n}$ tensors of type (1.1), then*

$$\begin{aligned}\delta^{i_1}_{j_1} \dots \delta^{i_n}_{j_n} (\underset{1}{a}{}^{j_1}_{i_1} \dots \underset{n}{a}{}^{j_n}_{i_n})_{\underset{3}{|}m} &= (\underset{1}{a}{}^{i_1}_{i_1} \dots \underset{n}{a}{}^{i_n}_{i_n})_{\underset{3}{|}m} \\ &= (\underset{1}{a}{}^{i_1}_{i_1} \dots \underset{n}{a}{}^{i_n}_{i_n})_{,m} + T^i_{jm} \left(\underset{1}{a}{}^j_i \underset{2}{a}{}^{i_2}_{i_2} \dots \underset{n}{a}{}^{i_n}_{i_n} + \underset{1}{a}{}^{i_1}_{i_1} \underset{2}{a}{}^j_i \underset{3}{a}{}^{i_3}_{i_3} \dots \underset{n}{a}{}^{i_n}_{i_n} + \dots + \underset{1}{a}{}^{i_1}_{i_1} \dots \underset{n-1}{a}{}^{i_{n-1}}_{i_{n-1}} \underset{n}{a}{}^j_i \right)\end{aligned} \tag{2.9}$$

Proof. For $n = 1, 2$ the equation (2.9) is valid on the base of (2.6,2.8). Suppose that it is valid also for $n-1$ factors. With respect of Leibniz rule, for the left side in (2.9) is obtained

$$\begin{aligned}
&\delta^{i_1}_{j_1}\dots\delta^{i_n}_{j_n}\Big[(\underset{1}{a}{}^{j_1}_{i_1}\dots\underset{n-1}{a}{}^{j_{n-1}}_{i_{n-1}})_{\underset{3}{|}m}\underset{n}{a}{}^{j_n}_{i_n}+\underset{1}{a}{}^{j_1}_{i_1}\dots\underset{n-1}{a}{}^{j_{n-1}}_{i_{n-1}}\underset{n}{a}{}^{j_n}_{i_n\underset{3}{|}m}\Big]\\
&=\Big[(\underset{1}{a}{}^{i_1}_{i_1}\dots\underset{n-1}{a}{}^{i_{n-1}}_{i_{n-1}})_{,m}+T^i_{jm}(\underset{1}{a}{}^{j}_{i}\underset{2}{a}{}^{i_2}_{i_2}\dots\underset{n-1}{a}{}^{i_{n-1}}_{i_{n-1}}+\dots+\underset{1}{a}{}^{i_1}_{i_1}\dots\underset{n-2}{a}{}^{i_{n-2}}_{i_{n-2}}\underset{n-1}{a}{}^{j}_{i})\Big]\underset{n}{a}{}^{i_n}_{i_n}+\underset{1}{a}{}^{i_1}_{i_1}\dots\underset{n-1}{a}{}^{i_{n-1}}_{i_{n-1}}(\underset{n}{a}{}^{i_n}_{i_n,m}+T^{i_n}_{j_nm}\underset{n}{a}{}^{j_n}_{i_n})\\
&=(\underset{1}{a}{}^{i_1}_{i_1}\underset{2}{a}{}^{i_2}_{i_2}\dots\underset{n}{a}{}^{i_n}_{i_n})_{,m}+T^i_{jm}(\underset{1}{a}{}^{j}_{i}\underset{2}{a}{}^{i_2}_{i_2}\dots\underset{n-1}{a}{}^{i_{n-1}}_{i_{n-1}}+\dots+\underset{1}{a}{}^{i_1}_{i_1}\dots\underset{n-1}{a}{}^{j}_{i}\underset{n}{a}{}^{i_n}_{i_n})+\underset{1}{a}{}^{i_1}_{i_1}\dots\underset{n-1}{a}{}^{i_{n-1}}_{i_{n-1}}\underset{n}{a}{}^{j_n}_{i_n}T^{i_n}_{j_nm}.
\end{aligned}$$

Exchanging in the last term $i_n \to i$, $j_n \to j$, we get the right side in (2.9).

