Positive Solution to a Generalized Lyapunov Equation via a Coupled Fixed Point Theorem in a Metric Space Endowed With a Partial Order

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Abstract. We consider the generalized continuous-time Lyapunov equation:

$$A^*XB + B^*XA = -Q,$$

where Q is an $N \times N$ Hermitian positive definite matrix and A, B are arbitrary $N \times N$ matrices. Under certain conditions, using a coupled fixed point theorem du to Bhaskar and Lakshmikantham combined with the Schauder fixed point theorem, we establish an existence and uniqueness result of Hermitian positive definite solution to such equation. Moreover, we provide an iteration method to find convergent sequences which converge to the solution if one exists. Numerical experiments are presented to illustrate our theoretical results.

1. Introduction

Consider the generalized continuous-time algebraic Lyapunov equation (GCALE)

$$A^*XB + B^*XA = -Q \tag{1}$$

with given matrices Q, A, B and unknown matrix X. Such equations play an important role in stability theory [6, 18], optimal control problems [14, 17] and balanced model reduction [16]. In [19], it is proved that (1) has a unique Hermitian, positive definite solution X for every Hermitian positive definite matrix Q if and only if all eigenvalues of the pencil $\lambda A - B$ are finite and lie in the open left half-plane. When $A = I$ (the identity matrix), (1) becomes the standard Lyapunov equation

$$XB + B^*X = -Q. \tag{2}$$

The classical numerical methods to solve (2) are the Bartels-Stewart method [1], the Hammarling method [10] and the Hessenberg-Schur method [9]. An extension of these methods to solve (1) with the assumption A is nonsingular, is given in [5, 7–9, 19]. Other approaches to solve (1) are the sign function method [4, 13, 15], the ADI method [3, 11, 20].

In this paper, we consider the equation (1), where Q is an $N \times N$ Hermitian positive definite matrix and A, B are arbitrary $N \times N$ matrices. Using Bhaskar-Lakshmikantham coupled fixed point theorem [2], we provide a sufficient condition that assures the existence and uniqueness of a Hermitian positive definite solution to (1). Moreover, we present an algorithm to solve this equation. Numerical experiments are given to illustrate our theoretical result.

2. Notations and Preliminaries

We shall use the following notations: $M(N)$ denotes the set of all $N \times N$ matrices, $H(N) \subset M(N)$ is the set of all $N \times N$ Hermitian matrices and $P(N) \subset H(N)$ (resp. $\overline{P(N)}$) is the set of all $N \times N$ positive definite matrices (resp. positive semidefinite matrices). Instead of $X \in P(N)$, we will also write $X > 0$. Furthermore, $X \geq 0$ means that X is positive semidefinite. As a different notation for $X - Y \geq 0$ ($X - Y > 0$), we will use $X \geq Y$ ($X > Y$). If $X, Y \in H(N)$ such that $X \leq Y$, then $[X, Y]$ will be the set of all $Z \in H(N)$ satisfying $X \leq Z \leq Y$. If $X, Y \in H(N)$ such that $X < Y$, then $]X, Y[$ will be the set of all $Z \in H(N)$ satisfying $X < Z < Y$. We denote by $\|\cdot\|$ the spectral norm, i.e., $\|A\| = \sqrt{\lambda^+(A^*A)}$, where $\lambda^+(A^*A)$ is the largest eigenvalue of A^*A. The $N \times N$ identity matrix will be written as I. It turns out that it is convenient here to use the metric induced by the trace norm $\|\cdot\|_1$. Recall that this norm is given by $\|A\|_1 = \sum_{j=1}^{n} s_j(A)$, where $s_j(A)$, $j = 1, \ldots, n$ are the singular values of A. In fact, we shall use a slight modification of this norm. For $Q \in P(N)$, we define $\|A\|_{1,Q} = \|Q^{1/2}AQ^{1/2}\|_1$. For any $U \in M(N)$, we denote by $\mathrm{Sp}(U)$ the spectrum of U, i.e., the set of its eigenvalues.

