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Some Remarks on Mathematical General Relativity Theory

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Abstract. This paper gives a brief survey of the development of general relativity theory starting from Newtonian theory and Euclidean geometry and proceeding through to special relativity and finally to general relativity and relativistic cosmology.

1. Geometry

Over 2000 years ago, the Greek geometer Euclid presented the mathematical community with, amongst other things, the details of the geometry known at that time. Euclid's "Elements" not only described a geometry which was, until the 19th century, the *only* geometry (and, for some, the *only possible* geometry), it also paved the way to the axiomatic method so powerfully employed by Hilbert at the end of the 19th century to set Euclid's work on a more secure foundation [1]. Of course, and before Hilbert's time, Lobachevski and Bolyai had already demonstrated that other geometries were possible and their independent creations of *non-Euclidean geometry*, were revealed in the period 1826-31 (and probably earlier by Gauss). However, apart from its main consequence of proving that Euclid's parallel postulate could not be derived from his other postulates, this alternative geometry does not seem (at least at the time) to have been treated with the importance it deserved. On the other hand, Euclid's geometry was used (as it still is) as a branch of applied mathematics and engineering. A history of such matters can be found in [2, 3].

Euclid's treatise was regarded for many years as possessing the highest standards of rigour (to which all others should aspire). Of course, modern mathematical logic demands more and deficiences in Euclid's axiomatic method were found. Just over 100 years ago, Hilbert [1] published his axiomatic account of Euclid's geometry basing it on a number of undefined primitive elements (points, lines and (if 3-dimensional) planes, together with an incidence relation to declare which points were on which lines). This was followed by five groups of axioms; (i) axioms of incidence, (ii) axioms of betweenness, (iii) axioms of congruence, (iv) the completeness axiom and (v) the Euclidean parallel axiom. Hilbert's axioms lead categorically to the usual Euclidean plane or Euclidean space, depending on the dimension, and, with a simple modification to the parallel axiom, categorically to the geometry of Lobachevski and Bolyai. Thus Euclidean geometry, in Hilbert's hands, was cleansed of its iniquities (mainly due to the belief that it accurately reflected the geometry of nature and thus its component parts (points, lines, etc,) were linked to physical objects) and became pure mathematics.

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2. Classical Newtonian Theory

Perhaps the first serious study of "natural philosophy" or "physics" came first with Galileo and, especially, with Newton. Restricting consideration to the description of the motion of particles, the "space" through which such particles were moving was naturally assumed, in some sense, to satisfy the rules of Euclidean geometry. Of course, one needed also to have some means of measuring the time taken by particles to move from one point to the next. Newton chose what is now known as "absolute time". This can be thought of in the following way. Let E be the *universe of events*, that is the set of all the events that could possibly be with an event being identified by its position in space and its (absolute) time of occurrence. Then, absolute time is a function $T : E \to \mathbb{R}$ and for $p \in E$, $T(p)$ is the absolute time of p. To set this up in a practical way, it is assumed that one can construct as many "good clocks" as required in E and with the property that a good clock at p reads time $T(p)$. The "simultaneity spaces" of the function T are the sets of the form $S_t \equiv T^{-1}\{t\}$ for some time $t \in \mathbb{R}$. To complete the picture one now assumes that each set S_t admits the structure of 3-dimensional Euclidean geometry in the sense of Hilbert and that it admits Cartesian coordinates x, y, z such that the lines and planes of Hilbert's structure are given by the usual linear relations between these coordinates with a unit of length chosen consistent with the usual one on each of these axes. Such a choice of space coordinates on each set S_t (together with the absolute time function) constitute an *observer* although, it must be added, not necessarily a particularly good one, since no mention has been made of how the sets S_t are "joined together" (see [5]). However, the important point that each observer assigns the same absolute time to each event in E, is noted.

