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Isoperimetric Inequality, F. Gehring's Problem on Linked Curves and Capacity

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[a] *To the memory on professor Vojin Dajović on the occasion 100 years since his birthday*

Abstract. In this mainly review paper, we discuss connections between F. Gehring's problem and some results of isoperimetric type. We also prove a few new results and give novelity at some places.

1. Introduction

In order to discuss Gehring's problem we first need some definitions.
For $a \in \mathbb{R}^n$, we denote by $S^n_r = S(a;r) = \{x : |x-a| = r\}$ and $B^n_r = B(a;r) = \{x : |x-a| < r\}$ the sphere and the ball in $\mathbb{R}^n$ of radius r with center at a respectively. We also use short notation $\mathbb{S}^n$ and $\mathbb{B}^n$ for $S(0;1)$ and $B(0;1)$ respectively.

A path in $\mathbb{R}^n$ is continuous mapping $\gamma : [a,b] \to \mathbb{R}^n$ of compact interval $[a,b] \subset \mathbb{R}$ into $\mathbb{R}^n$. By γ^* or $tr(\gamma)$ we denote the trace $\gamma^* = \{\gamma(t) : a \leq t \leq b\}$ of γ and its length by $|\gamma|$ or length(γ). Closed curves γ and γ_0 are linked if γ_0 is not homotopic 0 in $\mathbb{R}^3 \setminus \gamma^*$.

The following theorem was conjectured by Gehring [16, 24], Problem 7.22.

Theorem 1.1 (The problem of Gehring). *If γ and γ_0 are linked curves in $\mathbb{R}^3$ on distance* 1, *show that length of each of those curves is at least 2π.*

In this paper we will try to tell an interesting story about this conjecture. Theorem 1.1 attracted attention of mathematicians. It seems that several mathematicians have been working on this problem independently; including Gehring [25], M. Ortel, M. Mateljević, 1975, [32], R. Osserman [41, 42], Edelstein and Schwarz [18], Eremenko, O. Vinkovskii and I. Syutrik [19]. In section 3, short review of their works and further development (which include work of Jason Cantarella, Joe Fu, Rob Kusner, John M. Sullivan, and Nancy Wrinkle cf. [10–12]) are given. In section 2, we also sketch the proof given in [32] and give two generalizations of Theorem 1.1, Theorem 2.1 and Theorem 2.2 below.

Roughly speaking the paper can be divide in two parts; the first part consists of Section 2- 4, Section 8 and Section 9, and the rest of the paper is the second part (Sections 5-7), which is mainly independent of the first one. The author first wrote the first part. Then taking into account that minimal surfaces and isoperimetric inequalty on minimal surfaces are used for solution of Gehring's problem, the author decided to add the second part.

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In Section 5 and 6 we consider planar version of isoperimetric inequality, related to potential,Green's function,Robin constant,diameter, transfinite diameter, capacity of condenser, modulus and the extremal distance, capacity and area-modulus inequality. For example, Theorems 5.2 and 5.1 are new results (or at least have some novelty) as far as I know. In particular, the Carleman result, Theorem 6.2 attracts special attention: Among all ring domains with given area and with given area of the "holes" the domain bounded by two concentric circles gives the smallest value of $cap(D, F)$. For application in complex dynamics see Milnor [40]. Theorem 5.4 is a generalization of the Carleman result. It seems that it is a new result.

In Section 7 we consider planar versions of isoperimetric inequality, related to multiplicity (for mappings which are non-injective) and we announce Theorem 7.6.

In section 8(Appendix 1) we discuss some auxiliary results which include Fenchel's Theorem, the linking number of two paths, the geodesics on spheres etc.. Short review concerning connections between Borromean rings, the Gehring Link Problem and the new IMU logo and physical reality is given in Section 9(Appendix 2). The logo design is based on the Borromean rings, a famous topological link of three components, and the paper which is related to Gehring's problem, cf. [11].

2. Generalization of Gehring problem

For point x and set A define

$$c(x, A) = \{tx + (1-t)y : 0 \le t \le 1, y \in A\}.$$

Solution 1. A proof can be derived in few steps, cf [32]:

a) Show that for every $x \in \gamma^*$, $c(x, \gamma^*) \cap \gamma_0^* \neq \emptyset$
b) Let $x_* \in \gamma^*$ and $y_0 \in c(x_*, \gamma^*) \cap \gamma_0^*$. Then there exists $y_* \in \gamma^*$ and $0 < t < 1$ such that $y_0 = tx_* + (1-t)y_*$. Define $\Gamma = f \circ \gamma$, where

$$f(y) = y_0 + \frac{y - y_0}{\|y - y_0\|}.$$

c) Show that $|\Gamma| \le |\gamma|$.
d) Closed curve Γ belongs to the unit sphere $S(y_0, 1)$ and contains antipodal points $f(x_*)$ and $f(y_*)$ of the sphere $S(y_0, 1)$. Since geodesic line on the sphere are arcs of great circles, the length of Γ is at least 2π.

It seems that solution of Gehring's problem, first published in 1975, [32], see also 1976, [18]. On this and the other problems in complex analysis see in [2] and [5]. In [2] it is reported that Mateljević solved problem, but the [32] is not cited. In the second part of mentioned paper [2], a summary of W. K. Hayman's lecture, on progress on problems presented in two preceding report, is given. It is interesting that the above mentioned authors have overlooked the paper [32].

For example, in [11, 42] the solution is attributed to M.Ortel (unpublished). In [11], they write: "because Ortel's elegant solution was never published, we reproduce it here with his permission":

Solution 2. Fix any point $a \in A$; the cone on A from a is a disk spanning A. Since A and B are linked, B meets this disk at some point $b \in B$, lying on a chord of A. Because Dist(b, A) ≥ 1, projecting A to the unit sphere S around b does not increase its length. The projection is a closed curve joining two antipodal points on S, and so has length at least 2π. Solution 1 and Solution 2 are almost identical.

By d_e we denote Euclidean distance in $\mathbb{R}^n$. For given an arbitrary set E define

$$E_r = \{x \in \mathbb{R}^n \text{ for which there is } y \in E : |x - y| < r\}.$$

We call E_r "thickening" or "tube-domain" about E.

Let c be a path in $\mathbb{R}^3$ space. Set $r(x, y) = \sup\{d_e(z, tr(c)) : z \in [x, y]\}$ and define $\underline{r}(c) = \sup\{r(x, y) : x, y \in tr(c)\}$. We call $\underline{r}(c)$ the inner space radius of c.

Using approach as in the above proof of Theorem 1.1, we can get the following generalization:

Theorem 2.1. *Let γ be a path in $\mathbb{R}^3$ space. Then $|\gamma| \geq 2\pi \underline{r}(\gamma)$.*

There are points x_0, y_0 on γ and $z_0 \in [x_0, y_0]$ such that the curve γ has no points inside sphere $S_0 = S(z_0, r_0,$ where $r_0 = \underline{r}(c)$. Let γ' be the central projection of γ from z_0 onto the sphere S_0 around z_0. Then γ' passes through two diametrically opposite points of the sphere S_0 and thus, by Theorem 8.1, its length is at least $2\pi r_0$.

Theorem 2.2. *Let L_1 and L_2 be two paralell plane in in $\mathbb{R}^3$, R strip between them and γ^0 a curve in R joint L_1 and L_2. For $r > 0$, suppose that γ does not intersect tube γ^0_r and γ links γ^0_r in R. Then $|\gamma| \geq 2\pi r$.*

Fix a point M_0 on γ. Then one can find another point M_1 on γ such that the interval $[M_0, M_1]$ intersects γ^0. Indeed, otherwise we can deform γ to M_0 moving straight along these intervals $[M_0, M]$, $M \in tr(\gamma)$, and deformation will not cross γ^0. Let O be a point on $[M_0, M_1]$ that belongs to γ^0. Let γ' be the central projection of γ from O onto the unit sphere around O. Then γ' passes through two diametrically opposite points of the sphere and thus, by Theorem 8.1, its length is at least $2\pi r$.

3. Gehring problem and minimal surfaces

In [19], Eremenko sketched the following proof:
Let A and B be two closed curves in $\mathbb{R}^3$ which are linked. Suppose that the distance between A and B is 1. Prove that length of A (or B) is at least 2π.

Solution 3 (Sketch of the proof by A. Eremenko and O. Vinkovskii). Suppose that length(B) $< 2\pi$. Let F be the surface of minimal area whose boundary is B. (A continuous map from the closed unit disc to $\mathbb{R}^3$ such that the image of the unit circle is B, and which is minimizing the area). Then A must intersect F; let O be a point of intersection. Consider the sphere S of unit radius centered at O. The curve B must be outside this sphere. Let B' be the central projection from O of B on S. Then length(B') $\leq$ length(B) $< 2\pi$. So B' belongs to an open hemisphere by Lemma 8.1. This implies that B' and B belong to an open half-space which has O on its boundary. This contradicts the minimality of the area of F. Acording to Eremenko [19], he told this proof on the geometric seminar of I. N. Pesin in Lvov University in 1976 or 1977. A day later, one of the participants, a second-year undergraduate student I. Syutrik found the proof similar to the above proof of Theorem 1.1.

Shortly after this Eremenko found a published proof [18] based on the same idea as the first proof described above, but using the convex hull instead of the minimal surface. Further, we need a version of the isoperimetric inequality on minimal surfaces.

Theorem 3.1. *Suppose that domain D lying on minimal surfaces in $\mathbb{R}^n$ and that: (i) the boundary of D consists of a single rectifiable Jordan curve c. Then the isoperimetric inequality $L^2 \geq 4\pi A$ holds for domain D.*

¿From elementary properties of volume, it follows from $V(B^n_1) = \omega_n$ that $V(B^n_r) = \omega_n r^n$.

An important inequality in the theory of minimal surfaces is the following. Let M be an m-dimensional minimal submanifold in $\mathbb{R}^n$, and assume that the origin lies on M. Using the notation of 2, let $M_r = M \cap B^n_r$, and $V(r)$ m-dimensional measure of M_r.

Theorem 3.2. *If M has no boundary points in B^n_r, then $V(r) \geq \omega_m r^m$.*

Theorem 3.3. *Let C be a rectifiable Jordan curve in $\mathbb{R}^n$, and let B be a set in $\mathbb{R}^n$ which links C. Let L be the length of C and let r be the distance between B and C. Then $L \geq 2\pi r$. Equality holds only when C is a euclidean circle of radius r.*

We sketch a prof by R. Osserman [42].