The following lemmas will be useful later.

Lemma 2.1 (See [21]). *Let $A \geq 0$ and $B \geq 0$ be $N \times N$ matrices. Then*

$$0 \leq \mathrm{tr}(AB) \leq \|A\| \ \mathrm{tr}(B).$$

Here, for $T \in M(N)$, $\mathrm{tr}(T)$ denotes the trace of the matrix T.

Lemma 2.2 (See [12]). *Let $A \in H(n)$ such that $-I < A < I$. Then $\|A\| < 1$.*

Definition 2.3. *Let $(\Delta, \leq)$ be a partially ordered set. We say that a mapping $F : \Delta \times \Delta \to \Delta$ has the mixed monotone property if*

$$(X, Y), (J, K) \in \Delta \times \Delta, \ X \leq J, \ Y \geq K \Longrightarrow F(X, Y) \leq F(J, K).$$

The proof of our main result is based on the following coupled fixed point theorem posed in a metric space endowed with a partial order.

Lemma 2.4 (Bhaskar and Lakshmikantham [2]). *Let $(\Delta, \leq)$ be a partially ordered set and d be a metric on X such that (Δ, d) is a complete metric space. Let $F : \Delta \times \Delta \to \Delta$ be a continuous mapping. Suppose that*

(i) *F has the mixed monotone property;*

(ii) *there exists some $\delta \in [0, 1)$ such that*

$$d(F(x, y), F(u, v)) \leq \frac{\delta}{2}[d(x, u) + d(y, v)], \ x \geq u, \ y \leq v;$$

(iii) *there exists $(x_0, y_0) \in \Delta \times \Delta$ such that $x_0 \leq F(x_0, y_0)$, $y_0 \geq F(y_0, x_0)$;*

(iv) *for every $(x, y) \in \Delta \times \Delta$, there exists $z \in \Delta$ such that $x \leq z, y \leq z$.*

Then there exists a unique $x^* \in \Delta$ such that $F(x^*, x^*) = x^*$. Moreover,

(v) *the sequences $\{x_n\}$ and $\{y_n\}$ defined by*

$$x_{n+1} = F(x_n, y_n), \ y_{n+1} = F(y_n, x_n)$$

converge to x^*.

3. Main Results

Let

$$U = \frac{A - B + I}{\sqrt{2}}, \quad V = \frac{A + B + I}{\sqrt{2}}, \quad W = B - I.$$

Our first result is the following.

Theorem 3.1. *Suppose that there exists $(\widetilde{Q}, M) \in P(N) \times P(N)$ such that*

(a) $2(U\widetilde{Q}U^* + B\widetilde{Q}B^*) < \widetilde{Q}$;
(b) $2(V\widetilde{Q}V^* + W\widetilde{Q}W^*) < \widetilde{Q}$;
(c) $U^*MU + B^*MB < M - (V^*MV + W^*MW)$;
(d) $Q \in]U^*MU + B^*MB, M - (V^*MV + W^*MW)[$.

Then

(i) (1) *has one and only one solution $\widehat{X} \in P(N)$;*
(ii) $\widehat{X} \in [Q - (U^*MU + B^*MB), Q + (V^*MV + W^*MW)]$.

Proof. At first, observe that (1) is equivalent to

$$X = Q + (V^*XV + W^*XW) - (U^*XU + B^*XB). \tag{3}$$

Consider the continuous mapping $F : H(N) \times H(N) \to H(N)$ defined by

$$F(X, Y) = Q + (V^*XV + W^*XW) - (U^*YU + B^*YB), \quad \text{for all } X, Y \in H(N). \tag{4}$$

Then (3) is equivalent to

$$X = F(X, X). \tag{5}$$

Observe that F has the mixed monotone property. Indeed, for any $X, Y, J, K \in H(N)$ such that $X \leq J$ and $Y \geq K$, we have

$$\begin{aligned}
F(X, Y) &= Q + (V^*XV + W^*XW) - (U^*YU + B^*YB) \\
&\leq Q + (V^*JV + W^*JW) - (U^*KU + B^*KB) \\
&= F(J, K).
\end{aligned}$$

