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Exactness, Invertibility and the Love Knot

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Abstract. The "Love Knot" is a little brooch made out of three interlocking circular arcs; an image is incorporated in the familiar Venn diagram of three circular discs in the plane. The Love Knot offers a thread upon which to hang a little minuet between invertibility of the factors of a product, and concepts of "exactness"

1. Introduction

The traditional Venn diagram for three circular discs in the plane incorporates an image [10] of the "Love Knot" brooch; when the three sets satisfy a certain condition, their Venn diagram reduces to the Love Knot. Generally three circles divide the rest of the plane into eight connected components; one of them is unbounded and is the complement of the union of the three discs, while another of them is the intersection. It is an interesting exercise in elementary set theory to describe the remaining six. If however the three sets satisfy the curious condition, that each is a subset of the union of the other two, then three of these six components become empty: if

1.1 $$A \subseteq B \cup C\ ,\ B \subseteq C \cup A\ ,\ C \subseteq A \cup B\ ,$$

we shall say that the sets A, B, C conform to the love knot. In the same situation we shall also say that the propositions P, Q, R, where

1.2 $$P(x) \Longleftrightarrow x \notin A\ ,\ Q(x) \Longleftrightarrow x \notin B\ ,\ R(x) \Longleftrightarrow x \notin C\ ,$$

form a democracy:

1.3 $$Q(x)\&R(x) \Longrightarrow P(x)\ ;\ R(x)\&P(x) \Longrightarrow Q(x)\ ;\ P(x)\&Q(x) \Longrightarrow R(x)\ ;$$

each of them is a consequence of the conjunction of the other two. Equivalently, if E, F, G are three pairwise disjoint sets, a "democracy" can by realised as

1.4 $$P(x) \Longleftrightarrow x \in F \cup G\ ;\ Q(x) \Longleftrightarrow x \in G \cup E\ ;\ R(x) \Longleftrightarrow x \in E \cup F\ .$$

In terms of sets, with

1.5 $$A = \{x : \sim P(x)\}\ ;\ B = \{x : \sim Q(x)\}\ ;\ C = \{x : \sim R(x)\}\ ,$$

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each is a subset of the union of the other two. Sufficient for the democratic condition (1.3) is of course equivalence

1.6 $$P(x) \Longleftrightarrow Q(x) \& R(x) \ ;$$

for sets this means that one of the inclusions (1.1) is equality.

Democracy and the "Love Knot" are prevalent in elementary abstract algebra: two sided invertibility of products and their factors in semigroups, different kinds of "exactness" for pairs of ring elements, and the spectral theory of restrictions and quotients of a linear operator, all conform to the love knot. More exotic examples include the "Kato spectrum" of a Banach space operator, "Müller regularity", different kinds of "spectral permanence" for ring homomorphisms, and even politics! It is not usually easy to illustrate abstract mathematical ideas from the "real world"; but this is just (1.4): there are many countries whose government is shared between three political parties.

2. Invertibility

The love knot occurs frequently in elementary algebra: if A is a semigroup-with-identity, or more generally an abstract category then, writing

2.1 $$A^{-1} = A^{-1}_{left} \cap A^{-1}_{right} \ ,$$

where

2.2 $$A^{-1}_{left} = \{x \in A : 1 \in Ax\} \ , \ A^{-1}_{right} = \{x \in A : 1 \in xA\} \ ,$$

for the *invertible group* in A and with

2.3 $$J(x) \Longleftrightarrow x \in A^{-1} \ ,$$

then for arbitrary $(a, b) \in A^2$, the propositions

2.4 $$J(a) \ ; \ J(b) \ ; \ J(ba)$$

conform to the democratic pattern (1.3). The argument, considering both left and right inverses, is extremely simple:

2.5 $$\{a, b\} \subseteq A^{-1}_{left} \Longrightarrow ba \in A^{-1}_{left} \Longrightarrow a \in A^{-1}_{left} \ ,$$

while

2.6 $$\left(ba \in A^{-1}_{left} \ \& \ a \in A^{-1}_{right}\right) \Longrightarrow b \in A^{-1}_{left} \ .$$