Solution 4. Let S be a solution of Plateau's problem for C. That is, S is a simply-connected minimal surface spanning C. Since B and C are linked, it follows that $B \cap S \neq 0$. Let p be a point of $B \cap S$. By a translation we may assume that p is the origin. By hypothesis,the boundary of S (which is the curve C) lies outside of B_r^n. We may therefore apply Theorem 3.2 and conclude that the area A of S satisfies $A \geq \pi r^2$. But then the isoperimetric inequality for minimal surfaces (Theorem 3.2 above) implies $L^2 \geq 4\pi A \geq 4\pi^2 r^2$.

According to R. Osserman [42], Theorem 3.3 was conjectured by Gehring [24] and first proved by M. Ortel (unpublished). The proof given above was suggested by Osserman [2]. Another proof has been given by Edelstein and Schwarz [18], Gehring [25] has proved a higher-dimensional version of Theorem 3.3, but not with the best possible constant, which is presumably attained by linked spheres in orthogonal subspaces. A different proof which yields the best constant for a 2-sphere linked by a 1-sphere in $\mathbb{R}^4$ has been obtained by M. Gage [21].

3.1. Recent results

For further development see E. Bombieri and L. Simon [8], Jason Cantarella, Joe Fu, Rob Kusner, John M. Sullivan, and Nancy Wrinkle [11], Jason Cantarella, Robert B. Kusner, and John M. Sullivan [12] and L. Guth [26].

In the early 1970's, Bombieri and Simon [8] proved the following sharp inequality about the geometry of minimal surfaces in Euclidean space.

Theorem 3.4 (Bombieri-Simon radius inequality). *Suppose that $Z = Z^m$ is a closed submanifold of $\mathbb{R}^n$, and that Y^{m+1} is a minimal surface with $\partial Y = Z$. Suppose that Z has the same volume as a round m-sphere of radius R. Then for each point $y \in Y$, the distance from y to Z is at most R.*

Theorem 3.5 (Gehring link conjecture). *Suppose that $Z = Z^m$ and $W = W^{n-m-1}$ are linked submanifolds $\mathbb{R}^n$. If Z has the same volume as a round m-sphere of radius R, then the distance from Z to W is at most R.*

By the solution of the Plateau problem, there is a minimal surface Y with $\partial Y = Z$. Since Z and W are linked, Y must intersect W in some point $y \in W$. But by the radius inequality, the distance from y to Z is at most R. Gromov built an analogy between the Gehring link conjecture and the systolic problem. On the one hand, such an analogy sounds promising because both inequalities bound a 1-dimensional length (or distance) in terms of an m-dimensional volume.

4. Criticality for the Gehring link problem

The unique minimizing configuration for Gehrings problem is a Hopf link consisting of two congruent circles in perpendicular planes, each passing through the others center. This leads to a natural question: what are the length-minimizing shapes of other link types when the different components stay unit distance apart? This constraint prevents different components from crossing each other, but we cannot expect to fix the link type exactly. Instead, the natural setting for this problem is Milnor's notion of link homotopy: two links are link-homotopic if one can be deformed into the other while keeping different components disjoint. Clearly one link can be deformed into another while keeping all components at unit distance if and only if they are link homotopic. We will define the link-thickness of a link to be the minimum distance between different components. The problem we consider is then to minimize length in a link-homotopy class, subject to the constraint of fixed link-thickness. Equivalently, we could minimize the link-ropelength of the link, meaning the quotient of length over thickness.

A compact, oriented 1-manifold-with-boundary M is a finite union of components, each of which is homeomorphic to a circle $\mathbb{S}^1$ or an interval $[0, 1]$. A parametrized curve is a mapping from a compact, oriented 1-manifold- with-boundary M to $\mathbb{R}^3$. Two parametrized curves are equivalent if they differ by an orientation-preserving reparametrization (that is, by composition with an orientation- preserving self-homeomorphism of M). A curve L in $\mathbb{R}^3$ is an equivalence class of parametrized curves. We say L is closed when each component of its domain M is a circle, that is, when its boundary ∂L is empty. Definition The link-thickness LThi(L) of a curve L is the minimum distance between points on different components of L. This is the supremal ε for which the ($\varepsilon/2$- neighborhoods of the components of L are disjoint. So suppose we start with a closed curve L and we want to minimize length under the constraint that the link-thickness remains at least one. Since we can rescale any link to have LThi ≥ 1, this problem is the same as minimizing (link-) ropelength, the quotient of length by link-thickness. The thickness constraint naturally prevents different components from passing through each other, but does not prevent any given component from changing its knot type through self-intersections. This is the setting for Milnors work on link homotopy:

Here we give only the content of the abstract of [11]. In 1974, Gehring posed the problem of minimizing the length of two linked curves separated by unit distance. This constraint can be viewed as a measure of thickness for links, and the ratio of length over thickness as the ropelength. In the paper [11] they refine Gehrings problem to deal with links in a fixed link-homotopy class: they prove ropelength minimizers exist and introduce a theory of ropelength criticality. Their balance criterion is a set of necessary and sufficient conditions for criticality, based on a strengthened, infinite-dimensional version (Theorem 5.4) of the Kuhn-Tucker theorem. They use this to prove that every critical link is C^1 with finite total curvature. The balance criterion also allows them to explicitly describe critical configurations (and presumed minimizers) for many links including the Borromean rings. They also exhibit a surprising critical configuration for two clasped ropes: near their tips the curvature is unbounded and a small gap appears between the two components. These examples reveal the depth and richness hidden in Gehrings problem and our natural extension.

5. Isoperimetric inequality, diameter and capacity

The isoperimetric problem is to determine a plane figure of the largest possible area whose boundary has a specified length. If L is the circumference of a closed Jordan rectifiable curve γ in the plane and the area of a plane region it encloses $A = A(\gamma)$, then $A \leq L^2/4\pi$.

A version which includes self-intersecting curve is outlined in [34, 35].

Proposition 5.1. *[34] Let K be positively oriented unit circle and let a curve γ be defined by $w = \phi(e^{i\theta}), 0 \leq \theta \leq 2\pi$, is of bounded variation and $\phi(e^{i\theta}) \sim \sum_{n=-\infty}^{\infty} \hat{\phi}_n\, e^{-in\theta}$. Then the sign area bounded by this curve is $A(\phi) = \frac{i}{2}\int_K \phi\, d\overline{\phi}$.*

If we denote by $L = |\gamma|$ length of the curve γ, then by isoperimetric inequality, $A(\phi) \leq L^2/4\pi$.

By the next Proposition 5.1, $A(\phi) == \pi \sum\limits_{-\infty}^{\infty} n|\hat{\phi}_n|^2 < \infty$.

Let a curve γ be defined on $[0, 2\pi]$ and $\gamma \sim \sum_{n=-\infty}^{\infty} \hat{\gamma}_n\, e^{-in\theta}$. If γ is of bounded variation, closed and continuous curve, then the sign area $A(\gamma) = \frac{i}{2}\int_0^{2\pi} \gamma\, d\overline{\gamma} = \pi \sum\limits_{-\infty}^{\infty} n|\hat{\gamma}_n|^2 < \infty$.

Theorem 5.1. *Then*

$$A(\gamma) = \pi \sum_{-\infty}^{\infty} n|\hat{\gamma}_n|^2 < \infty \tag{1}$$

Set $g = P[\gamma]$. For $0 \leq r < 1$, $B(\gamma_r) = \frac{i}{2}\int_0^{2\pi} \gamma_r\, d\overline{\gamma} = \pi \sum\limits_{-\infty}^{\infty} nr^{|n|}|\hat{\gamma}_n|^2$ and $B(\gamma_r) \to A(\gamma)$ when $r \to 1_-$. Hence the series 6 is Abel summable. Since $n|\hat{\gamma}_n|^2 \to 0$, by Tauber's convergence theorem it is convergent in the ordinary sense as well.

If we set

$A_+(\gamma) = \pi \sum\limits_1^{\infty} n|\hat{\gamma}_n|^2$, $A_-(\gamma) = \pi \sum\limits_{-\infty}^{1} n|\hat{\gamma}_n|^2$, it is clear that

$A_+(\gamma) = A_+(\gamma) + A_-(\gamma)$ and $A_+(\gamma) \leq A_+(\gamma)$.

If γ is closed Jordan positively oriented curve, the oriented area is the same as the usual area of $Int(\gamma)$.

Let γ be closed Jordan curve and $\tilde{G} = Ext(\gamma)$. By *Riemann's theorem* there is conformal mapping

$$f(z) = \lambda\, z + a_0 + \frac{a_1}{z} + \cdots + \frac{a_k}{z^k} + \cdots \tag{4}$$

of $\mathbb{E}$ onto $\tilde{G}$. For $\rho \geq 1$, set $\gamma_\rho(t) = f(\rho e^{it})$, $s(\rho) = A(\gamma_\rho)$, $\tau(\rho) = \rho^{-2}s(\rho)$ and $|\gamma|_1^* = \inf_c \int_0^{2\pi} |\gamma(t) - c|dt$.

Theorem 5.2. *Under the above hypothesis*
(i.1) τ *is not decreasing in* $[1, \infty]$
(ii.1) $\tau(\rho) \to \pi|\lambda|^2$ *if* $\rho \to \infty$.
(iii.1) $A(\gamma) \leq \pi|\lambda|^2$
(iv.1) $|\lambda| \leq |\gamma|_1^*$
(v.1) $|\lambda| \leq L$*, where L is length of* γ.

For (iii.1) see also [35, 39].

Proof. Since

$$\tau(\rho) = \rho^{-2}s(\rho) = \pi \sum_{k=-\infty}^{1} k|a_k|^2\rho^{2k-2} = \pi|\lambda|^2 + \pi \sum_{k=-\infty}^{1} k|a_k|^2\rho^{2k-2}$$

we get (i.1) and (ii.1).
(iii.1) follows from $A_+(\gamma) = \pi|\lambda|^2$.
Using $\lambda = \int_0^{2\pi} \gamma(t)e^{-it}dt$ and
$i\lambda = \int_0^{2\pi} \gamma'(t)e^{-it}dt$ we find (iv.1) and (v.1) respectively. □

As a corollary of (iii.1) and (v.1), we get the isoperimetric inequality for simple curve: $4\pi A \leq L^2$.
The next example shows that the estimate (iv.1) can be better than (v.1). Let $r > 1$, $z_k = e^{ik2\pi/n}$, $w_k = rz_k$ and P_n polygon $z_1w_1z_2w_2\cdots z_{n-1}w_{n-1}z_nw_nz_1$, then $l(P_n) \to \infty$ and it is clear that $|P_n|_1^* \leq 2\pi r$.