Now, let $X, Y, J, K \in H(N)$ such that $X \geq J$ and $Y \leq K$. We have

$$\|F(X, Y) - F(J, K)\|_{1,\widetilde{Q}} = \operatorname{tr}\left(\widetilde{Q}^{1/2}(F(X, Y) - F(J, K))\widetilde{Q}^{1/2}\right)$$

$$= \operatorname{tr}\left(\widetilde{Q}^{1/2}(V^*(X - J)V + W^*(X - J)W + U^*(K - Y)U + B^*(K - Y)B)\widetilde{Q}^{1/2}\right)$$

$$= \operatorname{tr}\left(\widetilde{Q}^{1/2}(V^*(X - J)V + W^*(X - J)W)\widetilde{Q}^{1/2}\right)$$

$$+ \operatorname{tr}\left(\widetilde{Q}^{1/2}(U^*(K - Y)U + B^*(K - Y)B)\widetilde{Q}^{1/2}\right).$$

On the other hand, using Lemma 2.1, we obtain

$$\operatorname{tr}\left(\widetilde{Q}^{1/2}(V^*(X - J)V + W^*(X - J)W)\widetilde{Q}^{1/2}\right) = \operatorname{tr}\left(V\widetilde{Q}V^*(X - J) + W\widetilde{Q}W^*(X - J)\right)$$

$$= \operatorname{tr}\left(V\widetilde{Q}V^*(X - J)\widetilde{Q}^{1/2}\widetilde{Q}^{-1/2} + W\widetilde{Q}W^*(X - J)\widetilde{Q}^{1/2}\widetilde{Q}^{-1/2}\right)$$

$$= \operatorname{tr}\left(\widetilde{Q}^{-1/2}V\widetilde{Q}V^*(X - J)\widetilde{Q}^{1/2} + \widetilde{Q}^{-1/2}W\widetilde{Q}W^*(X - J)\widetilde{Q}^{1/2}\right)$$

$$= \operatorname{tr}\left(\widetilde{Q}^{-1/2}V\widetilde{Q}V^*\widetilde{Q}^{-1/2}\widetilde{Q}^{1/2}(X - J)\widetilde{Q}^{1/2} + \widetilde{Q}^{-1/2}W\widetilde{Q}W^*\widetilde{Q}^{-1/2}\widetilde{Q}^{1/2}(X - J)\widetilde{Q}^{1/2}\right)$$

$$\leq \left\|\widetilde{Q}^{-1/2}V\widetilde{Q}V^*\widetilde{Q}^{-1/2} + \widetilde{Q}^{-1/2}W\widetilde{Q}W^*\widetilde{Q}^{-1/2}\right\| \operatorname{tr}\left(\widetilde{Q}^{1/2}(X - J)\widetilde{Q}^{1/2}\right)$$

$$= \left\|\widetilde{Q}^{-1/2}(V\widetilde{Q}V^* + W\widetilde{Q}W^*)\widetilde{Q}^{-1/2}\right\| \operatorname{tr}\left(\widetilde{Q}^{1/2}(X - J)\widetilde{Q}^{1/2}\right)$$

$$= \left\|\widetilde{Q}^{-1/2}(V\widetilde{Q}V^* + W\widetilde{Q}W^*)\widetilde{Q}^{-1/2}\right\| \|X - J\|_{1,\widetilde{Q}} := \mathcal{E}_1.$$

Then

$$\text{tr}\left(\widetilde{Q}^{1/2}(V^*(X-J)V+W^*(X-J)W)\widetilde{Q}^{1/2}\right) \leq \mathcal{E}_1. \tag{6}$$

Similarly, we have

$$\text{tr}\left(\widetilde{Q}^{1/2}(U^*(K-Y)U+B^*(K-Y)B)\widetilde{Q}^{1/2}\right) \leq \mathcal{E}_2, \tag{7}$$

where

$$\mathcal{E}_2 := \left\|\widetilde{Q}^{-1/2}(U\widetilde{Q}U^*+B\widetilde{Q}B^*)\widetilde{Q}^{-1/2}\right\| \|K-Y\|_{1,\widetilde{Q}}.$$