In Newtonian theory, it is assumed that one can tell the difference between a "real" force, such as a gravitational or electromagnetic force, and an "inertial" force, such as arises from certain motions of the observer, and this gives rise to the idea of a "free" particle (one upon which no real forces act) and of an *inertial observer* (who experiences no inertial forces). Such assumptions derive from the Newtonian concept of *absolute space* (and, it must be admitted, are difficult to justify). Roughly speaking, an inertial observer chooses the Cartesian coordinates x, y, z in each S_t such that x, y, z, t, with t absolute time, first give a global chart for a manifold structure on E and second that the path of any free particle is given by $t \to \mathbf{r(t)} = (x(t), y(t), z(t))$ with x, y and z *linear* functions of absolute time t and $\mathbf{r(t)}$ the position vector of the particle (and so the particle velocity $\dot{\mathbf{r}}$ is a constant vector). If $x(t)$, $y(t)$ and $z(t)$ are constant functions the particle is at rest for this observer. It is also assumed that the collection of all such charts for these inertial observers gives rise to a smooth manifold structure for E. Thus, for inertial observers, the straight lines of Euclidean geometry in each S_t play a fundamental role in the physics in that they coincide, in the above sense, with the paths of free particles. Similar remarks apply also to their role in describing the paths of light within Maxwell's theory. Any particle P with mass m (this is the inertial mass of P—see the next paragraph) will, for any inertial observer, then satisfy Newton's second law $m\ddot{\mathbf{r}} = \mathbf{f}$ where a dot stands for a time derivative and $\mathbf{f}$ is the real force vector acting on P. As it stands this equation merely serves as a definition of the real force; its real importance arises when this force is independently specified as, for example, a gravitational force based on Newton's inverse square law.

There are infinitely many inertial observers and their coordinates are connected by the usual Galilean coordinate transformations. Then any two inertial observers are either mutually at rest or move with constant velocity one with respect to the other and thus view the physics of Newton's second law in essentially the same way. (An intuitive way of looking at this is to appeal to Newton's concept of "absolute space" and to think of one of these inertial observers as being at rest in absolute space and to ascribe inertial forces as arising for some other (non-inertial) observer due to the acceleration of this latter observer with respect to absolute space. In fact it does not matter which inertial observer is chosen to "represent" absolute space and, in this sense, there are many "absolute spaces".) This is part of the *Newtonian Principle of Relativity* this latter being, roughly speaking, the inability to distinguish one event in E from any other and the inability to distinguish one inertial frame from any other by means of *mechanical phenomena*. If one tried to extend these mechanical phenomena to include Maxwell's electromagnetic theory a special family of inertial observers would be singled out, any two of which are mutually at rest with respect to each other, and may be thought of as being at rest in the ether (absolute space?) or, equivalently, as that special family of inertial observers who, using their space and time coordinates, measure the speed of light to be the same in all directions. Clearly this fails for observers in motion with respect to this special family and thus no such extension of Newton's principle to electromagnetic phenomena is possible.

There is one rather special result from Newtonian theory which should be mentioned here. In the study, within Newtonian theory, of the gravitational interaction between two particles M and m if one focusses attention on m in the gravitational field of M one can identify three types of "mass" parameters for a particle; the "active" gravitational mass, m_{AG}, of m (the measure of m's power of gravitational attraction), the "passive" gravitational mass, m_{PG}, of m (m's susceptibility to being attracted) and the inertial mass, m_I, of m (the resistance of m to being accelerated), with similar symbols for M. Then Newton's second and third laws give in any inertial frame

$$\frac{GM_{AG}m_{PG}}{r^2} = m_I A \qquad \frac{GM_{AG}m_{PG}}{r^2} = \frac{Gm_{AG}M_{PG}}{r^2} \tag{1}$$

where G is Newton's gravitational constant, r is the distance between m and M measured from simultaneous positions of M and m and A is the acceleration of m in that frame. The second of these shows that $\frac{M_{AG}}{M_{PG}} = \frac{m_{AG}}{m_{PG}}$ and so, by the arbitrariness of M and m and the place and time of the interaction, the ratio of a particle's active and passive gravitational masses is the same for all particles and hence, by a suitable choice of units, one may set $m_{AG} = m_{PG} = m_G$ where m_G is the *gravitational* mass of m, and similarly for M. Regarding the first equation in (1) the experiments of Eotvos, Dicke and many others have shown, with an extraordinary degree of accuracy, that with M and r fixed in the above equation the acceleration A of m is independent of the particle m. This is easily checked to be equivalent to any such ratio $\frac{m_I}{m_G}$ being independent of m and hence to the ability to choose the units of inertial and gravitational mass such that any particle's inertial and gravitational masses are equal, $m_G = m_I$, (and hence only one mass parameter is required). In other words there is no unambiguous gravitational field of *force* at the place and time where m is but, rather, a well-defined *gravitational acceleration* at that event for this fixed gravitational field; any particle at this event would experience the same gravitational acceleration as m. Since Newton's second law is a second order differential equation, such an acceleration uniquely determines the subsequent motion of m given m's initial space-time place and initial velocity. In this sense, the gravitational force is indiscriminate, imparting the same acceleration to *all* bodies at a fixed space-time point in a fixed gravitational field. This is (one form of) the *Newtonian principle of equivalence* and appears as a theorem in Newton's theory.