(2.4) does not however hold separately for left or for right inverses; the love knot conclusion (2.4) is liable to fail if two-sided invertibility is replaced, in (2.3), by either left invertibility, right invertibility or *relative regularity*:

2.7 $$A^{\cap} = \{x \in A : x \in xAx\} \ .$$

Failure for left invertibility, and also for right invertibility, is clear if, for example ([3] (4.2.8)),

2.8 $$ba = 1 \neq ab \ ;$$

for relative regularity Caradus has ([3] Example 4.5, Example 4.6) examples which show

2.9 $$a \in A^{\cap} \not\Longrightarrow a^2 \in A^{\cap} \not\Longrightarrow a \in A^{\cap} \ .$$

3. Exactness

While perhaps not so familiar as two-sided invertibility, "exactness" is also a very simple concept. If A is in particular a ring, or more generally an additive category, we shall [3],[5],[9] describe the pair $(b, a) \in A^2$ as *splitting exact* if, whether or not the *chain condition*

3.1 $$ba = 0$$

is satisfied,

3.2 $$1 \in Ab + aA\ ,$$

weakly exact if, for arbitrary $(u, v) \in A^2$, there is implication

3.3 $$va = 0 = bu \Longrightarrow vu = 0\ ,$$

and *regular* if

3.4 $$\{a, b\} \subseteq A^\cap\ .$$

When A is an additive category then it is understood that (b, a) is *compatible* in the sense that the product is defined:

3.5 $$\exists\ ba \in A\ ,$$

and the implication (3.3) is subject to the existence of products vu,va,bu. Now we claim ([3] Theorem 1.6; [5],[6],[9],[12]) that, for chains $(b, a) \in A^2$, the conditions

3.6 *splitting exact, weakly exact, regular*

satisfy the democratic condition (1.3), and indeed (1.6); the argument is again straightforward: for example if there is weak exactness (3.3) then

$$(a = aa^\wedge a\ \&\ b = bb^\wedge b) \Longrightarrow (1 - aa^\wedge)(1 - b^\wedge b) = 0\ .$$

The chain condition cannot be omitted here: for example

$$b \in A^{-1}_{left} \Longleftrightarrow AND_{a\in A}\ (b, a)\ \textit{splitting exact}\ .$$

While relative regularity does not in general conform to the love knot, for splitting exact pairs we get the "regular" analogue of (2.4), and indeed more; if $(b, a) \in A^2$ satisfies the splitting exact condition (3.2), then [5],[9],[12] we have (1.6) equivalence

3.7 $$\{a, b\} \subseteq A^\cap \Longleftrightarrow ba \in A^\cap\ :$$

if $ba = bacba$ and $b'b + aa' = 1$ then

$$(1 - aa')a(1 - cba) = 0 = (1 - bac)b(1 - b'b)\ .$$

Conversely if $a = aa^\wedge a$ and $b = bb^\wedge b$ then exactness gives

$$baa^\wedge b^\wedge ba = b(aa^\wedge + b^\wedge b - 1)a = ba\ .$$

The same is true [9] for left invertibility, right invertibility, "monomorphism" and "epimorphism"; here $b \in A$ is a monomorphism if $(b, 0)$ is weakly exact, while $a \in A$ is an epimorphism if $(0, a)$ is weakly exact.

For left invertibility, right invertibility, monomorphism and epimorphism (but (2.9) not regularity), we can replace splitting exactness (3.2) by *commutivity*

3.8 $$ba = ab\ .$$

In the categories $A = L$ of linear mappings between vector spaces, or $A = BL$ of bounded operators between Banach spaces, we say that $(b, a) : Z \leftarrow Y \leftarrow X$ is *linearly exact* if there is inclusion

3.9 $$b^{-1}(0) \subseteq a(X)\ ;$$

for bounded linear operators there is also *normed exactness*, which means for (b, a) that there are $k > 0$ and $h > 0$ for which, for arbitrary compatible u, v,

3.10 $$\|vu\| \leq k\|v\|\,\|bu\| + h\|va\|\,\|u\|\ .$$

Evidently each of linear and normed exactness are intermediate between splitting and weak exactness.