Now, using (6) and (7), we get

$$\|F(X,Y)-F(J,K)\|_{1,\widetilde{Q}} \leq \frac{\delta}{2}\left(\|X-J\|_{1,\widetilde{Q}}+\|K-Y\|_{1,\widetilde{Q}}\right),$$

where

$$\delta = 2\max\left\{\left\|\widetilde{Q}^{-1/2}(V\widetilde{Q}V^*+W\widetilde{Q}W^*)\widetilde{Q}^{-1/2}\right\|, \left\|\widetilde{Q}^{-1/2}(U\widetilde{Q}U^*+B\widetilde{Q}B^*)\widetilde{Q}^{-1/2}\right\|\right\}.$$

Observe that from Lemma 2.2, (a) and (b), we have $\delta \in (0,1)$. On the other hand, Taking $X_0 = 0$ and $Y_0 = M$, from (c) and (d), we have

$$X_0 < F(X_0, Y_0) \quad \text{and} \quad Y_0 > F(Y_0, X_0). \tag{8}$$

Then we proved that all the conditions of Lemma 2.4 are satisfied. Note that conditions (iv) of Lemma 2.4 hold in the case $\Delta = H(N)$. We deduce that there exists a unique $\widehat{X} \in H(N)$ solution to (5).

Now, to establish our result, we need to prove that $\widehat{X} \in P(N)$. The Schauder fixed point theorem will be useful in this step. Define the mapping $G : [F(0,M), F(M,0)] \rightarrow H(N)$ by

$$G(X) = F(X,X), \ X \in [F(0,M), F(M,0)].$$

Note that from (8) and the mixed monotone property of F, we have

$$F(0,M) \leq F(M,0).$$

Let $X \in [F(0,M), F(M,0)]$, i.e.,

$$F(0,M) \leq X \leq F(M,0).$$

Using the mixed monotone property of F, we get

$$F(F(0,M), F(M,0)) \leq F(X,X) = G(X) \leq F(F(M,0), F(0,M)).$$

On the other hand, from (8), we have

$$0 < F(0,M) \quad \text{and} \quad M > F(M,0).$$

Again, using the mixed monotone property of F, we get

$$F(F(M,0), F(0,M)) \leq F(M,0) \quad \text{and} \quad F(F(0,M), F(M,0)) \geq F(0,M).$$

Then

$$F(0,M) \leq G(X) \leq F(M,0).$$

Thus we proved that $G([F(0,M), F(M,0)]) \subseteq [F(0,M), F(M,0)]$, i.e., G maps the compact convex set $[F(0,M), F(M,0)]$ into itself. Since G is continuous, it follows from Schauder's fixed point theorem that G has at least one fixed point in this set. However, fixed points of G are solutions to (1), and we proved already that (1) has a unique Hermitian solution. Then this solution must be in the set $[F(0,M), F(M,0)]$, i.e.,

$$\widehat{X} \in [Q-(U^*MU+B^*MB), Q+(V^*MV+W^*MW)] \subset P(N).$$

Then Theorem 3.1 is proved. □

Remark 3.2. *Note that in the proof of the above theorem, we cannot apply the Banach contraction principle to the mapping G. Indeed, the inequalities (6) and (7) are satisfied only for $X \geq J$ and $Y \leq K$. Recall that the proof of these inequalities is based on Lemma 2.1 which requires that $B \geq 0$.*

Theorem 3.3. *Suppose that all the assumptions of Theorem 3.1 hold. Let $\{X_n\}$ and $\{Y_n\}$ be the sequences defined by $X_0 = 0$ (zero matrix), $Y_0 = M$, and*

$$\begin{cases} X_{n+1} & = & Q + (V^*X_nV + W^*X_nW) - (U^*Y_nU + B^*Y_nB), \\ Y_{n+1} & = & Q + (V^*Y_nV + W^*Y_nW) - (U^*X_nU + B^*X_nB). \end{cases} \qquad (9)$$

Then

$$\lim_{n \to \infty} \|X_n - \widehat{X}\| = \lim_{n \to \infty} \|Y_n - \widehat{X}\| = 0.$$

Proof. It follows immediately from (v) of Lemma 2.4. □

Now, we present some consequences following from Theorems 3.1 and 3.3, when A, B are Hermitian matrices. Our first consequence is the following.