5.1. area-modulus inequality

For $0 < r < R$, let $A(r, R) = \{r < |z| < R\}$ be the annulus with inner radius r and other radius R.
A domain A is ring if A^c has exactly two components. By topology, ∂A has also two components C_1 and C_2. Denote by $\Gamma = \Gamma_A$ the collection of curves $\gamma \subset A$ connecting C_1 and C_2.
There is $A(r_1, r_2)$ and conformal maping ϕ of $A(r_1, r_2)$ onto A. Modulus of A is defined as

$$M(A) = \frac{\log(r_2/r_1)}{2\pi}.$$

Theorem 5.3. *Let* $F \subset D$ *and* $A = D \setminus F$ *topological annulus. Then*

$$e^{4\pi M(A)}area(F) \leq area(D). \tag{2}$$

If equality holds in (5.3), then A is a circular regular ring.

If we set $S_0 = areaF = \pi r_0^2$ and $S_1 = areaD = \pi r_1^2$, and $A_0 = A(r_0, r_1)$ then:

$$4\pi M(A) \leq \ln\frac{S_1}{S_0} = 4\pi\, M(A_0)\,.$$

Hence we rewrite Theorem 5.3 respectively in the form:
(I.1) $M(A) \leq M(A(r_1, r_2))$. We can also restate Theorem 5.3:
(I.1′) Consider the family of all doubly-connected plane domains bounded by an outer curve C_1 and an inner curve C_0. For each domain D, let A_i be the area bounded by C_i, $i = 0, 1$. Then among all domains conformally equivalent to a given one, the minimum of A_1/A_0 is attained by a circular annulus.
We first give here a proof due to Szegö (see [42] [1]) based on the isoperimetric inequality.
Let $r_0 < |z| < r_1$ be a given annulus, and let D be its image under a conformai map $f(z)$. Let $L(r)$ be the length of the image of $|z| = r$, and $A(r)$ the area enclosed. Then $4\pi A(r) \leq L^2(r) \leq 2\pi A'(r)/A(r)$ and $2/r \leq 2A'(r)/A(r)$, $r_0 < r < r_1$. Integrating from r_0 to r_1, yields $2\ln\frac{r_1}{r_0} \leq \ln\frac{A_1}{A_0}$, or $\frac{r_1^2}{r_0^2} \leq \frac{A_1}{A_0}$, which proves the theorem. □

Under the hypothesis of Theorem 5.3, there exists an annulus $A_r = \{r < |z| < 1\}$ and a conformal mapping $\phi : A_r \to A$, $\phi(z) = \sum a_kz^k$.

Let $\Gamma_\rho = \phi \circ K_\rho$ and $G_\rho = Int(\Gamma_\rho)$. Then

$$s(\rho) = area(G_\rho) = \sum k|a_k|^2\rho^{2k}.$$

Let

$$\tau(\rho) = \rho^{-2}s(\rho) = \sum k|a_k|^2\rho^{2k-2}.$$

Theorem 5.4. (i.2) *τ is increasing function.*
(ii.2) *In particular (2) holds. The equality holds in (5.3) if and only if $\phi(z) = a_0 + a_1 z$.*

Proof. Since $k(2k-2) \geq 0$, τ' is non negative and τ is increasing function and consequently $\tau(1) \geq \tau(r)$ and therefore

$$\frac{s(1)}{s(r)} \geq \frac{1}{r^2} = e^{2\ln\frac{1}{r}}. \tag{3}$$

Hence, since $M(A) = M(A_r) = \ln\frac{1}{r}/2\pi$, it follows $s(r)e^{4\pi mod(A)} \leq s(1)$. Since $s(1) = area(D)$ and $s(r) = area(F)$, this yields (2).

If equality holds in (5.3), then equality holds in (3). Hence $a_k = 0$, $k \neq 1$, and therefore $\phi(z) = a_0 + a_1 z$. □

If function ϕ is analytic on an annulus $A_r = \{r < |z| < 1\}$, then $\tau(\rho) = \rho^{-2}s(\rho)$ is not decreasing.

As a corollary $A(r_1, R_1)$ and $A(r_2, R_2)$ are conformally equivalent if and only if $R_1/r_1 = R_2/r_2$.

Let f be a holomorphic function on $A(r, R)$, $r < \rho < R$, $\Gamma_\rho = f \circ K_\rho$, where K_ρ positively oriented circle of radius ρ withe center at the origin.

Denote by $S(\rho) = S_f(\rho)$ the oriented area surrounded by Γ_ρ and set $\tau(\rho) = \tau_f(\rho) = \rho^{-2}S(\rho)$.

Theorem 5.5. (i.3) *τ_f is increasing on $[r, R]$, that is for $r < r_1 \leq R_1 < R$,*

$$\frac{R_1^2}{r_1^2} \leq \frac{S(R_1)}{S(r_1)}.$$

(ii.3) *Let A_1 be two ring domain in the plane and $\psi : A \to A_1$ be a covering with degree p.*
(iii.3) *Then $\tau_p(\rho) = \rho^{-2p}S(\rho)$ is increasing, that is for $r < r_1 \leq R_1 < R$,*

$$\frac{R_1^{2p}}{r_1^{2p}} \leq \frac{S(R_1)}{S(r_1)}.$$

(iv.3) *Let A and A_1 be two ring domains in the plane and let $\psi : A \to A_1$ be K-qr. Then $|deg\psi|M(A) \leq KM(A_1)$.*

The proof will appear in a forthcoming paper. If f is univalent, Theorem 5.5 (i.3) is reduced to Theorem A.

Beardon and Minda proved (Theorem 13.6, [6]):
If ψ is a holomorphic function. Then $|deg\psi|M(A) \leq M(A_1)$.
In addition, if ψ is a covering with degree p, then $M(A_1) = pM(A)$.

Note that if γ is a closed rectifiable curve of length $L = l(\gamma)$, then
$L \geq 2\pi cap\,\gamma$.

6. Capacity of condenser, modulus and the extremal distance

In this section we discus further result. A condenser in the complex plane $\mathbb{C}$ is a pair (D, K) where D is a proper subdomain of $\mathbb{C}$ and K is a compact subset of D. Let h be the solution of the generalized Dirichlet problem on $A = D \setminus K$ with boundary values 0 on ∂D and 1 on ∂K. The function h is the equilibrium potential of the condenser (D, K). The capacity of (D, K) is

$$Cap(D, K) = \int_A |\nabla h|^2 .$$

Milnor [40], p.232 attributes Theorem A to McMullen (McMullen inequality).

Let $F \subset D$ and $A = D \setminus F$ topological annulus in $\mathbb{C}$. On the other hand, Carleman proved (see [4] p.82-84, Theorem 2.14):

Theorem 6.1. *Among all ring domains with given area and with given area of the "holes" the domain bounded by two concentric circles gives the smallest value of $cap(D, F)$.*

Let A be a ring; then ∂A has two components say C_1 and C_2. Denote by $\Gamma = \Gamma_A$ the collection of curves $\gamma \subset A$ connecting C_1 and C_2. Let Γ^* be the collection of all curves that wind once around the annulus, separating C_1 from C_2. Since $cap(D, F) = M(\Gamma_A)$ and $M(A)M(\Gamma_A) = 1$, we have
(I.2) $\quad cap^{-1}(D, F) = M(A)$.

By this crucial equality, Theorems 5.3 and 6.2 are equivalent. Recall (i.3): *Let $F \subset D$ and $A = D \setminus F$ topological annulus in R^2, $S_1 = areaD = \pi r_0^2$ and $S_0 = areaF = \pi r_1^2$. Then* by (I.2) and Theorem 5.3, we have

$$4\pi cap^{-1}(D, F) \leq \ln \frac{S_1}{S_0}\,.$$

Therefore we can rewrite Theorems 5.3 and 6.2 respectively in the form:
(ii.3) $cap(D, F) \geq cap(B_{r_1}, B_{r_0})$,
(iii.3) $M(A) \leq M(A(r_0, r_1))$.

6.1. Further results

Since $M(A) = EL(\Gamma_A) = dist_A(D, F)$, Theorems 5.3 can be stated in terms of extremal length (the extremal distance in A between two sets D and F) (see subsection 6.3 for definitions):

Theorem 6.2. *Among all ring domains with given area and with given area of the "holes" the domain bounded by two concentric circles gives the largest value of the extremal distance in A between two sets D and F.*

C. Bandle proved a version for surfaces whose Gaussian curvature does not exceed a constant K_0 (see [4] p.82-84, Theorem 2.14).

The corresponding result, which is a generalization of Theorem A, if ϕ is not univalent (in particular p-valent) has been proved by author around 20 years ago. Mc Mullen found application of Theorem A in complex dynamics. In [45] the following result is proved: Let A be an annulus in $\mathbb{C}$ with fixed modulus $\mathrm{M}A$, and K the bounded component of $\mathbb{C} \setminus A$. If q is a meromorphic function in $A \cup K$ such that q has at most one simple pole, the pole (if any) is in K and $\int_{A\cup K} |q(z)| dxdy < \infty$, then

$$\int_K |q(z)| dxdy \leq e^{-2\pi \mathrm{M(A)}} \int_{A\cup K} |q(z)| dxdy\,.$$

If q has no pole, 2π can be replaced by 4π.

The proof is a word-to-word translation [40] Appendix B, Corollary B.9, McMullen inequality with Euclidean metric replaced by conformal metric $\sqrt{|q(z)|}|dz|$ induced from quadratic differential $q(z)dz^2$.
M. Papadimitrakis and S. Pouliasis, cf. [43] prove an inequality for the capacity of a condenser via a holo morphic function f, under a valency assumption on f, and they show that equality occurs if and only if f has finite constant valency. Also, they generalize a well known inequality for quasiregular mappings and give a necessary condition for the case of equality. Let p be a monic polynomial in one complex variable and K a measurable subset of the complex plane. In terms of the area of K, in [15] it is given an upper bound on the area of the preimage of K under p and a lower bound on the area of the image of K under p, (counted with multiplicity). Both bounds are sharp. The former extends an inequality of Polya. The proof uses Carleman's isoperimetric inequality for plane condensers. Also it is included a summary of the necessary potential theory. Let D be a domain in z-plane and consider a metric $ds = \rho|dz|$. Let γ be a closed Jordan rectifiable curve in D and G a plane region it encloses. We define

$$A(\rho) = A(\rho; G) = \iint\limits_G \rho^2(z)\, dxdy, \quad \text{and}$$

$$L(\rho) = L_\gamma(\rho) = \int\limits_\gamma \rho |dz|\,,$$

and call $A(\rho)$ and $L(\rho)$(respectively), ρ-area of G and ρ- length of γ respectively.