3. Special Relativity

A number of optical experiments conducted at the end of the 19th century started to worry the physics community. The most famous of them all was the Michelson-Morley experiment which was performed several times in this period. The important and simple consequence of this experiment can be summed up by the remarks that (i) it involved light beams and assumed the existence of the so-called "ether", and that (ii) if one dealt with the mathematics of the experiment using the Galilean transformations and traditional electromagnetic theory a contradiction was obtained. This contradiction could be removed by assuming that the speed of light is independent of the (inertial) observer measuring it but, of course, this contradicts the addition laws for relative velocity arising from the Galilean transformations. Other possible explanations were suggested but easily rejected. In fact, it is not obvious how this contradiction can be avoided without rejecting Newton's absolute time. Einstein (and without making any direct mention of the Michelson-Morley experiment) suggested that absolute time be rejected and that each inertial observer O has its own "personal" time with which to operate without essentially changing the classical idea of such an inertial observer [14]. This personal time is assumed to be in agreement, in an obvious sense, with good clocks *at rest* with respect to O. Further, if O and O' are any inertial observers with respective time coordinates $T_O : E \to \mathbb{R}$ and $T_{O'} : E \to \mathbb{R}$ and if a good clock at rest for O is chosen and put into coincidence with, and at rest with respect to, a good clock for O' these two clocks, whilst not necessarily showing the same time, would tick with the same *unit* of time. Thus an inertial observer O has a coordinate system consisting of its time coordinate T_O and space coordinates x, y, z set up, as in the previous Newtonian case, in the Euclidean simultaneity spaces $T_O^{-1}(t)$. Again this gives a global chart for E but the usual concept of "simultaneity" between observers is now lost. It is assumed that the collection of all such charts gives a smooth manifold structure for E.

Another assumption made is that if any inertial observer O computes the speed of any beam of light using its time coordinate T_O and its space coordinates as described above, together with Euclidean geometry, the result would be independent of the position and velocity of the source and direction of the beam of light, *and also of the observer O*. An important consequence of this is the following. Suppose $p \in E$ and O and O' are inertial observers for each of whom p has coordinates $(0,0,0,0)$. Let another event p' have coordinates x, y, z, t for O and x', y', z', t' for O'. Suppose that $x^2 + y^2 + z^2 - c^2t^2 = 0$ where c is the (unambiguous) speed of light. Then a photon of light could pass through p and p' for O and hence for O' also. Thus $x'^2 + y'^2 + z'^2 - c^2t'^2 = 0$. From this one may argue that a metric, denoted η, may be set up on the tangent space T_pE to E at each $p \in E$ and which, in the coordinates x, y, z, ct (noting the change of the time coordinate from t to ct), has (Lorentz signature with) components diag$(1,1,1,-1)$. This is the *Minkowski metric*. It then follows from the above photon argument that the collection of *null* vectors (the *null cone*) at p, $N_p \equiv \{v : \eta(v,v) = 0\}$, in each tangent space T_pE to E at p is independent of the inertial observer. Thus a consequence of losing the concept of absolute time is the gaining of this null cone structure on E. It turns out that the equivalent of those Galilean transformations which fix the space origin for some Newtonian inertial observer are those linear transformations which fix the space origin of O and, in the above sense, "preserve" the null cone. One is here inclined to stress the elegant work of Minkowski on which the 4−dimensional formulation of special relativity is heavily based (and which can be found in both the original German and in English translation in [7]).

Einstein added to these assumptions the (Einstein) principle of relativity which, like Newton's, assumes the indistinguishability of events in E but which extends Newton's by insisting that *no experiment whatsoever* can distinguish one inertial observer from another and the consequence of these assumptions is Einstein's Special Theory of Relativity. The "invariance" of the speed of light removes the problem arising from the Michelson-Morley experiment and also the ambiguity of the constant c occurring in Maxwell's equations for the electromagnetic field for *each* observer. The coordinate transformations between the inertial observers which result (the equivalent of the Galilean transformations) are the celebrated Lorentz transformations. A simple recalculation of the velocity addition laws for motion using the Lorentz transformations is then seen to be consistent with the assumption regarding the speed of light. One consequence of special relativity theory is not, as some authors would have us believe, that the ether is "disproved" but rather that the ether is abandoned since it does not seem to be an observable in physics.

