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Fixed Point Theorems for Occasionally Weakly Compatible Mappings, II

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Abstract. In this paper we point out that a number of fixed point papers, involving several maps, are special cases of a general result proved several years ago by the author and G. F. Jungck, and one proved by Aliouche and Popa [1].

Prior to 1968, the Banach contraction principle was the main tool used to obtain fixed points. Let X be a complete metric space. A map T satisfies the Banach contraction principle if

$$d(Tx, Ty) \leq kd(x, y)$$

for all $x, y \in X$, for some constant k satisfying $0 \leq k < 1$. The beauty of this result is that one can begin with any point $x_0 \in X$, form the sequence $\{x_n\}$, defined by $x_{n+1} = Tx_n$, and obtain convergence to the unique fixed point of T. The main drawback of this principle is that functions T must be Lipschitz continuous, with Lipschitz constant less than one. In 1968 Kannan [4] provided an example a map which is not continuous at every point of its domain, satisfying a contractive condition different from that of Banach, but which has the nice property that, beginning at any point x_0 of the space, the sequence $x_{n+1} = Tx_n$ converges to the unique fixed point of T. Following Kannan's paper there appeared a spate of papers, containing a wide variety of contractive conditions, all enjoying the property that the iteration $x_{n+1} = Tx_n$ converges to the unique fixed point. An examination of these papers shows that they all employ the same proof technique. The first step in every paper is to set $y = Tx$. Then one obtains an inequality of the form

$$d(Tx, T^2x) \leq kd(x, Tx).$$

Recognizing this commonality, Sehie Park [8] proved a general theorem, which included many of the theorems published in the 60's and 70's as special cases. His result is the following.

Theorem 0.1. *Lef f be a selfmap of a metric space (X, d). If there exists a point $u \in X$ and a $\lambda \in [0, 1)$ such that $\overline{O}(u)$, the closure of the orbit of u, is complete and*

$$(*) \quad d(fx, fy) \leq \lambda d(x, y)$$

holds for any $x, y = fx$ in $O(u)$, then $\{f^i u\}$ converges to some $\xi \in X$, and

$$d(f^i u, \xi) \leq \frac{\lambda^i}{1-\lambda} d(u, fu) \quad for \quad i \geq 1.$$

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Further, if f is orbitally continuous at ξ, or if $(*)$ holds for any $x, y \in \overline{O}(u)$, then ξ is a fixed point of f.

Park also proved a general theorem for the cases in which (*) is true for $\lambda = 1$ and strict inequality holds.

More recently, there has developed a large literature involving fixed points for contractive conditions in which some of the maps appear on both sides of the contractive inequality. A simple inequality of this type is

$$d(fx, gy) \leq k \max\{d(Sx, Ty), d(Sx, fx), d(Ty, gy)\},$$

where $0 \leq k < 1$, and f, g, S, T are selfmaps of some space X. In such situations it is necessary to also assume inclusion relations of the form $f(X) \subset T(X)$ and $g(X) \subset S(X)$, and to assume some kind of commutativity condition. Jungck, who proved the first theorem in this direction, assumed that the maps were pairwise commuting. Later authors defined weaker commutative type conditions. One of the more useful generalizations, also due to Jungck, is that of weakly compatible. Two maps are said to be weakly compatible if they commute at coincidence points.

Observing that every theorem of this type also follows the same proof pattern, Jungck and the author established a general theorem in this setting. Two maps are said to be occasionally weakly compatible (owc) if there is a point at which the maps commute. The following result is Theorem 1 of [3].

Theorem 0.2. *Let X be a set with a symmetric r. Suppose that f, g, S, T are selfmaps of X and that the pairs $\{f, S\}$ and $\{g, T\}$ are each owc. If*

$$r(fx, gy) < M(x, y) \tag{1}$$

for each $x, y \in X$ for which $fx \neq gy$, and where

$$M(x, y) := \max\{r(Sx, Ty), r(Sx, fx), r(Ty, gy), \\ r(S, gy), r(Ty, fx)\},$$

then there is a unique point $w \in X$ such that $fw = gw = w$ and a unique point $z \in X$ such that $gz = Tz = z$. Moreover, $z = w$, so that there is a unique common fixed point of f, g, S, and T.