The isoperimetric inequality for surfaces is closely related to their Gaussian curvature. Namely it is well known the following facts:

For every Riemannian metric ρ on a disk $\mathbb{B}^2$ with the Gauss curvature $K(x) \leq 0$ the euclidean isoperimetric inequality holds

$$4\pi A \leq L^2, \tag{4}$$

where $L(\rho)$ (ρ- length) is the circumference of a closed Jordan rectifiable curve γ in $\mathbb{B}^2$ and the area of a plane region it encloses $A(\rho) = A(\rho;\gamma)$.

A surface enjoys locally isoperimetric inequality (4) if and only if its Gaussian curvature is nonpositive (Beckenbach-T. Rado and Weil [14, Remark V.5.3.] and [48]).

6.2. Another proof of Theorem 5.3

A nice proof is presented in [40]. It is based on length-area-modulus inequality and the isoperimetric inequality. Namely, there exists a curve γ with winding number 1 about A whose length satisfies $L^2 \leq area(A)/mod(A)$. Since F is enclosed with this curve, by the isoperimetric inequality $area(F) \leq \frac{L^2}{4\pi}$. Hence

$$area(F) \leq \frac{area(D)}{1 + 4\pi mod(A)}. \tag{5}$$

Cut the annulus A up into n concentric nested annuli A_k such that $mod(A_k) = mod(A)/n$. Let F_k be the bounded component of the complement of A_k. Since $A_k \cup F_k = F_{k+1}$, then $area(F_{k+1})/area(F_k) \geq 1+4\pi mod(A)/n$ by (5), hence

$$area(D)/area(F) \geq (1 + 4\pi mod(A)/n)^n,$$

where the right -hand side converges to $e^{4\pi mod(A)}$ as $n \to \infty$. □

Let a curve γ be defined on $[0, 2\pi]$ and $\gamma \sim \sum_{n=-\infty}^{\infty} \hat{\gamma}_n e^{-in\theta}$. If γ is of bounded variation, closed and continuous curve, then the sign area $A(\gamma) = \frac{i}{2}\int_0^{2\pi} \gamma\, d\overline{\gamma}$. Then

$$A(\gamma) = \pi \sum_{-\infty}^{\infty} n|\hat{\gamma}_n|^2 < \infty \tag{6}$$

Set $g = P[\gamma]$. For $0 \leq r < 1$, $B(\gamma_r) = \frac{i}{2}\int_0^{2\pi} \gamma_r\, d\overline{\gamma} = \pi \sum\limits_{-\infty}^{\infty} nr^{|n|}|\hat{\gamma}_n|^2$ and $B(\gamma_r) \to A(\gamma)$ when $r \to 1_-$. Hence the series 6 is Abel summable. Since $n|\hat{\gamma}_n|^2 \to 0$, by Tauber's convergence theorem it is convergent in the ordinary sense as well.

6.3. Extremal length

In the theory of conformal and quasiconformal mappings, the extremal length of a collection of curves Γ is a conformal invariant of Γ. More specifically, suppose that D is an open set in the complex plane and Γ is a collection of paths in D and $f : D \to D'$ is a conformal mapping. Then the extremal length of Γ is equal to the extremal length of the image of Γ under f. For this reason, the extremal length is a useful tool in the study of conformal mappings. Extremal length can also be useful in dimensions greater than two, but the following deals primarily with the two dimensional setting.

To define extremal length, we need to first introduce several related quantities. Let D be an open set in the complex plane. Suppose that Γ is a collection of rectifiable curves in D. If $\rho : D \to [0,\infty]$ is Borel-measurable, then for any rectifiable curve γ we let

$$L_\rho(\gamma) := \int_\gamma \rho\, |dz|$$

denote the ρ-length of γ, where $|dz|$ denotes the Euclidean element of length. (It is possible that $L_\rho(\gamma) = \infty$.) What does this really mean? If $\gamma : I \to D$ is parameterized in some interval $I = [a, b]$, then $\int_\gamma \rho\,|dz|$ is the integral of the Borel-measurable function $\rho(\gamma(t))$ with respect to the Borel measure on I for which the measure of every subinterval $J \subset I$ is the length of the restriction of γ to J. In other words, it is the Lebesgue-Stieltjes integral $\int_I \rho(\gamma(t))\,ds_\gamma(t)$, where $s_\gamma(t) = \text{length}_\gamma(t)$ is the length of the restriction of γ to $[a, t] = \{s \in I : s \le t\}$. Also set

$$L_\rho(\Gamma) := \inf_{\gamma\in\Gamma} L_\rho(\gamma).$$

The area of ρ is defined as

$$A(\rho) := \int_D \rho^2\,dx\,dy,$$

and the extremal length of Γ is

$$EL(\Gamma) := \sup_\rho \frac{L_\rho(\Gamma)^2}{A(\rho)}\,,$$

where the supremum is over all Borel-measureable $\rho : D \to [0, \infty]$ with $0 < A(\rho) < \infty$. If Γ contains some non-rectifiable curves and Γ_0 denotes the set of rectifiable curves in Γ, then $EL(\Gamma)$ is defined to be $EL(\Gamma_0)$. The term modulus of Γ refers to $1/EL(\Gamma)$.

Let r_1 and r_2 be two radii satisfying $0 < r_1 < r_2 < \infty$. Let $\underline{A}$ be the annulus $\underline{A} = \underline{A}(r_1, r_2) := \{z \in \mathbb{C} : r_1 < |z| < r_2\}$ and let C_1 and C_2 be the two boundary components of $\underline{A}$: $C_1 := \{z : |z| = r_1\}$ and $C_2 := \{z : |z| = r_2\}$. Then one can show

$$EL(\Gamma^*) = \frac{2\pi}{\log(r_2/r_1)} = EL(\Gamma)^{-1},$$

where $\Gamma = \Gamma_{\underline{A}}$ and $\Gamma^* = \Gamma^*_{\underline{A}}$.

A domain A is ring if A^c has exactly two components. By topology, ∂A has also two components C_1 and C_2. Recall we denote by $\Gamma = \Gamma_A$ the collection of curves $\gamma \subset A$ connecting C_1 and C_2. Let $\Gamma^* = \Gamma^*_A$ be the collection of all curves that wind once around the annulus, separating C_1 from C_2. If ring A is conformally equivalent to a circular ring $\underline{A} = \underline{A}(r_1, r_2)$, since the modulus and the extremal length are conformal invariants, we find $M(A) = M(\underline{A})$. Hence

$M(A) = EL(\Gamma_A) = EL(\Gamma^*_A)^{-1} = \frac{\log(r_2/r_1)}{2\pi}$.

The extremal distance in D between two sets D_1 and D_2 in $\overline{D}$ (notation $d_D(D_1, D_2)$) is the extremal length of the collection of curves in D with one endpoint in one set and the other endpoint in the other set.

6.4. Potential

Let D be simply-connected domain f, $z_0 \in D$ be conformal of D onto $\mathbb{U}$. Then the function $g(z, z_0) = -\ln|f|$ is called Green's function of D. Let E be a continum and $D = \mathbb{C} \setminus E$. By *Riemann's mapping theorem* there is a conformal mapping

$$F(z) = \lambda\,z + \frac{a_1}{z} + \cdots + \frac{a_k}{z^k} + \cdots \tag{1}$$

of $\mathbb{E}$ on D. Then $d_\infty(E) = |\lambda|$. Let E be a compact connected set, let $G = E_\infty$ denote the unbounded component of its complement $E^c = \mathbb{C} \setminus E$ and $E_0 = E^c_\infty$. By *Riemann's mapping theorem* there is a conformal mapping

$$\phi(w) = \frac{w}{\lambda} + \frac{b_1}{w} + \cdots + \frac{b_k}{w^k} + \cdots \tag{2}$$

of E_∞ onto $\mathbb{E}$. Set $g(w, \infty) = \ln|\phi(w)|$. Then $g(w, \infty) = \ln|w| - \ln|\lambda| + o(1)$.

We call $g(w, \infty)$ the Green function, $\gamma = \gamma(E) = -\ln|\lambda|$ the Robin's constant and $\text{cap}(E) = |\lambda|$ the capacity of E. Note that $|\lambda| = e^{-\gamma}$.

We will adapt the above definitions to more general case. We first consider plane case.

Definition 6.1. *Let D be a domain in the extended z-plane. A function $g(z, z_0)$ is called Green's function of D if it has the following properties:*

1. *$g(z, z_0)$ is harmonic in D, except at the point z_0*
2. *$g(z, z_0) + \ln|z - z_0| = \gamma + \epsilon(z)$ is harmonic in a neighborhood of z_0, where γ is a constant and $\epsilon(z) \to 0$ for $z \to z_0$. The constant γ is known as the Robin constant.*
3. *As z tends to any point on the boundary of D, $g(z, z_0) \to 0$*

If $z_0 = \infty \in D$, we adapt definition such that $g(z, \infty) - \ln|z|$ is harmonic in a neighborhood of $z_0 = \infty$.

1. If D is a bounded domain for which the Dirichlet problem is solvable, then the Green's function for D exists. Let G be the solution with boundary data $h(\zeta) = \ln|\zeta - z_0|$. Then $g(z, z_0) = G - \ln|z - z_0|$ is the Green's function for D.
2. If D is a domain bounded by a finite number of Jordan arcs, then the Green's function for D exists.
3. We consider now the case of an arbitrary compact set E.