2.3 Similarly to Theorem 2.1 it can be proved the next theorem.

Theorem 2.2. *If $\delta^{i_1}_{j_1}, \dots, \delta^{i_n}_{j_n}$ are δ-sumbols and $t^{i_1\dots i_n}_{j_1\dots j_n}$ tensor of type (n, n) then we have*

$$\delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\dots\delta^{i_n}_{j_n}t^{j_1j_2\dots j_n}_{i_1i_2\dots i_n\underset{3}{|}m}=t^{i_1i_2\dots i_n}_{i_1i_2\dots i_n\underset{3}{|}m}=t^{i_1i_2\dots i_n}_{i_1i_2\dots i_n,m}+T^i_{jm}\left(t^{ji_2\dots i_n}_{ii_2\dots i_n}+t^{j_1jj_3\dots j_n}_{i_1ii_2\dots i_n}+\dots+t^{i_1i_2\dots i_{n-1}j}_{i_1i_2\dots i_{n-1}i}\right) \tag{2.10}$$

3. The case when δ-symbols are under covariant derivatives

3.1 It is easy to see that $\delta^i_{j\underset{1}{|}m} = \delta^i_{j\underset{2}{|}m} = 0$. But, we have

$$\delta^i_{j\underset{3}{|}m} = \delta^i_{j,m} + L^i_{pm}\delta^p_j - L^p_{mj}\delta^i_p = 0 + T^i_{jm},$$

$$\delta^i_{j\underset{3}{|}m} = T^i_{jm} \tag{3.1}$$

In the same manner is

$$\delta^i_{j\underset{4}{|}m} = T^i_{mj} = -\delta^i_{j\underset{3}{|}m} \tag{3.2}$$

The equation (3.1) is an important result in the structure $(L_N, \underset{3}{|})$, if δ-symbol is under derivative. For example

$$(\delta^i_ju^j)_{\underset{3}{|}m} = \delta^i_{j\underset{3}{|}m}u^j + \delta^i_ju^j_{\underset{3}{|}m} \underset{(3.1)}{=} T^i_{jm}u^j + u^i_{\underset{3}{|}m},$$

i.e.

$$(\delta^i_ju^j)_{\underset{3}{|}m} = u^i_{\underset{3}{|}m} + T^i_{jm}u^j \underset{(2.1)}{=} u^i_{\underset{1}{|}m} + T^i_{jm}u^j, \tag{3.3}$$

but

$$\delta^i_ju^j_{\underset{3}{|}m} = \delta^i_ju^j_{\underset{1}{|}m} = u^i_{\underset{1}{|}m} = u^i_{,m} + L^i_{pm}u^p \tag{3.3'}$$

In the same way

$$(\delta^j_iv_j)_{\underset{3}{|}m} = v_{i\underset{3}{|}m} + T^j_{im}v_j \underset{(2.1)}{=} v_{i\underset{2}{|}m} + T^j_{im}v_j, \tag{3.4}$$

and also

$$\delta^j_iv_{j\underset{3}{|}m} = \delta^j_iv_{j\underset{2}{|}m} = v_{i\underset{2}{|}m} = v_{i,m} - L^p_{mi}v_p. \tag{3.4'}$$

Let us consider some more cases when δ-symbols are under derivatives.

$$(\delta^i_j u^j v_i)_{\underset{3}{|}} = \delta^i_{j\underset{3}{|}m} u^j v_i + \delta^i_j (u^j v_i)_{\underset{3}{|}m} \underset{(3.1),(2.4)}{=} T^i_{jm} u^j v_i + (u^i v_i)_{,m} + T^i_{jm} u^j v_i,$$

$$(\delta^i_j u^j v_i)_{\underset{3}{|}m} = (u^i v_i)_{,m} + 2T^i_{jm} u^j v_i \tag{3.5}$$

Further, we have

$$\begin{aligned}(\delta^i_p \delta^q_i u^p v_q)_{\underset{3}{|}m} &= (\delta^i_p u^p)_{\underset{3}{|}m} \delta^q_i v_q + \delta^i_p u^p (\delta^q_i v_q)_{\underset{3}{|}m} \underset{(3.3,4)}{=} (u^i_{\underset{3}{|}m} + T^i_{pm} u^p) v_i + u^i (v_{i\underset{3}{|}m} + T^q_{im} v_q)\\ &= (u^i_{,m} + L^i_{pm} u^p + T^i_{pm} u^p) v_i + u^i (v_{i,m} - L^p_{mi} v_p + T^q_{im} v_q) =\\ &= (u^i v_i)_{,m} + T^i_{pm} u^p v_i + T^i_{pm} u^p v_i + T^q_{im} u^i v_q,\end{aligned}$$

and after exchanging some mute indices

$$(\delta^i_p \delta^q_i u^p v_q)_{\underset{3}{|}m} = (u^i v_i)_{,m} + 3T^i_{jm} u^j v_i. \tag{3.6}$$

