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Some Applications of Srivastava's Theorem Involving a Certain Family of Generalized and Extended Hypergeometric Polynomials

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Abstract. Recently, Srivastava *et al.* [H. M. Srivastava, M. A. Chaudhry and R. P. Agarwal, The incomplete Pochhammer symbols and their applications to hypergeometric and related functions, *Integral Transforms Spec. Funct.* **23** (2012), 659–683] introduced and initiated the study of many interesting fundamental properties and characteristics of a certain pair of potentially useful families of the so-called generalized incomplete hypergeometric functions. Ever since then there have appeared many closely-related works dealing essentially with notable developments involving various classes of generalized hypergeometric functions and generalized hypergeometric polynomials, which are defined by means of the corresponding incomplete and other novel extensions of the familiar Pochhammer symbol. Here, in this sequel to some of these earlier works, we derive several general families of hypergeometric generating functions by applying Srivastava's Theorem. We also indicate various (known or new) special cases and consequences of the results presented in this paper.

1. Introduction, Definitions and Preliminaries

Generating functions and basic (or q-) generating functions, especially for sequences involving (for example) the generalized hypergeometric function ${}_rF_s$ and the generalized basic (or q-) hypergeometric function ${}_r\Phi_s$ with r numerator and s denominator parameters, play an important rôle in the investigation of various useful properties of the sequences which they generate. They are used in finding many properties, characteristics and formulas for the generated numbers and polynomials in a wide variety of research subjects in (for example) modern combinatorics. For a systematic introduction to, and for several interesting (and useful) applications of the various methods of obtaining linear, bilinear and bilateral (or mixed multilateral) generating functions for a fairly wide variety of sequences of special functions (and polynomials) in one, two and more variables, among much abundant available literature, we refer the interested reader to the extensive work presented in the monograph on this subject by Srivastava and Manocha [12].

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About three decades ago, numerous general families of generating functions as well as their *basic* (or q-) extensions for various polynomial systems in one, two and more variables were derived by Srivastava (see, for details, [9]; see also [12, p. 142 *et seq.*]). We choose to recall here one of Srivastava's results as Theorem 1 below (see [9, p. 331, Equation (2.2)] and [12, p. 144, Equation 2.6 (28)]).

Theorem 1. *Let* $\{\Theta_n\}_{n\in\mathbb{N}_0}$ *and* $\{\Phi_{n,k}\}_{n,k\in\mathbb{N}_0}$ *denote, respectively, suitably bounded single and double sequences of essentially arbitrary complex parameters. Then*

$$\sum_{n=0}^{\infty} \frac{\prod_{j=1}^{l}(c_j)_n}{\prod_{j=1}^{m}(d_j)_n} \frac{\Theta_n\, t^n}{n!} \sum_{k=0}^{[n/N]} \frac{(-n)_{Nk} \prod_{j=1}^{m}(1-d_j-n)_k}{\prod_{j=1}^{l}(1-c_j-n)_k} \frac{\Phi_{n,k}\, z^k}{k!}$$

$$= \sum_{n,k=0}^{\infty} \frac{\prod_{j=1}^{l}(c_j)_n}{\prod_{j=1}^{m}(d_j)_n} \Theta_{n+Nk}\, \Phi_{n+Nk,k} \frac{t^n}{n!} \frac{\left(z\left\{(-1)^{l-m+1}\, t\right\}^N\right)^k}{k!} \tag{1}$$

$(l, m \in \mathbb{N}_0;\ N \in \mathbb{N})$,

provided that each member of the assertion (1) *exists*, $[\kappa]$ *being the greatest integer in* $\kappa \in \mathbb{R}$.