We assume first that the complement of E is bounded by finite number of piecewise analytic Jordan curves and denote the unbounded component by $G = E_\infty$. It is known that G has a Green function g which is harmonic in G, vanishes on ∂G and has asymptotic behavior at ∞ of the form

$$g(z) = \ln|z| + \gamma + \epsilon(z),$$

where γ is a constant and $\epsilon(z) \to 0$ for $z \to \infty$. The constant γ is known as the Robin constant. In general, we can find an "exhaustion" of G by regular domains $G_1 \subset G_2 \subset G_3 \subset \ldots G_n \ldots$; such that $G_n \to G$. Each G_n has a Green's function g_n and a Robin's constant γ_n. By the maximum principle, $g_n(z, z_0) \leq g_{n+1}(z, z_0)$, $z, z_0 \in G_n$ and therefore $\gamma_n \leq \gamma_{n+1}$. Hence, for a given pair z, z_0, $\lim_{n\to\infty} g_n(z, z_0) = g(z, z_0)$ (possibly$= \infty$) and $\lim_{n\to\infty} \gamma_n = \gamma$ exist.

The constant γ is known as the Robin constant.

For the moment we shall call $\mathrm{cap}(E) = e^{-\gamma}$ the Robin-capacity of E.

We write $P(E)$ for the class of measures $\mu \geq 0$ with support contained in E and by $P_0(E)$ we denote subclass of it consisting of μ with $\mu(E) = 1$.

Consider a positive mass distribution μ on the compact set E. We define the logarithmic potential of μ

$$p(z) = \int_E ln \frac{1}{|z - \zeta|} d\mu(\zeta) \tag{7}$$

We set $V_\mu = \sup_z p(z)$. Note that p is a supperharmonic and harmonic on E^c. If ν is another mass distribution, we define

$$I(\mu, \nu) = \int_E p(z) d\nu(z)$$

and write $I(\mu)$ instead of $I(\mu, \mu)$; it is the energy integral of μ.

Define $\gamma_0(E) = inf I(\mu)$ and $\gamma_*(E) = inf V_\mu$, ($\mu \in P(E)$; $\mu(E) = 1$).

For the moment we shall call $e^{-\gamma_0}$ the energy-capacity of E.

Suppose that $\mu \in P(E)$ and $\mu(E) = 1$.

There is $a > 0$ such that $\big|\ln|1 - \zeta/z|\big| \leq a/|z|$ for every $\zeta \in E$.

Since $-\ln|z - \zeta| = -\ln|z| - \ln|1 - \zeta/z|$, we find $p_\mu(z) = -\ln|z| + o(1)$.

If also $\nu \in P(E)$ and $\nu(E) = 1$, then $h(z) = p_\mu(z) - p_\nu(z)$ is harmonic function at ∞ and $h(\infty) = 0$.

We assume first that the complement of E is bounded by finite number of piecewise analytic Jordan curves and denote the unbounded component by G.

If we set $h(\infty) = 0$ function h is harmonic on $G_0 = G \cup \infty$ and it follows by an application of the maximum principle on G_0 that $\max h \geq 0$ on ∂G.

If there is a positive mass distribution μ_* on the compact set E such that the logarithmic potential p_* of μ_* is a constant V on E, then μ_* minimizes V_μ.

Theorem 6.3. *Among all distribution with total mass $\mu(E) = 1$, there is one $\mu_* \in P_0(E)$ that minimizes V_μ.*

If the complement of E is bounded by finite number of piecewise analytic Jordan curves, then the logarithmic potential p_* of μ_* is equal Robin's constant γ on E.

The distribution μ_* is known as the equilibrium distribution and its logarithmic potential p_* as the equilibrium potential.

Theorem 6.4 ([1, 20]). *Among all distribution with total mass $\mu(E) = 1$, there is one that minimizes V_μ. The same distributions minimizes $I(\mu)$, and two minima are equal.*

In particular, the Robin capacity equal to the energy capacity.

Definition 6.2. *If* $\min V_\mu = V$*, we call* $cap(E) = e^{-V}$ *the capacity (logarithmic) of E.*

Proof. We assume first that the complement is bounded by finite number of piecewise analytic Jordan curves and denote the unbounded component by $G = E_\infty$. It is known that G has a Green function g which is harmonic in G, vanishes on ∂G and has asymptotic behavior at ∞ of the form

$$g(z) = \ln|z| + \gamma + \epsilon(z),$$

where γ is a constant and $\epsilon(z) \to 0$ for $z \to \infty$. The constant γ is known as the Robin constant.

Define a positive mass distribution μ_* by setting

$$\mu_*(e) = -\frac{1}{2\pi}\int_{e\cap\partial G}\frac{\partial g}{\partial n}|dz|$$

for any Borel set e.

For $\zeta \in G$, Green's formula yields

$$g(\zeta) - \gamma = \frac{1}{2\pi}\int_{\partial G} ln\frac{1}{|z-\zeta|}\frac{\partial g}{\partial n}|dz| \tag{8}$$

formula shows that potential p_* of μ_* satisfies $p_*(\zeta) = \gamma - g(\zeta)$ for $\zeta \in G$. Since Green's formula can also be applied when ζ is an exterior point of G, we find that $p_*(\zeta) = \gamma$ on E.

Let μ be another positive mass distribution with total mass 1 and let p be its potential and $h = p - p_*$.

If we set $h(\infty) = 0$ function h is harmonic on $G \cup \infty$ and it follows by the maximum principle that $V_\mu \geq V_{\mu_*} = \gamma$. □

We give a physical interpretation: Professor R. Shankar, Department of Physics, Yale University, electrodinamic, gave comments (explained why the equilibrium potential is constant on E): If the potential varied, there would be a field (which is its gradient) and current would flow. That is not equilibrium. So what happens is that current flows till there is no reason to flow, i.e when the field is zero and hence V is constant.

The logarithmic capacity of a compact set E in the complex plane is given by $\gamma(E) = e^{-V(E)}$, (1) where

$$V(E) = \inf_\nu \int_{E\times E} ln\frac{1}{|u-v|}d\nu(u)d\nu(v), (2)$$

and ν runs over each probability measure on E. The quantity $V(E)$ is called the Robin's constant of E and the set E is said to be polar if $V(E) = +\infty$ or equivalently, $\gamma(E) = 0$.

6.5. The transfinite diameter

Let E be compact set in $\mathbb{C}$. We define

$$V(z_1, z_2, \cdots, z_n) = \prod_{k,j=1,k<j}^{n} (z_k - z_j),$$

where $z_1, z_2, \cdots, z_n \in E$, and V_n as maximum modula of $|V(z_1, z_2, \cdots, z_n)|$, when $z_1, z_2, \cdots, z_n \in E$. The diameter of order n is defined as

$$d_n = V_n^{\frac{2}{n(n-1)}}.$$

d_n is not increasing; d_2 is diameter of E. The number $d = d_\infty(E) = \lim_{n\to\infty} d_n$ is called the transfinite diameter of E.

Note that $d_\infty \le d_2$ and the transfinite diameter of a set is equal to that of its boundary.

Theorem 6.5. *The capacity of a closed bounded set is equal to its transfinite diameter.*

Let $I = [a, b]$. Then $d(I) = \frac{|a-b|}{4}$.

Theorem 6.6. *Let E be a compact in $\mathbb{C}$ and d_∞ the transfinite diameter of E. Then*

$$m(E) \le \pi d_\infty^2 .$$

Let E be a compact and E' orthogonal projection on a line. Then $|E'| \le 4d_\infty(E)$.

Theorem 6.7. *Let E be a compact connected set, let $G = E_\infty$ denote the unbounded component of its complement $E^c = \mathbb{C} \setminus E$ and $E_0 = E_\infty^c$. By Riemann's mapping theorem there is a conformal mapping*

$$F(z) = \lambda z + \frac{a_1}{z} + \cdots + \frac{a_k}{z^k} + \cdots \tag{4}$$

of $\mathbb{E}$ on G. Then $d_\infty(E) = d_\infty(E_0) = |\lambda|$.

Proof. Let $\phi = F^{-1}$. Then

$$\phi(w) = \frac{w}{\lambda} + \frac{b_1}{w} + \cdots + \frac{b_k}{w^k} + \cdots \tag{?4}$$

and $g(w, \infty) = \ln|\phi(w)|$ is Green function. Since $g(w, \infty) = \ln|w| - \ln|\lambda| + o(1)$, the Robin's constant $\gamma = \gamma(E) = -\ln|\lambda|$ and therefore $\text{cap}(E) = |\lambda|$. Hence Theorem 6.5 gives the desired result. □

$\phi(z) = \frac{1}{2}(z + 1/z)$ maps $\mathbb{E}$ onto $\mathbb{C} \setminus I$, where $I = [-1, 1]$. Hence $d_\infty(I) = 1/2$. The analytic capacity $\gamma(K)$. Let K be a compact set in $\mathbb{C}$ and let $\Omega(K) = K_\infty$ be the connected component of $\mathbb{C} \setminus K$ containing the point at infinity. The analytic capacity $\gamma(K)$ is defined by $\gamma(K) = \sup_{|f|_\infty \le 1} |f'(\infty)|$. Here f is a holomorphic function in $\Omega(K)$ whose expansion at infinity is given by $f(z) = f'(\infty)/z + a_2/z^2 + \ldots$ and $|f|_\infty$ denotes the supnorm of f.

Hayman proved (see [20], Chapther VIII) the following: Let F be meromorphic in $\mathbb{E}$, and

$$F(z) = \lambda z + \frac{b_1}{z} + \cdots + \frac{b_k}{z^k} + \cdots \tag{4}$$

near ∞, $\lambda \ne 0$. If E is the omitted set of F, then

$$\text{cap}(E) \le |\lambda| .$$

7. Multiplicity and isoperimetric inequality

Recall $\mathbb{U}_r = \{|z| < r\}$. Suppose that f is analytic in $\mathbb{U}$ and $D(r, f) = f(\mathbb{U}_r)$.

Let $D_\infty(r, f)$ denote the unbounded component of $D^c(r, f)$ and $C(r) = C(r, f)$ the boundary of $D_\infty(r, f)$. By $a(r) = a(r, f)$ and $l(r) = l(r, f)$ we denote the area of the set $D(r, f)$ and the length of the curve $C(r, f)$; the length of multiply covered arcs of $C(r)$ are counted only once. As a corollary of the isoperimetric inequality, we have

A-1 $4\pi a(r) \le l^2(r)$.

In [3, 27] it is proved:

A-2 If $f(z) = \sum_{k=1}^{\infty} a_k z^k$ in $\mathbb{U}$ and $0 < r < 1$, then $\pi \sum_{k=1}^{\infty} |a_k|^2 r^{2k} \le a(r)$.