4. General Relativity Theory

Special relativity is a theory of empty space (vacuum) except possibly for the existence of (weak) electromagnetic fields and a few tests particles. It regards space-time mathematically, from the point of view of any inertial observer, as the manifold $M = \mathbb{R}^4$ with a global metric η with components η_{ab} =diag$(1,1,1,-1)$. The nature of the Lorentz transformations shows that (M, η) is homogeneous and isotropic admitting, since $\mathbb{R}^4$ is simply connected in its usual topology, the maximal 10−dimensional Lie algebra of Killing vector fields. It is a kinematical theory of (empty) space-time based on the properties of the speed of light. Special relativity theory has turned out to be very successful and, further, suggests how a more comprehensive theory sufficient to embrace gravitational fields may be constructed from it. This is Einstein's general theory of relativity. It is, however, first noted that Euclidean geometry is important in the setting up of both classical Newtonian theory and special relativity. This is clearly reflected in the Euclidean nature of the simultaneity spaces in each of these theories. (It could have been that "nature" had chosen some other geometry for each of these simultaneity spaces to determine the paths of free particles and light beams!) Second, the choice of Euclidean geometry in these simultaneity spaces is, together with the assumed "time homogeneity", responsible for the space-time homogeneity property in each theory since 3−dimensional Euclidean geometry itself has a maximal 6−dimensional Killing algebra.

The Newtonian principle of equivalence and the experiments of Eotvos et al referred to earlier reveals the indiscriminate nature of gravitational fields and their ability to impart an unambiguous acceleration to particles coming under their influence. But the so-called "fictitious" (inertial) forces generated in (non-inertial) reference frames accelerating with respect to inertial frames are similarly indiscriminate, imparting this same acceleration to all observed free particles. At a space-time event, no experiment seems to be able to distinguish between these two types of forces. This is the basis behind Einstein's lift experiment at that event. This suggests that a theory of gravitation may be developed which benefits from the ideas of special relativity and which incorporates this feature of indistinguishability within it. Thus one may try to simulate the acceleration of a particle in a gravitational field for some inertial observer O at some space-time point by a non-inertial observer O' in a region free of gravitational fields whose acceleration with respect to some inertial observer is precisely the negative of this acceleration. The coordinate transformation from O to O' at this event then changes the Minkowski metric components *at that event* from $\mathrm{diag}(1, 1, 1, -1)$ to some other symmetric array g_{ab} in O' and which is also of Lorentz signature. If this is done for all points (and which requires, in general, a different coordinate transformation at each point) the quantity g_{ab} thus obtained may be regarded, in some sense, as reflecting the gravitational field experienced by O. But g is not merely η in another coordinate system but is, in general, a quite different, Lorentz "metric" in the coordinate domain of O' and which may have a curvature tensor derived from its Levi-Civita connection which is non-zero. With this somewhat simplistic, physical reasoning, general relativity attempts to describe the gravitational field within a 4−dimensional manifold (the universe of events, E) admitting a Lorentz metric. One immediate advantage of this approach is that such a metric has, in general, lost the symmetry (homogeneity) properties inherent in Newtonian and special relativity theory and can thus accommodate more general gravitational fields lacking any form of symmetry. Indeed, the homogeneity and isotropy inherent in the background (Euclidean) geometry of classical Newtonian theory and special relativity theory and which are, in part, responsible for the relativity principles of Newton and Einstein, are now seen to be serious restrictions on the use of such geometry to describe gravitation.