Corollary 3.4. *Suppose that*

(a) $A, B \in H(N)$;
(b) $2(UQU + BQB) < Q$;
(c) $2(VQV + WQW) < Q$.

Then

(i) *(1) has one and only one solution $\widehat{X} \in P(N)$;*
(ii) $\widehat{X} \in [Q - 2(UQU + BQB), Q + 2(VQV + WQW)]$.
(iii) *Moreover, let (X_n) and (Y_n) the sequences defined by $X_0 = 0$, $Y_0 = 2Q$, and*

$$\begin{cases} X_{n+1} & = & Q + (VX_nV + WX_nW) - (UY_nU + BY_nB), \\ Y_{n+1} & = & Q + (VY_nV + WY_nW) - (UX_nU + BX_nB). \end{cases}$$

We have

$$\lim_{n \to \infty} \|X_n - \widehat{X}\| = \lim_{n \to \infty} \|Y_n - \widehat{X}\| = 0.$$

Proof. It follows from Theorems 3.1 and 3.3 with $\widetilde{Q} = Q$ and $M = 2Q$. □

Corollary 3.5. *Suppose that*

(a) $A, B \in H(N)$;
(b) $2(U^2 + B^2) < I$, $\quad 2(V^2 + W^2) < I$;
(c) $U^2 + B^2 < Q < I - (V^2 + W^2)$.

Then

(i) *(1) has one and only one solution $\widehat{X} \in P(N)$;*
(ii) $\widehat{X} \in [Q - (U^2 + B^2), Q + (V^2 + W^2)]$.
(iii) *Moreover, let (X_n) and (Y_n) the sequences defined by $X_0 = 0$, $Y_0 = I$, and*

$$\begin{cases} X_{n+1} & = & Q + (VX_nV + WX_nW) - (UY_nU + BY_nB), \\ Y_{n+1} & = & Q + (VY_nV + WY_nW) - (UX_nU + BX_nB). \end{cases}$$

We have

$$\lim_{n \to \infty} \|X_n - \widehat{X}\| = \lim_{n \to \infty} \|Y_n - \widehat{X}\| = 0.$$

Proof. It follows from Theorems 3.1 and 3.3 with $\widetilde{Q} = M = I$. □

4. Numerical Experiments

In this section, we present some numerical experiments to check the convergence of the proposed algorithm (9). We take $X_0 = 0$ and $Y_0 = M \in P(N)$. For each iteration i, we consider the residual errors

$$E_i(X) = \|X_i - (Q + (V^*X_iV + W^*X_iW) - (U^*X_iU + B^*X_iB))\|,$$

$$E_i(Y) = \|Y_i - (Q + (V^*Y_iV + W^*Y_iW) - (U^*Y_iU + B^*Y_iB))\|$$

and

$$E_i = \max\{E_i(X), E_i(Y)\}.$$

All programs are written in MATLAB version 7.1.

First example. We consider (1) with

$$Q = \begin{pmatrix} 2 & 0.02 & 0.05 \\ 0.02 & 2 & 0.02 \\ 0.05 & 0.02 & 2 \end{pmatrix}, \quad A = \begin{pmatrix} -0.95 & 0.001 & 0.001 \\ 0.001 & -0.95 & 0.001 \\ 0.001 & 0.001 & -0.95 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 0.54 & -0.002 & -0.002 \\ -0.002 & 0.54 & -0.002 \\ -0.002 & -0.002 & 0.54 \end{pmatrix}.$$

In this case, we have $A, B \in H(3)$,

$$\mathrm{Sp}\Big(Q - 2(UQU + BQB)\Big) = \Big\{0.3345, 0.3379, 0.3880\Big\}$$

and

$$\mathrm{Sp}\Big(Q - 2(VQV + WQW)\Big) = \Big\{0.4532, 0.4573, 0.4612\Big\},$$

which imply that conditions (a)-(c) of Corollary 3.4 are satisfied.