Here, and in what follows, we denote by $\mathbb{R}$ and $\mathbb{C}$ the sets of real and complex numbers, respectively,

$$\mathbb{N}_0 := \mathbb{N} \cup \{0\} \quad (\mathbb{N} := \{1, 2, 3, \cdots\}) \qquad \text{and} \qquad \mathbb{Z}_0^- := \mathbb{Z}^- \cup \{0\} \quad (\mathbb{Z}^- := \{-1, -2, -3, \cdots\}),$$

a generalized binomial coefficient $\binom{\lambda}{\mu}$ may be defined (for real or complex parameters λ and μ) by

$$\binom{\lambda}{\mu} := \frac{\Gamma(\lambda+1)}{\Gamma(\mu+1)\Gamma(\lambda-\mu+1)} =: \binom{\lambda}{\lambda-\mu} \qquad (\lambda, \mu \in \mathbb{C}), \tag{2}$$

so that, in the special case when $\mu = n \quad (n \in \mathbb{N}_0)$, we have

$$\binom{\lambda}{n} = \frac{\lambda(\lambda-1)\cdots(\lambda-n+1)}{n!} = \frac{(-1)^n\,(-\lambda)_n}{n!} \qquad (n \in \mathbb{N}_0), \tag{3}$$

where $(\lambda)_\nu \quad (\lambda, \nu \in \mathbb{C})$ denotes the Pochhammer symbol given, in general, by

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0;\ \lambda \in \mathbb{C} \setminus \{0\}) \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (\nu = n \in \mathbb{N};\ \lambda \in \mathbb{C}), \end{cases} \tag{4}$$

it being understood *conventionally* that $(0)_0 := 1$ and assumed *tacitly* that the Γ-quotient exists (see, for details, [12, p. 21 *et seq.*]).

In the widely-scattered literature on the subject of this paper (see, for example, [2, 3, 7, 11, 14–17]) one can find several interesting generalizations of the familiar (Euler's) gamma function $\Gamma(z)$, as well as the corresponding generalizations and extensions of the Beta function $B(\alpha, \beta)$ ([5, Chapter 1] and [18, Chapter 12]), the hypergeometric functions ${}_1F_1$ and ${}_2F_1$, and the generalized hypergeometric functions ${}_rF_s$ of r numerator and s denominator parameters. For example, for an appropriately bounded sequence $\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}$

of essentially arbitrary (real or complex) numbers, Srivastava *et al.* [13, p. 243 *et seq.*] recently considered the function $\Theta(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}; z)$ given by

$$\Theta(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}; z) = \begin{cases} \sum_{\ell=0}^{\infty} \kappa_\ell \dfrac{z^\ell}{\ell!} & (|z| < R;\ R > 0;\ \kappa_0 := 1) \\[2ex] \mathfrak{M}_0\, z^\omega\, \exp(z) \left[1 + O\left(\dfrac{1}{|z|}\right)\right] & (|z| \to \infty;\ \mathfrak{M}_0 > 0;\ \omega \in \mathbb{C}) \end{cases} \tag{5}$$

for some suitable constants $\mathfrak{M}_0$ and ω depending essentially upon the sequence $\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}$. Then, in terms of the function $\Theta(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}; z)$ defined by (5), Srivastava *et al.* [13] introduced the following remarkably deep generalization of the extended gamma function $\Gamma_{\mathfrak{p}}(z)$ (see also [3] *and* the references cited therein) by

$$\Gamma_{\mathfrak{p}}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(z) := \int_0^\infty t^{z-1}\, \Theta\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}; -t - \frac{\mathfrak{p}}{t}\right) \mathrm{d}t \tag{6}$$

$$\Big(\Re(z) > 0;\ \Re(\mathfrak{p}) \geqq 0\Big),$$

provided that the defining integral in (6) exists.

The main purpose of this paper is to apply Srivastava's Theorem (that is, Theorem 1 above) with a view to deriving various interesting classes of generating functions for a family of the generalized and extended hypergeometric polynomials associated with the function ${}_p\mathcal{F}_q$ defined by [15, p. 2203, Equation (64)] (see also [7])

$${}_p\mathcal{F}_q\left[\begin{array}{c} (a_0; \mathfrak{p}, \{\kappa_\ell\}_{\ell\in\mathbb{N}_0}), a_2, \cdots, a_p; \\ \\ b_1, \cdots, b_q; \end{array} z\right] := \sum_{n=0}^{\infty} \frac{(a_0; \mathfrak{p}, \{\kappa_\ell\}_{\ell\in\mathbb{N}_0})_n\ (a_2)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!} \tag{7}$$