By A-1 and A-2, we find

A-3 $4\pi^2 \sum_{k=1}^{\infty} |a_k|^2 r^{2k} \le 4\pi a(r) \le l^2(r)$, $0 < r < 1$.

In [33] it is proved:

A-4 Suppose that f is analytic in $\mathbb{U}$. Then

(a) $\pi |a_n|^2 r^{2n} \le a(r, f)$, $n \ge 1$.

(b) $2\pi |a_n| r^n \le l(r, f)$, $n \ge 1$.

The isoperimetric inequality A-1 shows that (a) implies (b).

Note that A-3 gives a significant improvement of A-4 (a).

By $L(r) = L(r, f)$ and $A(r) = A(r, f)$ we denote the length of the curve $K(r) = K(r, f) : w = f(re^{it})$, $0 \le t \le 2\pi$, and the area of the set $D(r)$ counting multiplicity, respectively.

It is known that

$$L(r) = r \int_0^{2\pi} |f'(re^{it})| dt, \quad A(r) = \int_{U_r} |f'(z)|^2 dx\, dy\,, \tag{9}$$

A-5 $4\pi A(r) \le L^2(r)$, $r \in (0, 1)$, and

A-6 $L^2(r) - 4\pi A(r)$ is non-decreasing in $r \in (0, 1)$.

Question 1. Are there several-dimension generalization of the above statements?

Question 2. Let $f_0 : \mathbb{T} \to \mathbb{C}$ be a curve and $\underline{C} = C(f_0)$ the family of all reparametrization of f_0 and $H = H(f_0) = \{P[f] : f \in \underline{C}\}$ the corresponding family of harmonic mappings. Describe solutions of the problem

(A) $\inf_{f \in H}(\int_{\mathbb{U}} |\nabla f|^2 dxdy)$.

Are the solutions holomorphic functions ?

An extension of the area theorem

$\mathrm{E}^* = E(f) = \mathbb{C} \setminus f(\mathbb{E})$

Theorem 7.1 ([37]). *Let*

$$f(z) = \lambda z + \frac{b_1}{z} + \cdots + \frac{b_k}{z^k} + \cdots \tag{2a}$$

be analytic on $\mathbb{E}$

a) $area(\mathrm{E}^*) \ge \pi\, (|\lambda|^2 - \sum_1^\infty k\, |b_k|^2\,)$.

b) $area(\mathrm{E}^*) \le \pi\, |\lambda|^2$ *Equaliy holds in (a) iff f is univalent. Equaliy holds in (b) iff $f(z) = \lambda z$.*

Finally we state a generalization of the area theorem to analytic functions.

Theorem 7.2 ([37]). *Let $w = f(z) = \lambda z + \frac{a_1}{z} + \cdots + \frac{a_n}{z^n} + \cdots$ be an analytic function on $E = \{z : |z| > 1\}$ and let $G = \mathbb{C} \setminus f(E)$ be the omitted set. Then*

$$\pi\Big(|\lambda|^2 - \sum_{k=1}^{\infty} k|a_k|^2\Big) \le area(G). \tag{B1}$$

Equality holds if and only if f is a univalent function on E.

Proof. Let K_ρ be the circle $|z| = \rho$ with positive orientation and let γ_ρ be the curve defined by the equation $w = f_\rho(e^{it}) = f(\rho e^{it}), 0 \le t \le 2\pi$.

For given $w \neq \infty$ let $n(w)$ be the number of roots of $f(z) = w$ in $|z| > \rho$. Assume that $f \neq w$ on K_ρ and $\lambda \neq 0$.Since f has a pole of order 1 at ∞, we have $f(z) \neq w$ in $|z| \ge r$ for a large r and consequently, by the argument principle,

$$n(w) = \frac{1}{2\pi i} \int_{K_r - K_\rho} \frac{f'(z)}{f(z) - w} dz = 1 - \chi(\gamma_\rho, w), \tag{B2}$$

where $\chi = \chi(\gamma_\rho, w)$ is the winding number (or index) of the curve γ_ρ with respect to the point w. By the analytic Green's theorem (see, for example [**Po**]), the area

$$I_\rho = \frac{1}{2\pi i} \int_{\gamma_\rho} \bar{w} dw = \frac{1}{\pi} \int\int_{\mathbb{R}^2} \chi(\gamma_\rho, w) du dv. \tag{B3}$$

Let G_ρ be the set omitted by f on $E_\rho = \{|z| > \rho\}$. By (1) $w \in G_\rho$ if and only if $\chi(\gamma_\rho, w) = 1$. Also, it follows from (1) that $\chi(\gamma_\rho, w)$ is an integer less than or equal to zero if $w \notin \bar{G}_\rho$. This together with (B3) gives

$$\pi I_\rho \le area(G_\rho). \tag{B4}$$

Direct calculation as in the proof of area theorem gives (B1).

By the isoperimetric inequality $area(G) \le \pi cap^2(G)$. By Hayman $cap(E) \le |\lambda|$ and therefore (b).

For the case of equality see [37].

Let $K \subset \mathbb{C}$ be compact and let $M(K)$ denote the class of all meromorphic functions f of $\mathbb{U}$ into K_∞. Let f_0 be the Ahlfors function of K. For $g \in M(K)$ and $h = f_0 \circ g$. Since $h'(0) = f_0'(\infty)/\lambda$, where $\lambda = \hat{g}(-1)$, by Schwarz's lemma $|f_0'(\infty)| \le |\lambda|$. There is a branch g_0 of f_0^{-1}; then $f_0'(\infty) = \hat{g}_0(-1)$. Hence

$\inf\{|\hat{g}(-1)| : g \in M(K)\} = |f_0'(\infty)| = \mathrm{cap}(K)$

There is a covering map f of $\mathbb{E}_\infty$ onto K_∞, which is locally univalent. Hence

$$f(z) = \lambda z + \frac{b_1}{z} + \cdots + \frac{b_k}{z^k} + \cdots \tag{2a}$$

and

$$A(\gamma_\rho) = \pi\Big(|\lambda|^2 \rho^2 - \sum_{k=1}^{\infty} k|a_k|^2 \rho^{-2k}\Big) \le area(G). \tag{B1}$$

Hayman (see [20], Chapther VIII), proved:

Theorem 7.3. *Let F be meromorphic in* $\mathbb{E}$, *and*

$$F(z) = \lambda\, z + \frac{b_1}{z} + \cdots + \frac{b_k}{z^k} + \cdots \tag{4}$$

near ∞, $\lambda \neq 0$. *If E is the omitted set of F, then*

$$cap(E) \le |\lambda|\,.$$

Let $f_0 : E(F) \to \mathbb{U}$ be analytic. Set $J(z) = 1/z$, $g = F \circ J$ and $h = f_0 \circ g$. Since $h'(0) = f_0'(\infty)/\lambda$, by Schwarz's lemma

$|f_0'(\infty)| \le |\lambda|$.

Let $K \subset \mathbb{C}$ be compact and $\mathcal{H}_0(K^c) = \{f : \ f \in \mathcal{H}^\infty(\mathbf{C} \setminus K),\ \|f\|_\infty \le 1,\ f(\infty) = 0\}$. Then its analytic capacity is defined to be

$$\gamma(K) = \sup\{|f'(\infty)| : \ f \in \mathcal{H}^\infty(\mathbf{C} \setminus K),\ \|f\|_\infty \le 1,\ f(\infty) = 0\}\,. \tag{10}$$

Here $f'(\infty) := \lim_{z\to\infty} z\,(f(z) - f(\infty))\,, f(\infty) := \lim_{z\to\infty} f(z)$. For each compact $K \subset \mathbb{C}$, there exists a unique extremal function, i.e. $f \in \mathcal{H}_0(K^c)$ such that $f'(\infty) = \gamma(K)$. This function is called the Ahlfors function of K. Its existence can be proved by using a normal family argument involving Montel's theorem.

We close this section with short discussion concerning Removable sets and Painlevé's problem. The compact set K is called removable if, whenever G is an open set containing K, every function which is bounded and holomorphic on the set $G \setminus K$ has an analytic extension to all of G. By Riemann's theorem for removable singularities, every singleton is removable. This motivated Painlevé to pose a more general question in 1880: "Which subsets of $\mathbb{C}$ are removable?" It is easy to see that K is removable if and only if $\gamma(K) = 0$. However, analytic capacity is a purely complex-analytic concept, and much more work needs to be done in order to obtain a more geometric characterization.

7.1. The oriented area

In topology, a curve is defined as follows. Let I be an interval of real numbers (i.e. a non-empty connected subset of $\mathbb{R}$). Then a curve γ is a continuous mapping $\gamma : I \to X$, where X is a topological space, and $tr(\gamma) = \{\gamma(t) : t \in I\}$. Suppose we are given a closed, oriented curve in the xy- plane. We can imagine the curve as the path of motion of some object, with the orientation indicating the direction in which the object moves. Then the winding number of the curve is equal to the total number of counterclockwise turns that the object makes around the origin.

When counting the total number of turns, counterclockwise motion counts as positive, while clockwise motion counts as negative. For example, if the object first circles the origin four times counterclockwise, and then circles the origin once clockwise, then the total winding number of the curve is three.

The winding number of a closed curve γ in the plane around a given point $z \notin tr(\gamma)$, $n_\gamma(z)$, is an integer representing the total number of times that curve travels counting orientation around the point. The winding number depends on the orientation of the curve, and is negative if the curve travels around the point clockwise.

Let γ be a rectifiable closed curve of length L given by equation $z = z(t)$, $-\pi \le t \le \pi$. We define the oriented area enclosed by γ with

$$A(\gamma) = \frac{1}{2i}\int_\gamma \overline{\gamma}\, d\gamma \ = \frac{1}{2i}\int_{-\pi}^{\pi} z'(t)\,\overline{z}(t)\,dt\,.$$

If $D_k = \{z : n_\gamma(z) = k\}$, then

A-7 $A(\gamma) = \sum kA(D_k)$ and $\sum |k| A(D_k) \le L^2/4\pi$. Let $F = F(\gamma)$ be a family of all curves γ_1 such that $tr(\gamma_1) = tr(\gamma)$ and $n_{\gamma_1}(z) \le |n_\gamma(z)|$ for every $z \notin tr(\gamma)$. We can define a curve γ_0 such that $tr(\gamma_0) = tr(\gamma)$ and $n_{\gamma_0}(z) = |n_\gamma(z)|$ for every $z \notin tr(\gamma)$.