It is interesting to compare (3.5,6) with (2.4). Analogically to the case (2.7), we have

$$(\delta^i_j t^j_i)_{\underset{3}{|}m} = \delta^i_{j\underset{3}{|}m} t^j_i + \delta^i_j t^j_{i\underset{3}{|}m} \underset{(3.1),(2.7)}{=} T^i_{jm} t^j_i + t^i_{i,m} + T^i_{jm} t^j_i,$$

$$(\delta^i_j t^j_i)_{\underset{3}{|}m} = t^i_{i,m} + 2T^i_{jm} t^j_i. \tag{3.7}$$

Let us examine the case that corresponds to (2.8).

$$\begin{aligned}(\delta^i_j \delta^k_l a^j_i b^l_k)_{\underset{3}{|}m} &= \left[(\delta^i_j a^j_i)(\delta^k_l b^l_k)\right]_{\underset{3}{|}m} = (\delta^i_j a^j_i)_{\underset{3}{|}m} \delta^k_l b^l_k + \delta^i_j a^j_i (\delta^k_l b^l_k)_{\underset{3}{|}mm}\\ &\underset{(3.7)}{=} (a^i_{i,m} + 2T^i_{jm} a^j_i) b^k_k + a^i_i (b^k_{k,m} + 2T^k_{jm} b^j_k)\\ &= (a^i_i b^k_k)_{,m} + 2T^i_{jm} a^j_i b^k_k + 2T^k_{jm} a^i_i b^j_k,\end{aligned}$$

$$(\delta^i_j \delta^k_l a^j_i b^l_k)_{\underset{3}{|}m} = (a^i_i b^k_k)_{,m} + 2T^i_{jm} (a^j_i b^k_k + b^j_i a^k_k), \tag{3.8}$$

which we can compare with (2.8).
The next theorem is related to the some general case when δ-symbols are under derivative $\underset{3}{|}$.

Theorem 3.1. *Covariant derivative of the* 3^{rd} *kind* $(\underset{3}{|})$ *for* δ*-symbol is given with (3.1), i. e. it is expressed by the torsion. Generally is*

$$(\delta^{i_1}_{j_1} \delta^{i_2}_{j_2} \ldots \delta^{i_n}_{j_n} \underset{1}{a}{}^{j_1}_{i_1} \underset{2}{a}{}^{j_2}_{i_2} \ldots \underset{n}{a}{}^{j_n}_{i_n})_{\underset{3}{|}m} = (\underset{1}{a}{}^{i_1}_{i_1} \underset{2}{a}{}^{i_2}_{i_2} \ldots \underset{n}{a}{}^{i_n}_{i_n})_{,m} + 2T^i_{jm} \left(\underset{1}{a}{}^{j}_{i} \underset{2}{a}{}^{i_2}_{i_2} \ldots \underset{n}{a}{}^{i_n}_{i_n} + \underset{1}{a}{}^{i_1}_{i_1} \underset{2}{a}{}^{j}_{i} \underset{3}{a}{}^{i_3}_{i_3} \ldots \underset{n}{a}{}^{i_n}_{i_n} + \cdots + \underset{1}{a}{}^{i_1}_{i_1} \ldots \underset{n-1}{a}{}^{i_{n-1}}_{i_{n-1}} a^j_i \right) \tag{3.9}$$

Proof. Based on (3.7,8), the equation (3.9) is valid for $n = 1, 2$ and the proof can be realized by total induction. □

4. Conditions for vanishing the term with torsion

We saw that by application of the 3^{rd} and the 4^{th} kind of covariant derivative on total contraction in $(L_N, \underset{3}{|})$ it appears the term containing the torsion. The task is to consider the question when that term vanish, i. e. one obtains partial derivative of scalar quantity.