in terms of the following generalization of the Pochhammer symbol $(\lambda)_\nu$ $(\lambda, \nu \in \mathbb{C})$ defined by (4) [15, p. 2203, Equation (63)]:

$$(\lambda; \mathfrak{p}, \{\kappa_\ell\}_{\ell\in\mathbb{N}_0})_\nu := \frac{\Gamma_{\mathfrak{p}}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(\lambda+\nu)}{\Gamma_{\mathfrak{p}}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(\lambda)} \qquad (\lambda, \nu \in \mathbb{C}), \tag{8}$$

provided that the series on the right-hand side of (7) converges. Indeed, whenever one or the other of the numerator parameters $a_2, \cdots, a_p$ in (7) is a *nonpositive* integer, the definition (7) will define the corresponding family of generalized hypergeometric polynomials. Various (known or new) special cases and consequences of the generating functions presented in this investigation will also considered.

We remark in passing that the special case of the generalized and extended hypergeometric function ${}_p\mathcal{F}_q$ defined by (7) when

$$\kappa_\ell \equiv 1 \qquad (\ell \in \mathbb{N}_0)$$

was considered recently in another paper by Srivastava *et al.* [10].

2. Generating Functions for the Associated Class of Generalized and Extended Hypergeometric Polynomials

First of all, it should be pointed out that in results such as Srivastava's generating function (1) asserted by Theorem 1, an *empty* product is interpreted (as usual) to be 1. Thus, for example, it is always understood that

$$\prod_{j=1}^{l}(c_j)_n = 1 \quad \text{when} \quad l = 0 \qquad \text{and} \qquad \prod_{j=1}^{m}(d_j)_n = 1 \quad \text{when} \quad m = 0.$$

Now, with a view to applying Theorem 1 to a certain family of the generalized and extended hypergeometric polynomials which are associated naturally with the generalized and extended hypergeometric functions ${}_p\mathcal{F}_q$ defined by (7), we set

$$\Phi_{n,k} = \frac{\prod_{j=1}^{r}(g_j+n)_{Lk}}{\prod_{j=1}^{s}(h_j+n)_{Mk}} \; \frac{(a_1;\mathfrak{p},\{\kappa_\ell\}_{\ell\in\mathbb{N}_0})_k \;(a_2)_k\cdots(a_p)_k}{(b_1)_k\cdots(b_q)_k} \qquad (p,q,r,s\in\mathbb{N}_0;\; L,M\in\mathbb{N}). \tag{9}$$

For convenience, we denote the array of N parameters

$$\frac{\lambda}{N}, \frac{\lambda+1}{N}, \cdots, \frac{\lambda+N-1}{N} \qquad (\lambda\in\mathbb{C};\; N\in\mathbb{N})$$

by $\Delta(N;\lambda)$ and the array of Nr parameters

$$\frac{\lambda_j}{N}, \frac{\lambda_j+1}{N}, \cdots, \frac{\lambda_j+N-1}{N} \qquad (\lambda_j\in\mathbb{C};\; j=1,\cdots,r;\; N\in\mathbb{N})$$

by $\Delta(N,r;\lambda)$, the array being empty when $N=0$ (and indeed also when $r=0$), so that

$$\Delta(N,1;\lambda_j) = \Delta(N;\lambda_1).$$

We are thus led eventually to the following family of generating functions for the associated class of generalized and extended hypergeometric polynomials.

Theorem 2. *Let $\{\Theta_n\}_{n\in\mathbb{N}_0}$ denote a suitably bounded sequence of essentially arbitrary complex parameters. Then the following generating functions hold true for the associated class of generalized and extended hypergeometric polynomials*:

$$\sum_{n=0}^{\infty} \frac{\prod_{j=1}^{l}(c_j)_n}{\prod_{j=1}^{m}(d_j)_n} \frac{\Theta_n\, t^n}{n!}\ {}_{p+Lr+(m+1)N}\mathcal{F}_{q+Ms+Nl}$$