4. Skew exactness

One sided invertibility can be expressed in terms of exactness: necessary and sufficient for $b \in A^{-1}_{left}$ is that

4.1 $$(b, 0)\ \textit{is splitting exact}\ .$$

More generally it is necessary and sufficient for the left invertibility of $b \in A$ that there is $a \in A$ for which (b, a) is splitting exact and also *left skew exact*, in the sense ([7];[2] §10.9) that

4.2 $$a \in Aba\ :$$

indeed ([2] Theorem 10.9.4) if $b'b + aa' = 1$ and

4.3 $$a = cba$$

then

4.4 $$(1 - cb)(1 - b'b) = 0\ .$$

Similarly right invertibility for $a \in A$ is equivalent to splitting exactness together with *right skew exactness*

4.5 $$b \in baA\ ;$$

if $b'b + aa' = 1$ and $b = bad$ then

4.6 $$(1 - aa')(1 - ad) = 0\ .$$

In the category of linear mappings between vector spaces, or bounded operators between Banach spaces, with linear exactness (3.9), left skew exactness takes the form

4.7 $$b^{-1}(0) \cap a(X) = \{0\}\ ,$$

equivalently ([2] Theorem 10.9.1)

4.8 $$(ba)^{-1}(0) \subseteq a^{-1}(0)\ .$$

Right skew exactness takes the form

4.9 $$b^{-1}(0) + a(X) = Y\ ;$$

equivalently

4.10 $$b(Y) \subseteq (ba)(X)\ .$$

The normed linear analogue of left skew exactness takes the form

4.11 $$\|aw\| \leq k\|baw\|\ .$$

5. Invariant subspaces

"Invariant subspaces" for linear operators are also simple and fundamental. If $A = L(X)$ is the semigroup of homomorphisms $a : X \to X$ on an *abelian group* X then an *invariant subgroup* for $a \in A$ is $Y \subseteq X$ for which there is inclusion

5.1 $$a(Y) \subseteq Y\ ;$$

then we may write

5.2 $$a_Y : Y \to Y$$

for the *restriction* map, and

5.3 $$a_{/Y} : X/Y \to X/Y$$

for the induced *quotient* map $x + Y \mapsto ax + Y$. Now the conditions, on $a \in A$,

5.4 $$a \in L(X)^{-1}\ ;\ a_Y \in L(Y)^{-1}\ ;\ a_{/Y} \in L(X/Y)^{-1}$$

conform [1];[2] §3.11 to the democratic condition (1.3). Again the argument is very simple, involving the one-one and onto conditions; again the conclusion does not hold separately for one-one and for onto. Similar argument [13] shows that if $a^2 = 0$ then the exactness of

$$(a, a)\ ;\ (a_Y, a_Y)\ ;\ (a_{/Y}, a_{/Y})$$

conforms to the love knot. While we have been unable to find a common ancestor for the product and the invariant subspace love knots, they do seem to have a child together. When in particular the invariant subgroup $Y \subseteq X$ is *complemented* then $a \in L(X)$ can be represented as a *triangular matrix*: generally if A and B are rings (more generally, additive categories) and M and N bimodules over A and B, and if

5.5 $$T = \begin{pmatrix} a & m \\ n & b \end{pmatrix} \in G = \begin{pmatrix} A & M \\ N & B \end{pmatrix}\ ,$$

then, provided

$$n = 0 \in N\ ,$$

the conditions

5.6 $$a \in A^{-1}\ ;\ b \in B^{-1}\ ;\ T \in G^{-1}$$

conform [4],[11] to the democratic condition (1.3).