Let us start from (2.4), that is

$$\delta^i_j (u^j v_i)_{\underset{3}{|}m} = (u^i v_i)_{\underset{3}{|}m} = (u^i v_i)_{,m} + T^i_{jm} u^j v_i \tag{4.1}$$

and examine when will be

$$T^i_{jm} u^j v_i = 0 \tag{4.2}$$

Here we have the addition in relation to $i, j,$ and m determines the number of equations. Because in $L_N \quad i, j, m = 1, 2, \dots, N,$ we have in (4.2) N equations with

$$S_m = N\binom{N}{2},$$

unknowns having in mind that wrt j, m it exists an antisymmetry. As the rank of this system is $r \le N$, it can be determined mostly N unknowns T^i_{jm}, the rest $N\binom{N}{2} - r \ge N\binom{N}{2} - N$ can be taken arbitrary, i. e. they do not have to be zero. So, it be $T \neq 0$ and in the same time (4.2) will be satisfied.

Example 4.1. *For $N = 2$ the equation (4.2) gives*

$$T^1_{21} u^2 v_1 + T^2_{21} u^2 v_2 = 0, \; T^1_{21} u^1 v_1 + T^2_{12} u^1 v_2 = 0,$$

which is reduces to one equation

$$T^1_{12} v_1 + T^2 12 v_2 = 0 \quad \Leftrightarrow \quad T^2_{12} = T^1_{21} \frac{v_1}{v_2},$$

where T^1_{21} is arbitrary. Therefore, $T \neq 0$. Here $S_2 = 2, r = 1$.

Example 4.2. *For $N = 3$ from (4.2) for $m = 1, 2, 3$ we get*

$$\begin{aligned}
T^1_{21} u^2 v_1 + T^1_{31} u^3 v_1 + T^2_{21} u^2 v_2 + T^2_{31} u^3 v_2 + T^3_{21} u^2 v_3 + T^3_{31} u^3 v_3 = 0 \\
T^1_{12} u^1 v_1 + T^1_{32} u^3 v_1 + T^2_{12} u^1 v_2 + T^2_{32} u^3 v_2 + T^3_{12} u^1 v_3 + T^3_{32} u^3 v_3 = 0 \\
T^1_{13} u^1 v_1 + T^1_{23} u^2 v_1 + T^2_{13} u^1 v_2 + T^2_{23} u^2 v_2 + T^3_{13} u^1 v_3 + T^3_{23} u^2 v_3 = 0
\end{aligned} \tag{4.3}$$

To obtain the rank of this system, we have for its matrix

$$\begin{aligned}
\mathcal{M} &= \begin{bmatrix} -u^2v_1 & -u^3v_1 & 0 & -u^2v_2 & & 0 & -u^2v_3 & -u^3v_3 & 0 \\ u^1v_1 & 0 & -u^3v_1 & u^1v_2 & 0 & -u^3v_2 & u^1v_3 & 0 & -u^3v_3 \\ 0 & u^1v_1 & u^2v_1 & 0 & u^1v_2 & u^2v_2 & 0 & u^1v_3 & u^2v_3 \end{bmatrix} \\
&\sim \begin{bmatrix} u^2v_1 & u^3v_1 & 0 & u^2v_2 & u^3v_2 & 0 & u^2v_3 & u^3v_3 & 0 \\ 0 & u^1v_1 & u^2v_1 & 0 & u^1v_2 & u^2v_2 & 0 & u^1v_3 & u^2v_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}
\end{aligned}$$

So, reduced system is

$$\begin{aligned}
u^2 v_1 T^1_{12} + u^3 v_1 T^1_{13} + u^2 v_2 T^2_{12} + u^3 v_2 T^2_{12} + u^2 v_3 T^3_{12} + u^3 v_3 T^3_{13} = 0 \\
u^1 v_1 T^1_{13} + u^2 v_1 T^1_{23} + u^1 v_2 T^2_{13} + u^2 v_2 T^2_{23} + u^1 v_3 T^3_{13} + u^2 v_3 T^2_{23} = 0
\end{aligned} \tag{4.4}$$

The rank of the system (4.3) is $r = 2$, and from (4.4) we can find two unknowns, e. g. T^1_{12}, T^1_{13}, while the rest 7 quantities T^i_{jm} can be taken arbitrary, that is they can be $\neq 0$ and then $T \neq 0$ and in (2.4) $(u^i v_i)_{\underset{3}{|}m} = (u^i v_i)_{,m}$.