The following result is Theorem 2.1 of [7].

Theorem 0.3. *Let f and g be weakly reciprocally continuous self-mappings of a complete metric space (X, d) satisfying*

(C1) $f(X) \subset g(X)$,

(C2) *there exists a number $h \in (0, 1)$ such that*

$$d(fx, fy) \leq h \max\{d(gx, gy), d(fx, gx), d(fy, gy), [d(fx, gy) + d(fy, gx)]/2\} \tag{2}$$

for any $x, y \in X$.

If f and g are either compatible or R-weakly commuting of type (A_g) or R-weakly commuting of type (A_f) or R-weakly commuting of type (P), then f and g haver a unique common fixed point.

Theorem 2.1 of [6] states the same result as Theorem 0.3, except the contractive inequality does not contain the term $[d(fx, gy) + d(fy, gx)]/2$.

Clearly (2) is a special case of (1) with $g = f, S = T = g$. In the course of the proof of Theorem 2.1 of [6] is is shown that f and g have a coincidence point. Therefore they are owc and the conclusion follows from Theorem 0.2.

The following is Theorem 2.1 of [5].

Theorem 0.4. *Let f, g, h and k be self-maps defined on a metric space (X, d) satsfying the following conditions:*

(i) $f(X) \subseteq h(X), g(X) \subseteq k(X)$,

(ii) *For all $x, y \in X$, there exists a right continuous function $\psi : [0, \infty) \to [0, \infty), \psi(0) = 0$ and $\psi(s) < s$ for $s > 0$ such that*

$$\int_0^{d(fx,gy)} \phi(t)dt \leq \psi\Big(\int_0^{M(x,y)} \phi(t)dt\Big), \tag{3}$$

where $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ *is a Lebesgue integrable mapping which is summable, nonnegative and such that:*

$$\int_0^{\epsilon} \phi(gt)dt > 0 \quad for\, each \quad \epsilon > 0,$$

$$M(x, y) = \max\Big\{d(kx, hy), d(kx, fx), d(gy, hy), \\ \frac{1}{2}[d(kx, gy) + d(hy, fx)]\Big\}$$

(1) *The pairs* (f, k) *or* (g, h) *satisfy the* $(E.A.)$ *property.*
(2) *The pairs* (f, k) *and* (g, h) *are weakly compatible.*
If $fk(X)$ *is closed, then* f, g, h *and* k *have a unique common fixed point.*

From the fact that $\psi(s) < s$,inequality (3) implies taht

$$\int_0^{d(fx,gy)} \phi(t)dt < \int_0^{M(x,y)} \phi(t)dt,$$

which in turn implies that

$$d(fx, gy) < M(x, y),$$

which is a special case of (1) with $S = k, T = h$. In the course of proving Theorem 4 it is shown that (f, h) and (g, k) have coincidence points. Therefore they are owc and the result follows from Theorem 0.2.

Theorem 3.1 of [5] is the same as Theorem 0.4, except that condition (1) has been replaced by the statement that the pairs (f, k) or (g, h) satisfy the (CLR_f) or the (CLR_g)property. Again, in the course of proving the theorem it is shown that the pairs (f, h) and (g, k) have coincidence points.

The symbol Φ denotes the class of functions $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that ϕ is nondecreasing and $\phi(t) < t$ for each $t > 0$. The following result is also a special case of Theorem 0.2.