We announce the following result:

Theorem 7.4. *Let $f : \mathbb{U} \to \mathbb{C}$ be harmonic, continuous on $\overline{U}$ and γ curve defined by $f(e^{it})$, $0 \le t \le 2\pi$. If γ is rectifiable curve of length L, then*

$$\int_U (|g'|^2 - |h'|^2)dxdy = \int_C n_\gamma(w)dyudv = A(\gamma) \le \frac{L^2}{4\pi}\,. \tag{11}$$

M. Pavlović has informed me that he also proved this result independently. In [29], D.Kalaj, M. Marković and M. Mateljević proved:

Theorem 7.5. (i.1) *If Σ is a simply-connected harmonic surface which allows a regular harmonic parametrization, then the Gaussian curvature of Σ is nonpositive.*
(i.2) (Isoperimetric inequality for harmonic surfaces). If $\Sigma \subset \mathbb{R}^n$ is a harmonic Jordan surface with the rectifiable boundary γ, then the classical isoperimetric inequality holds:

$$4\pi A(\Sigma) \le L^2(\gamma)\,. \tag{12}$$

Using the Wirtinger inequality, we prove:

Theorem 7.6. *Let*

$$f(z) = z + \frac{a_1}{z} + \cdots + \frac{a_k}{z^k} + \cdots, \quad g(z) = z + \frac{b_1}{z} + \cdots + \frac{b_k}{z^k} + \cdots. \tag{2}$$

be holomorphic function on $\mathbb{B}'$ *and* $h = (f, g)$ *and* $S \subset \mathbb{C}^2$ *surface such that* $S \cup h(\overline{\mathbb{U}}')$ *is simple connected. Then* *area*$(S) \geq \pi\,(|a_1|^2 + |b_1|^2 - \sum_1^\infty k\,(|a_k|^2 + |b_k|^2)\,)$.

7.2. the duality relation and Hardy spaces

In [35, 36], we study applications of the isoperimetric inequality to some extremal problem in Hardy spaces. Applying the duality relation in some cases we get better estimates from those which were obtained by the isoperimetric inequality. In this subsection we present some results obtained in [35, 36]. For given $f, h \in H^1$ we define $L(f, h) = L_h(f) = \pi^{-1} \int_{\mathbb{D}} f\,\overline{h}\,dxdy$. For given $h \in H^1$ we can consider L_h as linear functional on H^1. In [36], we prove that

1) $L(f, h) \leq \|f\|_1 \|h\|_1$ with equality if and only if $f = \alpha(1 - az)^{-2}$, $h = \lambda f$ $(\alpha, \lambda \in C, |a| < 1)$.
2) $|L_h| \leq |h|_1$ and
3) if $f \in H^p$ and $|a| < 1$, then $f(a) \leq (1 - |a|^2)^{-1/p} |f|_p$, cf. also [30, 31, 47].
4) If $f' \in H^1$ and $F = f'$, then $|L_F(F)| \leq |F|_1 |f|_\infty$ with equality if and only if f is a finite Blaschke product.

Lemma 7.1. *If* $f, h \in H^1$ *and* k *is function defined by* $k(z) = \int_0^z h(w)dw$, $z \in \mathbb{D}$, *then*

$$L(f, h) = \pi^{-1} \int_{\mathbb{D}} f\,\overline{h}\,dxdy = \frac{1}{2\pi i} \int_C f\,\overline{k}\,dz, \tag{13}$$

where C *is positively oriented boundary of the unit disk.*

Combining Lemma 7.1 and the duality relation([17], p.130) with the inequality 1) we obtain:
If $f \in H^1$ and $k = \int_0^z h$, then

$$\sup_{f \in H^1, \|f\|_1 \leq 1} \pi^{-1} \left| \int_{\mathbb{D}} f\,\overline{h}\,dxdy \right| = \min_{g \in H^\infty} \|\overline{k} - g\|_\infty \leq \|h\|_1 \tag{14}$$

It is clear that that this inequality improves 1).
Let $\gamma(t) = u(t) + i\,v(t)$, $\alpha \leq t \leq \beta$, be an absolutely continuous path with $u(\alpha) = u(\beta) = 0$. Then (see [39])

$$\min_{x \in \mathbb{R}} \int_\alpha^\beta |u(t) - x||v'(t)|\,dt \leq \frac{1}{4\pi} L^2 \tag{15}$$

8. Appendix 1

In this section, for the sake of convenience of the reader we sketch a proof of the following theorem about geodesics on a sphere and Fenchel's Theorem about total curvature.

Theorem 8.1. *The geodesics on* $\mathbb{S}^n$ *are precisely the great circles, that is, the intersections of* $\mathbb{S}^n$ *with the planes through the center of* $\mathbb{S}^n$.

Proof. Reflection through a plane E^2 is an isometry $I : \mathbb{S}^n \to \mathbb{S}^n$ whose fixed point set is $C = \mathbb{S}^n \cap E^2$. Let x and y be two points of C with a unique geodesic C' of minimal length between them. Then, since I is an isometry, the curve $I(C')$ is a geodesic of the same length as C' between $I(x) = x$ and $I(y) = y$. Therefore $C' = I(C')$. This implies that $C' \subset C$. Finally, since there is a great circle through any point of $\mathbb{S}^n$ in any given direction, these are all the geodesics. Antipodal points on the sphere have a continium of geodesics of minimal length between them. All other pairs of points have a unique geodesic of minimal length between them, but an infinite family of non-minimal geodesics, depending on how many times the geodesic goes around the sphere and in which direction it starts. By the same reasoning every meridian line on a surface of revolution is a geodesic. The geodesics on a right circular cylinder Z are the generating lines, the circles cut by planes perpendicular to the generating lines, and the helices on Z.

8.1. Fenchel's Theorem

If X is a topological space, a path (curve) in X is a continuous mapping f of a compact interval $[\alpha, \beta] \subset \mathbb{R}$ (here $\alpha < \beta$) into X. Strictly speaking, a curve $\gamma = [f]$ is class of equivalence of path f. Often, it is convenient to identify γ with a parametrization f. We call $[\alpha, \beta]$ the parameter interval of f (γ) and denote the range of γ by $\mathrm{tr}(\gamma)$. Thus γ is a mapping, and $\mathrm{tr}(\gamma)$ is the set of all points $\gamma(t)$, for $\alpha \leq t \leq \beta$. Suppose γ is a rectifiable, oriented curve in $\mathbb{R}^n$ and f a parameterization of γ on $I = [0, 1]$. Then the function $s = s_f$, defined for $\varphi \in I = [0, 1]$ by $s(\varphi)$ to be the length of the restriction $\gamma|[0, \varphi]$, is called the arc-length (natural) parameter; and function $\bar{f}$, defined by $\bar{f}(s) = f(t)$, is called arc-length (natural) parameterization of f. Note that $s = s_f$ is an increasing continuous function from $[0, 1]$ onto $[0, l]$, where l is the length of γ. In addition, if γ is differentiable, then $s(\varphi) = \int_0^\varphi |f'(t)|dt$.

Now, suppose that γ is a rectifiable, oriented, differentiable curve given by its arc-length parameterization $g = \bar{f}$. We define the tangent $T_f(t) = f'(t)$ and the unit tangent (the tangential indicatrix) $I_f(t) = \frac{T_f(t)}{|T_f(t)|} = \frac{f'(t)}{|f'(t)|}$, and for $p = g(s) = f(t)$, $T(p) = T_g(s) = g'(s)$. Since $I_f(t)$ and $T_g(s)$ are the unit vector, we have $I_f(t) = T(s)$, where we write shortly $T(s)$ for $T_g(s)$.

Hence $I'_f(t) = T'(s)s'(t)$.

Then $|g'(s)| = 1$ and $s = \int_0^s |g'(t)|dt$, for all $s \in [0, l]$.

If γ is a twice-differentiable curve, then the curvature of γ at a point $p = g(s)$ is given by $\kappa_\gamma(p) = |g''(s)|$; when it is convenient we write $\kappa(s)$ instead of $\kappa_\gamma(p)$.

Since $|g'(s)| = 1$ then $< g''(s), g'(s) >= 0$. In particular, if γ is in $\mathbb{C}$, we find

$$g'(s) = \alpha i g''(s),\ \alpha \in \mathbb{R}, \text{ and } |\alpha| = \kappa_\gamma^{-1}(p). \tag{16}$$

For a C^1 curve c in $\mathbb{R}^n$ of length L we define the total curvature by $\kappa_{tot} = \kappa_{tot}(c) = \int_0^L \kappa(s)ds$ and the average curvature by $\kappa_{av} = \kappa_{tot}/L$.

Recall, the unit tangent vectors emanating from the origin form a curve I_c, given by $I_c(t) = T_c(t)/|T_c(t)|$, on the unit sphere called the tangential indicatrix of the curve c. To calculate the length of the tangent indicatrix, we form the integral of $\kappa_c(t) = |T'(t)| = \kappa(s(t))s'(t)$, where s is the arc-length parameter, with respect to t; $|I_c| = \int_0^1 |I'_c(t)|dt$.

Since $\kappa(s)ds = \kappa(s(t))s'(t)dt$, $|I_c| = \int_0^L |T'(s)|ds = \int_0^1 \kappa(s(t))s'(t)dt = \int_0^L \kappa(s)ds$, so the length of curve I_c is κ_{tot}. Thus, this significant integral is the total curvature of the curve c.

Theorem 8.2 (Fenchel's Theorem). *The total curvature of a closed space curve c is greater than or equal to 2π.*

Thus, Fenchel's theorem (Werner Fenchel, 1929) states that the average curvature of any closed convex plane curve is $\frac{2\pi}{L}$, where L is the perimeter.

More generally, for an arbitrary closed curve in space the average curvature is $\geq \frac{2\pi}{L}$ with equality holding only for convex plane curves. The proof of Fenchel's Theorem given by R. Horn in 1971, depends on Lemma 8.1.

Lemma 8.1. *Let g be a closed curve on the unit sphere with length $L < 2\pi$. Then there is a point m on the sphere that is the north pole of a hemisphere containing g.*

Proof of Fenchel's Theorem. Let $m \in \mathbb{S}^2$ be an arbitrary point. Set $f(t) = c(t) \cdot m$. Then $0 = f'(t) = c'(t) \cdot m = s'(t)T(t) \cdot m$, so there are at least two points (maximum and minimum of f) on the curve such that the tangential image is perpendicular to m. Therefore the tangential indicatrix of the closed curve c is not contained in a hemisphere, so by the lemma, the length of any such indicatrix is greater than 2π. Therefore the total curvature of the closed curve c is also greater than or equal to 2π.