5. Commentary

Two connections are used in the works of many authors. In the work of T. Otsuki [1] and others two connections are used: $'\Gamma$-for contravariant, and $''\Gamma$ for covariant indices. These connections are not obligatory to be symmetric and are related by so named Otsuki relation ([1], eq. (3.13))

$$P^i_{j,k} +'' \Gamma^i_{pk} P^p_j -' \Gamma^p_{jk} P^i_p = 0, \tag{5.1}$$

where P is a tensor field of the type (1.1) ($\det P^i_j \neq 0$). In Otsuki space so called basic differentiation is defined e. g. as follows

$$t^i_{j|m} = t^i_{j,m} +' \Gamma^i_{pm} t^p_j -'' \Gamma^p_{jm} t^i_p, \tag{5.2}$$

and we have

$$\delta^i_{j|m} =' \Gamma^i_{jm} -'' \Gamma^i_{jm} \tag{5.3}$$

Obviously, in our case is

$$L^i_{jm} =' \Gamma^i_{jm}, \quad L^i_{mj} ='' \Gamma^i_{jm} \tag{5.4}$$

and we have

$$\left.\begin{array}{l} 'L^i_{jm} = L^i_{jm} =' \Gamma^i_{jm} \\ ''L^i_{jm} = L^i_{mj} =' \Gamma^i_{mj} \end{array}\right\} \quad \Rightarrow \quad ''L^i_{jm} =' L^i_{mj},$$

which corresponds to (5.1).

By a choice of a connection covariant derivative is defined. Here we see that the contrary is valid: by choice of covariant derivative $(\underset{3}{|}, \underset{4}{|})$ the connection is defined. As we have proved in [7], eq.(2.14), in the structure $(L_N, \underset{3}{|}, \underset{4}{|})$ the induced connection of a submanifold is symmetric.

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References

[1] T. Otsuki, *On general connections I*, Math. J. Okayama Univ., 9, (1959-60), 99-144.
[2] M. Prvanović, *On a special connection of an Otsuki space*, Tensor, N.S., vol. 37, (1982), 237-243.
[3] M. Prvanović, *Prostranstvo Ocuki-Nordena*, Izv. VUZ, Matematika 7, (1984), 59-63.
[4] A.Moor, *Otsukische Räume mit einem Zweifach rekursenten metrischen Grundtensor*, Periodica Math. Hungarica, vol. 13(2), (1982), 129-135.
[5] S. Minčić, *Ricci type identities for basic differentiation and curvature tensors in Otsuki spaces*, N. Sad J. Math, vol. 31, No 2, (2001), 73-87.
[6] S. Minčić, *Independent curvature tensors and pseudotensors of spaces with non-symmetric affine connection*, Coloq. mat. Sor. J. Bolyai, 31, Diff. geometry Budapest, (1979), 446-460.
[7] S. Minčić, *Derivation equations of submanifolds in an asymmetric affine connection space*, Krag. J. Math. vol. 35, No 2, (2011), 265-276.
[8] S. Minčić, M. Stanković, Lj. Velimirović, *Generalized Riemannian spaces and spaces of non-symmetric affine connection*, Fac. of Sc. and Math., Niš, 2013.
[9] A. P. Norden, *Prostranstva affinnoy svyaznosti*, Moskva, 1950.
[10] M. S. Stanković, M. Lj. Zlatanović, , Lj. S. Velimirović, *Equitorsion holomorphically projective mappings of generalized Kahlerian space of the first kind,* Czechoslovak Mathematical Journal, Vol. 60, No. 3, (2010), 635-653.
[11] M. S. Stanković, *On a canonic almost geodesic mappings of the second type of affine spaces*, FILOMAT 13, (1999), 105-114.
[12] M. S. Stanković, *Special equitorsion almost geodesic mappings of the third type of non-symmetric affine connection spaces*, Applied Mathematics and Computation, 244, (2014), 695-701.
[13] M. Lj. Zlatanović, I. Hinterleitner , M. Najdanović, *Geodesic mapping onto Kahlerian space of the first kind,* Czech. Math. Journal, Vol. 64, No. 4, (2014), 1113-1122.
[14] M. Lj. Zlatanović, I. Hinterleitner, M. Najdanović, *On equitorsion concircular tensors of generalized Riemannian spaces*, Filomat, Vol. 28, No. 3., (2014), 463-471.