Theorem 0.5. *Let* A, B, S *and* T *be selfmaps of a complete metric space* (X, d). *Suppose that any one of the maps* A *or* B *is continuous, satisfying*
(1) $A(X) \subset T(X)$ *and* $B(X) \subset S(X)$,
(2)

$$\begin{aligned} d^{2p}(Ax, By) \leq\ & a\phi_0(d^{2p}(Sx, Ty)) + (1 - a)\max\{\phi_1(d^{2p}(Sx, Ty)), \\ & \phi_2(d^q(Sx, Ax)d^{q'}(Ty, By)), \phi_3(d^r(Sx, By)d^{r'}(Ty, Ax)) \\ & \phi_4(d^s(Sx, Ax)d^{s'}(Ty, Ax)), \phi_5(d^{\ell}(Sx, By)d^{\ell'}(Ty, By))\} \end{aligned}$$

for all $x, y \in X$, *where* $\phi_i \in F, i = 1, 2, 3, 4, 5, 0 \leq a \leq 1, 0 < p, q, \ell, \ell', r, r', s, s' \leq 1$, *such that*

$$2p = q + q' = r + r' = s + s' = \ell + \ell'.$$

(3) *If the pairs* $\{A, S\}$*and* $\{B, T\}$ *are weakly compatible, then* A, B, S *and* T *have a unique common fixed point* $y' \in X$. *Further,* y' *is the common fixed point of* A *and* S *and of* B *and* T.

Inequality (3) implies that

$$\begin{aligned} d^{2p}(Ax, By) \leq \max\{&\phi_0(d^{2p}(Sx, Ty)), \phi_1(d^{2p}(Sx, Ty)), \phi_2(d^q(Sx, Ax)d^{q'}(Ty, By)), \\ &\phi_3(d^r(Sx, By)d^{r'}(Ty, Ax)), \phi_4(d^s(Sx, Ax)d^{s'}(Ty, Ax)), \\ &\phi_5(d^{\ell}(Sx, By)d^{\ell'}(Ty, By))\} \end{aligned} \tag{4}$$

Note that

$$d^q(Sx,Ax)d^{q'}(Ty,By)) \leq [\max\{d(Sx,Ax),d(Ty,By)\}]^q[\max\{d(Sx,Ax),d(Ty,By)\}]^{q'}$$
$$= [\max\{d(Sx,Ax),d(Ty,By)\}]^{2p}. \tag{5}$$

Similarly,

$$d^r(Sx,By)d^{r'}(Ty,Ax)) \leq [\max\{d((Sx,By),d(Ty,Ax)\}]^{2p}, \tag{6}$$

$$(d^s(Sx,Ax)d^{s'}(Ty,Ax)) \leq [\max\{d(Sx,Ax),d(Ty,Ax)\}]^{2p}, \tag{7}$$

and

$$d^\ell(Sx,By)d^{\ell'}(Ty,By)) \leq [\max\{d(Sx,By),d(Ty,By)\}]^{2p}. \tag{8}$$

Using the fact that $\phi_i \in \Phi$, and substituting (3) - (6) into (2) gives

$$d^{2p}(Ax,By) < [\max\{d(Sx,Ty),d(Ty,By),d(Sx,Ax),d(Ty,Ax),d(Sx,By)\}]^{2p};$$

i.e.,

$$d(Ax,By) < \max\{d(Sx,Ty),d(Ty,By),d(Sx,Ax),d(Ty,Ax),d(Sx,By)\}, \tag{9}$$

which is inequality (1) with $f = A, g = B$.

In the course of the proof of Theorem 0.5 it is shown that the pairs $\{A,S\}$ and $\{B,T\}$ have conicidence points. Therefore they are owc and the result follows from Theorem 0.2.

Theorem 1 of [10] is the same as Theorem 4, except that the condition of continuity of A or B is dropped. Therefore this result also is a special case of Theorem 0.2.

A more general result than Theorem 0.2 is the following, which is Theorem 3.1 of [1], where F_6 is the set of functions from $\mathbb{R}^6_+ \to \mathbb{R}$ with $t_3 + t_4 \neq 0$.