Let us consider the iterative method (9) with $X_0 = 0$ and $Y_0 = 2Q$. After 100 iterations, one gets the following approximation to the positive definite solution $\widehat{X}$:

$$\widehat{X} \approx X_{100} = Y_{100} = \begin{pmatrix} 1.9495 & 0.0288 & 0.0142 \\ 0.0288 & 1.9496 & 0.0288 \\ 0.0142 & 0.0288 & 1.9495 \end{pmatrix}.$$

We obtained $E_{100} = 1.9215 \times 10^{-13}$.

Second example. We consider (1) with

$$Q = \begin{pmatrix} 0.4 & 0.01 & 0.02 & 0.03 & 0.04 \\ 0.01 & 0.4 & 0.01 & 0.02 & 0.03 \\ 0.02 & 0.01 & 0.4 & 0.01 & 0.02 \\ 0.03 & 0.02 & 0.01 & 0.4 & 0.01 \\ 0.04 & 0.03 & 0.02 & 0.01 & 0.4 \end{pmatrix}, \quad A = \begin{pmatrix} -0.95 & 0.001 & 0.001 & 0.001 & 0.001 \\ 0.001 & -0.95 & 0.001 & 0.001 & 0.001 \\ 0.001 & 0.001 & -0.95 & 0.001 & 0.001 \\ 0.001 & 0.001 & 0.001 & -0.95 & 0.001 \\ 0.001 & 0.001 & 0.001 & 0.001 & -0.95 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 0.44 & -0.02 & -0.02 & -0.02 & -0.02 \\ -0.02 & 0.44 & -0.02 & -0.02 & -0.02 \\ -0.02 & -0.02 & 0.44 & -0.02 & -0.02 \\ -0.02 & -0.02 & -0.02 & 0.44 & -0.02 \\ -0.02 & -0.02 & -0.02 & -0.02 & 0.44 \end{pmatrix}.$$

In this case, we have $A, B \in H(5)$,

$$\mathrm{Sp}\Big(I - 2(U^2 + B^2)\Big) = \Big\{0.4078, 0.4078, 0.4078, 0.4078, 0.64716\Big\},$$

$$\mathrm{Sp}\Big(I - 2(V^2 + W^2)\Big) = \Big\{0.0094, 0.1577, 0.1577, 0.1577, 0.1577\Big\},$$

$$\mathrm{Sp}\Big(Q - (U^2 + B^2)\Big) = \Big\{0.0516, 0.0882, 0.0963, 0.0984, 0.3049\Big\},$$

$$\mathrm{Sp}\Big(I - (V^2 + W^2) - Q\Big) = \Big\{0.0231, 0.1844, 0.1865, 0.1949, 0.2312\Big\},$$

which imply that conditions (a)-(c) of Corollary 3.5 are satisfied.

Considering the iterative method (9) with $X_0 = 0$ and $Y_0 = I$, after 82 iterations one gets an approximation to the positive definite solution $\widehat{X}$ given by

$$\widehat{X} \approx X_{82} = Y_{82} = \begin{pmatrix} 0.4895 & 0.0429 & 0.0541 & 0.0658 & 0.0781 \\ 0.0429 & 0.4878 & 0.0418 & 0.0535 & 0.0658 \\ 0.0541 & 0.0418 & 0.4873 & 0.0418 & 0.0541 \\ 0.0658 & 0.0535 & 0.0418 & 0.4878 & 0.0429 \\ 0.0781 & 0.0658 & 0.0541 & 0.0429 & 0.4895 \end{pmatrix}.$$

We obtained $E_{82} = 7.0549 \times 10^{-16}$.

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