$$\left[\begin{array}{r} \Delta(N;-n), \Delta(L,r;g_j+n), \Delta(N,m;1-d_j-n), \left(a_1;\mathfrak{p},\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right), a_2,\cdots,a_p; \\ \Delta(M,s;h_j+n), \Delta(N,l;1-c_j-n), b_1,\cdots,b_q; \end{array} \left(\frac{L^{Lr}}{M^{Ms}\, N^{(l-m-1)N}}\right) z\right]$$

$$= \sum_{n,k=0}^{\infty} \frac{\prod_{j=1}^{l}(c_j)_n}{\prod_{j=1}^{m}(d_j)_n}\, \Theta_{n+Nk}\, \frac{\prod_{j=1}^{r}(g_j+n+Nk)_{Lk}}{\prod_{j=1}^{s}(h_j+n+Nk)_{Mk}}\, \frac{\left(a_1;\mathfrak{p},\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)_k\ (a_2)_k\cdots(a_p)_k}{(b_1)_k\cdots(b_q)_k}$$

$$\cdot \frac{t^n}{n!}\, \frac{\left(z\left\{(-1)^{l-m+1}\, t\right\}^N\right)^k}{k!} \tag{10}$$

$$\Big(l,m,p,q,r,s\in\mathbb{N}_0;\ L,M,N\in\mathbb{N}\Big),$$

provided that both sides of the assertion (10) *exist.*

Several interesting corollaries and consequences of the generating function (10) asserted by Theorem 2 are worthy of mention here. First of all, if we set

$$\Theta_n = (\lambda)_n \quad (n\in\mathbb{N}_0), \quad l=m=r-1=s=0 \quad (g_1=\lambda) \qquad \text{and} \qquad z\mapsto \frac{z}{N^N\, L^L},$$

we find from (10) that

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!}\ {}_{p+N+L}\mathcal{F}_q\left[\begin{array}{r} \Delta(N;-n), \Delta(L;\lambda+n), \left(a_1;\mathfrak{p},\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right), a_2,\cdots,a_p; \\ b_1,\cdots,b_q; \end{array} z\right] t^n$$

$$= (1-t)^{-\lambda}\ {}_{p+N+L}\mathcal{F}_q\left[\begin{array}{r} \Delta(N+L;\lambda), \left(a_1;\mathfrak{p},\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right), a_2,\cdots,a_p; \\ b_1,\cdots,b_q; \end{array} \left(\frac{(N+L)^{N+L}}{N^N\, L^L}\right)\frac{z(-t)^N}{(1-t)^{N+L}}\right] \tag{11}$$

$$\Big(|t|<1;\ \lambda\in\mathbb{C};\ N,L\in\mathbb{N}\Big).$$

Secondly, in its special case when

$$\Theta_n \equiv 1 \quad (n\in\mathbb{N}_0) \qquad \text{and} \qquad l-1=m=r=s=0 \quad (c_1=\lambda),$$

the generating function (10) immediately yields

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!}\ {}_{p+N}\mathcal{F}_{q+N}\left[\begin{array}{r} \Delta(N;-n), \left(a_1;\mathfrak{p},\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right), a_2,\cdots,a_p; \\ \Delta(N;1-\lambda-n), b_1,\cdots,b_q; \end{array} z\right] t^n$$

$$= (1-t)^{-\lambda}\ {}_{p}\mathcal{F}_q\left[\begin{array}{r} \left(a_1;\mathfrak{p},\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right), a_2,\cdots,a_p; \\ b_1,\cdots,b_q; \end{array} zt^N\right] \tag{12}$$

$$\Big(|t| < 1;\ \lambda \in \mathbb{C};\ N \in \mathbb{N}\Big).$$

Thirdly, we set

$$\Theta_n = (\lambda)_n \quad (n \in \mathbb{N}_0), \quad l = m = r = s = 0 \qquad \text{and} \qquad z \mapsto \frac{z}{N^N}$$

in the generating function (10). We thus obtain

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!}\ {}_{p+N}\mathcal{F}_q \left[\begin{array}{r} \Delta(N;-n), \left(a_1; \mathfrak{p}, \{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right), a_2, \cdots, a_p; \\ b_1, \cdots, b_q; \end{array} z \right] t^n$$

$$= (1-t)^{-\lambda}\ {}_{p+N}\mathcal{F}_q \left[\begin{array}{r} \Delta(N;\lambda), \left(a_1; \mathfrak{p}, \{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right), a_2, \cdots, a_p; \\ b_1, \cdots, b_q; \end{array} z\left(-\frac{t}{1-t}\right)^N \right] \tag{13}$$

$$\Big(|t| < 1;\ \lambda \in \mathbb{C};\ N \in \mathbb{N}\Big).$$

In its *further* special case when $L = N$, the generating function (11) can easily be simplified to the following form:

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!}\ {}_{p+2N}\mathcal{F}_q \left[\begin{array}{r} \Delta(N;-n), \Delta(N;\lambda+n), \left(a_1; \mathfrak{p}, \{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right), a_2, \cdots, a_p; \\ b_1, \cdots, b_q; \end{array} z \right] t^n$$

$$= (1-t)^{-\lambda}\ {}_{p+2N}\mathcal{F}_q \left[\begin{array}{r} \Delta(2N;\lambda), \left(a_1; \mathfrak{p}, \{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right), a_2, \cdots, a_p; \\ b_1, \cdots, b_q; \end{array} \frac{z(-4t)^N}{(1-t)^{2N}} \right] \tag{14}$$

$$\Big(|t| < 1;\ \lambda \in \mathbb{C};\ N \in \mathbb{N}\Big).$$

Lastly, By virtue of the following limit formula:

$$\lim_{|\lambda|\to\infty} \left\{ (\lambda)_n \left(\frac{z}{\lambda}\right)^n \right\} = z^n = \lim_{|\mu|\to\infty} \left\{ \frac{(\mu z)^n}{(\mu)_n} \right\} \qquad (\lambda, \mu \in \mathbb{C};\ n \in \mathbb{N}_0), \tag{15}$$

in the limit case when t is replaced by t/λ and $|\lambda| \to \infty$, we find from the generating function (13) that

$$\sum_{n=0}^{\infty} {}_{p+N}\mathcal{F}_q \left[\begin{array}{r} \Delta(N;-n), \left(a_1; \mathfrak{p}, \{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right), a_2, \cdots, a_p; \\ b_1, \cdots, b_q; \end{array} z \right] \frac{t^n}{n!}$$

$$= e^t\ {}_{p}\mathcal{F}_q \left[\begin{array}{r} \left(a_1; \mathfrak{p}, \{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right), a_2, \cdots, a_p; \\ b_1, \cdots, b_q; \end{array} z\,(-t)^N \right] \tag{16}$$

$$\Big(|t| < 1;\ N \in \mathbb{N}\Big),$$

which, for $N = 1$, may be compared with the known results [14, p. 126, Equations (30) and (31)] and [15, pp. 2200–2201, Equations (49) and (50)].

Such hypergeometric generating functions as (for example) (11) to (14) and (16) above were derived, in a markedly different manner by using combinatorial identities, by Srivastava and Cho [17]. As a matter of fact, the following interesting generalization of the hypergeometric generationg function (12) of the type analogous to (11) can also be found in the work of Srivastava and Cho [17]:

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!}\ {}_{p+N}\mathcal{F}_{q+N+L}\left[\begin{array}{c} \Delta(N;-n),\left(a_1;\mathfrak{p},\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right),a_2,\cdots,a_p; \\ \Delta(N+L;1-\lambda-n),b_1,\cdots,b_q; \end{array} z\right] t^n$$
$$= (1-t)^{-\lambda}\ {}_{p}\mathcal{F}_{q+L}\left[\begin{array}{c} \left(a_1;\mathfrak{p},\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right),a_2,\cdots,a_p; \\ \Delta(L;1-\lambda),b_1,\cdots,b_q; \end{array} \frac{(N+L)^{N+L}}{N^N\,L^L}\, zt^N\,(1-t)^L\right] \tag{17}$$
$$\Big(|t|<1;\ \lambda\in\mathbb{C};\ N,L\in\mathbb{N}\Big),$$

which, in the *exceptional* case when $L = 0$, yields the hypergeometric generating function (12).