The next result is the following, from [11]

Theorem 0.6. *Let $f, g, S,$ and T be selfmaps of a symmetric space (X,d) satisfying the following condition:*

$$F(d(Sx,Ty),d(fx,gy),d(fx,Sx),d(gy,Ty),d(fx,Ty),d(Sx,gy)) \leq 0 \tag{10}$$

for all $x,y \in X$ if $d(fx,Sx) + d(gy,Ty) \neq 0$, where $F \in F_6$, or

$$d(Sx,Ty) = 0 \quad if \quad d(fx,Sx) + d(gy,Ty) = 0.$$

Suppose that the pairs (S,f) and (T,g) are owc. Then f, g, S and T have a unique common fixed point in X.

We now indicate some results which are special cases of Theorem 0.6. The first is Theorem 1 of [2], where F denotes the set of functions $f : \mathbb{R}^5_+ \to \mathbb{R}_+$ which are upper-semicontinuous and non-decreasing in each coordinate variable.

Theorem 0.7. *Let X be uniformly convex and K a nonempty closed subset of X. Let A, S,and T be three selfmaps of K satisfying the following conditions:*

(1) *S and T are continuous $AK \subset SK \cap TK$,*

(2) *$\{A,S\}$ and $\{A,T\}$ are weakly commuting pairs on K,*

(3) *there exists a function $f \in F$ such that, for every $x, y \in K$:*

$$\|Ax,Ay\| \leq f(\|Sx-Ty\|,\|Sx-Ax\|,\|Sx-Ay\|,\|Ty-Ax\|,\|Ty-Ay\|)$$

where f has additional requirements:

(a) for $t > 0$, $f(t,t,0,\alpha t,t) \leq \beta t$ and $f(t,t,\alpha t,0,t) \leq \beta t$, $\beta < 1$ for $\alpha < 2$ and $\beta = 1$ for $\alpha = 2, \alpha,\beta \in \mathbb{R}^+$,

(b) $f(t,0,t,t,0) < t$ for $t > 0$.

Then there exists a point $u \in K$ such that
(c) *u is the unique common fixed point of A, S and T.*
(d) *For any $x_0 \in K$, the sequence $\{Ax_n\}$ defined by*

$$Tx_{2n} = Ax_{2n-1}, Sx_{2n+1} = Ax_{2n}, \quad for \quad n = 0, 1, 2, \cdots (3),$$

converges strongly to u.

Inequality (3) of Theorem 6 is a special case of (8) with $S = T = A, f = S$ and $g = T$. In the course of proving Theorem 0.7 it is shown that there is a point u which is a common coincidence point of A, S and T. Since weakly commuting is a special case of weakly compatible, the maps are now owc. The result then follows from Theorem 0.5.

The second result is from [9].

Theorem 0.8. *Let X and K be as in Theorem 4. Let A, B, S and T be mappings on K satisfying:*
(1) *one of A, B, S and T is continuous and $AK \subset TK, BK \subset SK$,*
(2) *$\{A, S\}$ and $\{B, T\}$ are compatible of type (A), ′*
(3) *there exists a function $f \in F$ such that, for every $x, y \in K$*

$$\|Ax - By\| \leq f(\|Sx - Ty\|, \|Sx - Ax\|, \|Sx - By\|.\|Ty - Ax\|, \|Ty - By\|),$$

where f satisfies the conditions (a) and (b) of Theorem 4. Then there exists a point $u \in K$ such that
(a) *u is the unique common fixed point of A, B, S and T,*
(b) *for any $x_0 \in K$, the sequence $\{y_n\}$ defined by*

$$y_{2n} = Sx_{2n} = Bx_{2n-1},$$

$$y_{2n+1} = Tx_{2n+1} = Ax_{2n}, \quad n = 1, 2, 3, \ldots$$

converges strongly to u.

Using the details that showed that Theorem 0.6 is a special case of Theorem 0.5, one only needs to note that compatibility of type (A) is a special case of owc.

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