The triangular matrix love knot (5.6) is also essentially a consequence of the product love knot (2.4):

5.7 $$\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\ .$$

6. Kato non singularity

The work of Tosio Kato on Banach space operators is not usually encountered in an elementary context, although some of the discussion is simple enough. We shall say [5] that a ring element $a \in A$ is *Kato invertible* if it is relatively regular and satisfies the Saphar "hyper-exactness" condition that, for arbitrary $n \in \mathbb{N}$,

6.1 $$(a, a^n)\ ,\ equivalently\ (a^n, a)\ ,\ \textit{is splitting exact}\ ;$$

since, for splitting exact pairs $(b, a) \in A^2$, relative regularity conforms to the love knot, and more particularly (1.6), it follows that [5],[9],[12]

Kato invertibility is a Müller regularity .

If in particular $A = L(X)$ is the linear operators on a vector space, or $A = B(X)$ the bounded operators on a Banach space, we can replace splitting exactness in (6.1) by linear exactness (3.8). More generally we shall [6] declare $a \in A = B(X)$ to be *Kato non singular* if we replace relative regularity by closed range. The closed range condition, in conjunction with hyper-exactness, conforms to the love knot, and indeed (1.6): generally, for bounded linear operators $a : X \to Y$ and $b : Y \to Z$, there is [6] by the Open Mapping Theorem, implication

6.2 $$b(Y), b^{-1}(0) + a(X) \textit{ closed} \Longrightarrow ba(X) \textit{ closed} \Longrightarrow b^{-1}(0) + a(X) \textit{ closed} .$$

The implication (3.2) $\Longrightarrow$ (3.7) says [9],[10] that

6.3 $$A^{\cap} \subseteq A \textit{ is a non commutative Müller regularity} ;$$

the left and the right invertibles and the mono- and epi-morphisms have the same status. In $B(X)$, by (6.2), the same is true of the closed range operators; it follows [6],[9],[12] that

Kato non singularity is a Müller regularity .

Here we are describing as a "non commutative Müller regularity" a subclass $H \subseteq A$ with the property that, whenever $(b, a) \in A^2$ is splitting exact, there is two-way implication

6.4 $$\{a, b\} \subseteq H \Longleftrightarrow ba \in H .$$

This is evidently a more primitive version of the concept of *regularity* upon which Vladimir Müller bases his abstract spectral theory [12].

7. Spectral permanence

"Spectral permanence" is also not usually discussed in an elementary context; again the basic idea is extremely simple. If $T : A \to B$ is a semigroup homomorphism, or more generally a functor between categories, then there is inclusion

7.1 $$T(A^{-1}) \subseteq B^{-1} ,$$

equivalently

7.2 $$A^{-1} \subseteq T^{-1}B^{-1} .$$

If there is equality in (7.2), so that

7.3 $$T^{-1}B^{-1} \subseteq A^{-1} ,$$

we shall [8],[14] say that $T : A \to B$ has *spectral permanence*. If in (7.3) we can replace invertibility by relative regularity we shall [8],[14] say that T has *generalized permanence*. Generally, in a semigroup A we shall say that $a \in A$ is *simply polar*, equivalently "group invertible", written

7.4 $$a \in SP(A) ,$$

provided it has a commuting generalized inverse, $c \in A$ for which

7.5 $$a = aca , \ ca = ac .$$

Generally there is inclusion

7.6 $$T\ SP(A) \subseteq SP(B)\ ;$$

we shall say that T has *simple permanence* when there is equality

7.7 $$T^{-1}SP(B) \subseteq SP(A)\ .$$

For Banach algebra homomorphisms $T : A \to B$, although not homomorphisms between general rings, the conditions

spectral permanence, simple permanence, one-one

conform [8] to the love knot. There are possibly more love knots here: if $S : A \to B$ and $T : B \to D$ are ring homomorphisms with spectral permanence then the product $ST : A \to D$ will also have spectral permanence, which in turn will confer spectral permanence on $T : A \to B$. Conversely if the pair (S, T) is exact then maybe the spectral permanence of ST will also imply the spectral permanence of S.

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