3. Concluding Remarks and Observations

In our present investigation, we have successfully applied Srivastava's Theorem (that is, Theorem 1 above) with a view to deriving several general families of generating functions for the generalized and extended hypergeometric polynomials associated with the function ${}_p\mathcal{F}_q$ defined by (7). We have also indicated various (known or new) special cases and consequences of our main result (10) asserted by Theorem 2. Here, in this concluding section, we choose to record the following *further* special cases of the hypergeometric generating functions (12), (13), (14) and (16) when $N = 1$:

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!}\ {}_{p+1}\mathcal{F}_{q+1}\left[\begin{array}{c} -n,\left(a_1;\mathfrak{p},\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right),a_2,\cdots,a_p; \\ 1-\lambda-n,b_1,\cdots,b_q; \end{array} z\right] t^n$$
$$= (1-t)^{-\lambda}\ {}_{p}\mathcal{F}_{q}\left[\begin{array}{c} \left(a_1;\mathfrak{p},\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right),a_2,\cdots,a_p; \\ b_1,\cdots,b_q; \end{array} zt\right] \tag{18}$$
$$\Big(|t|<1;\ \lambda\in\mathbb{C}\Big),$$

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!}\ {}_{p+1}\mathcal{F}_{q}\left[\begin{array}{c} -n,\left(a_1;\mathfrak{p},\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right),a_2,\cdots,a_p; \\ b_1,\cdots,b_q; \end{array} z\right] t^n$$
$$= (1-t)^{-\lambda}\ {}_{p+1}\mathcal{F}_{q}\left[\begin{array}{c} \lambda,\left(a_1;\mathfrak{p},\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right),a_2,\cdots,a_p; \\ b_1,\cdots,b_q; \end{array} -\frac{zt}{1-t}\right] \tag{19}$$
$$\Big(|t|<1;\ \lambda\in\mathbb{C}\Big),$$

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \, {}_{p+2}\mathcal{F}_q \left[\begin{array}{r} -n, \lambda+n, (a_1; \mathfrak{p}, \{\kappa_\ell\}_{\ell\in\mathbb{N}_0}), a_2, \cdots, a_p; \\ \\ b_1, \cdots, b_q; \end{array} z \right] t^n$$

$$= (1-t)^{-\lambda} \, {}_{p+2}\mathcal{F}_q \left[\begin{array}{r} \frac{1}{2}\lambda, \frac{1}{2}\lambda+\frac{1}{2}, (a_1; \mathfrak{p}, \{\kappa_\ell\}_{\ell\in\mathbb{N}_0}), a_2, \cdots, a_p; \\ \\ b_1, \cdots, b_q; \end{array} -\frac{4zt}{(1-t)^2} \right] \tag{20}$$

$$\Big(|t| < 1; \ \lambda \in \mathbb{C}\Big)$$

and

$$\sum_{n=0}^{\infty} {}_{p+1}\mathcal{F}_q \left[\begin{array}{r} -n, (a_1; \mathfrak{p}, \{\kappa_\ell\}_{\ell\in\mathbb{N}_0}), a_2, \cdots, a_p; \\ \\ b_1, \cdots, b_q; \end{array} z \right] \frac{t^n}{n!}$$

$$= e^t \, {}_{p}\mathcal{F}_q \left[\begin{array}{r} (a_1; \mathfrak{p}, \{\kappa_\ell\}_{\ell\in\mathbb{N}_0}), a_2, \cdots, a_p; \\ \\ b_1, \cdots, b_q; \end{array} -zt \right] \tag{21}$$

$$\Big(|t| < 1\Big),$$

respectively.

We conclude our investigation by remarking further that the cases of the hypergeometric generating functions (12), (13), (14) and (16), as well as their specialized or limit cases (18) to (21), when

$$\kappa_\ell \equiv 1 \qquad (\ell \in \mathbb{N}_0)$$

were given by Srivastava *et al.* [10, p. 490, Theorem 6 and Corollary 6]. In fact, as already pointed out by Srivastava and Manocha [12, Chapter 2, Section 2.6], all these hypergeometric generating functions stem essentially from the works of Brafman [1], Chaundy [4] and Rainville (see, for example, [8] and [6, p. 267, Equation 19.10(25)]; see also [12, p. 141]).

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