Corollary 1. *If, for a closed curve c, we have*
(i) $\kappa_c(t) \leq 1/R$ for all t, then
(ii) the curve has length $L \geq 2\pi R$.

By Fenchel's Theorem, $2\pi \le |I_c|$. On the other hand $|I_c| = \int_0^1 |T'(t)|dt = \int_0^1 \kappa(s(t))s'(t)dt = \int_0^L \kappa(s)ds$ and therefore by the hypothesis (i), $|I_c| \le L/R$. Hence, we get (ii).

There is a generalization of Lemma 8.1:

Lemma 8.2. *Any closed loop of length strictly less than 2π in the sphere $\mathbb{S}^{2n-1}$ must lie inside an open hemisphere, and so cannot be the boundary of any minimal surface spanning the unit ball and containing the origin.*

Gauss's integral definition

Given two non-intersecting differentiable curves $\gamma_1, \gamma_2 : S^1 \to \mathbb{R}^3$, define the Gauss map Γ from the torus to the sphere by

$$\Gamma(s,t) = \frac{\gamma_1(s) - \gamma_2(t)}{|\gamma_1(s) - \gamma_2(t)|}.$$

Pick a point in the unit sphere, v, so that orthogonal projection of the link to the plane perpendicular to v gives a link diagram. Observe that a point (s,t) that goes to v under the Gauss map corresponds to a crossing in the link diagram where γ_1 is over γ_2. Also, a neighborhood of (s,t) is mapped under the Gauss map to a neighborhood of v preserving or reversing orientation depending on the sign of the crossing. Thus in order to compute the linking number of the diagram corresponding to v it suffices to count the signed number of times the Gauss map covers v. Since v is a regular value, this is precisely the degree of the Gauss map (i.e. the signed number of times that the image of Γ covers the sphere). Isotopy invariance of the linking number is automatically obtained as the degree is invariant under homotopic maps. Any other regular value would give the same number, so the linking number doesn't depend on any particular link diagram.

This formulation of the linking number of γ_1 and γ_2 enables an explicit formula as a double line integral, the Gauss linking integral:

$$\text{linking number } = \frac{1}{4\pi}\oint_{\gamma_1}\oint_{\gamma_2} \frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3} \cdot (d\mathbf{r}_1 \times d\mathbf{r}_2). \tag{17}$$

This integral computes the total signed area of the image of the Gauss map (the integrand being the Jacobian of Γ) and then divides by the area of the sphere (which is 4π).

8.2. Minimal surfaces

Minimal surfaces can be defined in several equivalent ways in $\mathbb{R}^3$. The fact that they are equivalent serves to demonstrate how minimal surface theory lies at the crossroads of several mathematical disciplines, especially differential geometry, calculus of variations, potential theory, complex analysis and mathematical physics.

Local least area definition: A surface $M \subset \mathbb{R}^3$ is minimal if and only if every point $p \in M$ has a neighborhood with least-area relative to its boundary.

Note that this property is local: there might exist other surfaces that minimize area better with the same global boundary.

Variational definition:A surface $M \subset \mathbb{R}^3$ is minimal if and only if it is a critical point of the area functional for all compactly supported variations.

This definition makes minimal surfaces a 2-dimensional analogue to geodesics.

Soap film definition: A surface $M \subset \mathbb{R}^3$ is minimal if and only if every point $p \in M$ has a neighborhood D_p which is equal to the unique idealized soap film with boundary ∂D_p.

Mean curvature definition: A surface $M \subset \mathbb{R}^3$ is minimal if and only if its mean curvature vanishes identically.

A direct implication of this definition is that every point on the surface is a saddle point with equal and opposite principal curvatures.

Differential equation definition: A surface $M \subset \mathbb{R}^3$ is minimal if and only if it can be locally expressed as the graph of a solution of

$$(1 + u_x^2)u_{yy} - 2u_x u_y u_{xy} + (1 + u_y^2)u_{xx} = 0\,.$$

Douglas and simultaneously Rado solved the famous problem of Plateau,namely, that every Jordan wire in $\mathbb{R}^n$ bounds at least one disc-type surface of least area.

9. Appendix 2

9.1. Borromean rings, the Gehring Link Problem and the IMU logo

The Borromean rings consist of three topological circles which are linked and form a Brunnian link (i.e., removing any ring results in two unlinked rings). In other words, no two of the three rings are linked with each other as a Hopf link, but nonetheless all three are linked.

As you can see, it consists of three circles, linked so that they cannot be pulled apart. But no individual circle links with any one other, it is only the figure as a whole which cannot be disentangled.

This is an example of what mathematicians call a "link with three strands". The study of such links is part of Knot Theory, a mathematical topic which studies the form of knots and links. At the present time it does not study some matters of interest to a practical user such as the size of the knot when made up of string or rope, or its suitability as a knot for a particular and practical task. Nonetheless, the theory has amazing relations with topics such as polymer theory and theoretical physics. For more information try looking at the Knot Plot Site (outside link). The article [9] shows how Borromean squares exist, and have been made by Robinson (sculptor), who has also given other forms of this structure, see [50] for further information and literature, and [46] for visualization. Although the typical picture of the Borromean rings (above right picture) may lead one to think the link can be formed from geometrically ideal circles, they cannot be. (Freedman & Skora 1987) proves that a certain class of links, including the Borromean links, cannot be exactly circular. Alternatively, this can be seen from considering the link diagram: if one assumes that circles 1 and 2 touch at their two crossing points, then they either lie in a plane or a sphere. In either case, the third circle must pass through this plane or sphere four times, without lying in it, which is impossible; see [28],(Lindström & Zetterström 1991). A realization of the Borromean rings as ellipses 3*D* image of Borromean Rings It is, however, true that one can use ellipses (right picture). These may be taken to be of arbitrarily small eccentricity; i.e. no matter how close to being circular their shape may be, as long as they are not perfectly circular, they can form Borromean links if suitably positioned; as an example, thin circles made from bendable elastic wire may be used as Borromean rings.

This article explains why Borromean links cannot be exactly circular.

B.Lindström and H.O.Zetterström, cf. [28], proved that "Borromean circles are impossible": three flat circles cannot construct them, but by triangles they can. The Australian sculptor J.Robinson assembled three flat hollow triangles to form a structure (called Intuition), topologically equivalent to Borromean rings. Their cardboard model collapses under its own weight, to form a planar pattern. The Borromean rings are a hyperbolic link: the complement of the Borromean rings in the 3-sphere admits a complete hyperbolic metric of finite volume. The canonical (Epstein-Penner) polyhedral decomposition of the complement consists of two regular ideal octahedra. The volume is $16\Lambda(\pi/4) = 7.32772...$ where Λ is the Lobachevsky function.

The International Mathematical Union (IMU) has adopted the new logo above, see [49], as announced on 22 August 2006 at the opening ceremony of the International Congress of Mathematicians (ICM 2006) in Madrid. It was the winner of an international competition announced by the IMU in 2004.

The logo was designed by John Sullivan, Professor of Mathematical Visualization at the Technical University of Berlin (TU Berlin) and at the DFG Research Center MATHEON, and adjunct professor at the University of Illinois, Urbana (UIUC), with help from Prof. Nancy Wrinkle of Northeastern Illinois University.

The logo design is based on the Borromean rings, a famous topological link of three components. The rings have the surprising property that if any one component is removed, the other two can fall apart (while all three together remain linked). This so-called Brunnian property has led the rings to be used over many centuries in many cultures as a symbol of interconnectedness, or of strength in unity.

Although the Borromean rings are most often drawn as if made from three round circles, such a construction is mathematically impossible.

The IMU logo instead uses the tight shape of the Borromean rings, as would be obtained by tying them in rope pulled as tight as possible. Mathematically, this is the length-minimizing configuration of the link subject to the constraint that unit-diameter tubes around the three components stay disjoint. This problem and its solution are described in the paper [11].

Although this critical configuration is quite close to one made of convex and concave circular arcs, its actual geometry is surprisingly intricate. Each component is planar and piecewise smooth, with the shapes of many of the 14 pieces described by elliptic integrals. The improvement over the similar piecewise circular configuration leads to a savings of length of less than one tenth of one percent! (The paper cited above first noticed a similar surprise in the simple clasp: one rope attached to the floor clasped around another attached to the ceiling. There as well, the minimizing shapes for the ropes are quite complicated, leaving a small gap between the thick tubes right at the tip.)

The tight configuration of the Borromean rings has pyritohedral symmetry (3*2 in the Conway/Thurston orbifold notation), and the IMU logo uses a view along a three-fold axis of rotation symmetry. Instead of the thick tubes, which would touch one another all along their lengths, thinner tubes are drawn, allowing a better view of the link.

Sullivan says the new logo "represents the interconnectedness not only of the various fields of mathematics, but also of the mathematical community around the world. " Together with Charles Gunn of TU Berlin, he has made a 5-minute computer-graphics video The Borromean Rings: a new logo for the IMU for presentation at the ICM opening ceremony, see [50] ICM 2006.

The video, also viewable online, includes other views of the tight Borromean rings, rendered for instance as woven rope or transparent soap film.

9.2. *Connections with physics*

A quantum-mechanical analog of Borromean rings is called a halo state or an Efimov state (the existence of such states was predicted by physicist Vitaly Efimov, in 1970). A team of physicists led by Randy Hulet of Rice University in Houston finally achieved the trio of particles, and published their findings in the online journal Science Express.

"It's an amazing effect, really," Hulet said. "A lot of people didn't believe [Efimov] at first. It was a very strange prediction."

The theory is unique because it's a solution to a special case of what's called the "three-body" problem. Scientists have solved the "two-body" problem - that is, they have calculated exactly how two objects should move based on their starting positions, masses and velocities. Scientists can also calculate this scenario for many masses, but a pure solution to the general three-body problem has been elusive.

"Physicists can handle two-body problems quite well, and many-body problems fairly well, but when there are just a few objects, like the three bodies in these Efimov trimers, there are just too many variables," Hulet said. The Efimov calculation is not the solution to the general case, but rather a solution to a specific case of three bodies. Thus, discovering a real-life example of three particles fulfilling his prediction is an important step to learning more about few-body physics. See also papers by C.Moskowitz and K. Tanaka cited in [50].

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