The central ideas of general relativity [15] are, first and because of the assumed physical inability to distinguish between inertial forces and gravitational forces, there is no concept of an inertial observer. Also, just as in special relativity theory, absolute time is abandoned but now, in addition, absolute space is also rejected. The idea of space-time being a manifold with its attendant atlas of coordinate systems appeals since no distinction is now made between one coordinate system (observer) and another, on physical grounds (but, possibly, on grounds of convenience). In this sense, all observers are "equivalent". Second, Einstein assumed that a "space-time" consisted of a 4−dimensional manifold admitting a Lorentz metric, that is a pair (M, g), with the geometry determined by g as, in some sense, representing the gravitational field and with g being subject to *Einstein's field equations* which will be introduced later. Here the unmistakable influence of Minkowski is vital (see [7]). Third and in analogy with Newton's second law being a second order differential equation (section 2), Einstein assumed that the paths of particles (their *world lines*) falling freely in a gravitational field should be the (second order) timelike geodesics arising from the Levi-Civita connection of g. Such paths are uniquely determined, at least locally, by the particle's initial position and (space-time) direction (4−velocity), the latter being equivalent to the particle's initial velocity. This is the *Einstein principle of equivalence* and appears in general relativity as an assumption rather than a theorem, as in Newtonian theory. Thus the force concept is abandoned, being replaced, as far as particle motion is concerned, by the gravitational field giving the geometry which, in turn, determines the timelike geodesics which the particles follow [15].

For the determination of the metric g Einstein supplied field equations and which, apparently and perhaps not surprisingly, took him a significant length of time to finalise. Recalling what was said in the first point of the previous paragraph, Einstein required these field equations to be the same for all coordinate systems (observers). This is usually referred to as *Einstein's principle of general covariance* and more will be said about it later. (The benefit to Einstein from the earlier pioneering work of Riemann [4] and others is clear.) Consider, first, the situation for a *vacuum* region of space-time (one in which there is no actual gravitating matter but which is under the influence of such matter). This covers the important problem of the dynamics of the solar system outside the sun and in which the planets are considered as point particles. The field equations in the vacuum case are simply that the Ricci tensor *Ricc* resulting from the curvature tensor associated with the Levi-Civita connection of g is zero, that is

$$Ricc = 0 \tag{2}$$

(or, in components, $R_{ab} = 0$).These are ten non-linear, second order, partial differential equations for finding (or at least restricting) g. Since they are tensor equations, the above demand that such field equations be "covariant" is clearly satisfied. Assuming the sun to be spherically symmetric and static, Einstein was able to find an approximate solution of these equations representing the gravitational field of the sun. From this he was able to explain the problem (which had defeated astronomers for some time) regarding the precession of the perihelion of planet Mercury and to introduce two further concepts; the shift of the frequency of spectral lines in a gravitational field and the bending of light rays passing through a gravitational field. Each of these phenomena has been verified experimentally. In 1916, K Schwarzshild was able to find the corresponding *exact* solution for such a (spherically symmetric, static), vacuum gravitational field and which depended on the fact that it admitted a 4−dimensional Killing algebra representing its spherical symmetric and static nature. (The static assumption could have been dropped as was shown later by Birkhoff.)

If the vacuum assumption is dropped, one requires some extra terms in the field equations to represent the matter content. Maxwell had, of course, already constructed his (3−dimensional, symmetric) electromagnetic *stress-energy tensor* and Minkowski had enlarged this to a (space-time, symmetric) electromagnetic energy-momentum tensor for Maxwell's equations within special relativity. Einstein extended this tensor to his general, symmetric, type $(0, 2)$, second order, energy-momentum tensor T, with components T_{ab} and which is supposed to represent the matter content of the (relevant part of the) universe and then wrote down his field equations in the general form

$$G \equiv Ricc - \frac{R}{2}g = \kappa T \qquad (G_{ab} \equiv R_{ab} - \frac{R}{2}g_{ab} = \kappa T_{ab}) \tag{3}$$

where G is the Einstein tensor, $R \equiv R_{ab}g^{ab}$ is the Ricci scalar, and κ is the Einstein gravitational constant. Since $\kappa T_{ab}g^{ab} = -R$ these equations are easily shown to reduce to the vacuum field equations $Ricc = 0$ if $T \equiv 0$ on M and, in general, are to be solved once the exact form of the components of energy-momentum T_{ab} are given. (Sometimes, for cosmological solutions, an extra term Λg is included in the left hand side of (2) and (3) for some (so-called, cosmological) constant Λ). Examples of T for perfect fluid and electromagnetic sources of gravitation were, in fact, given by Einstein and the former was used by him in his first attempt at a cosmological model within general relativity theory [11]. If approximation techniques are used for weak and slowly changing gravitational fields in Einstein's theory (so that, for example, one assumes possible a choice of coordinates in which the metric g may be written as some small deviation from the Minkowski metric η), one may recover the equations of classical gravitational theory. The Einstein field equations given here were painstaking developed by Einstein. In fact, the number of options for the left hand side of (3) are severely restricted as was argued by Eddington [9] and established much later by Lovelock [10]. For some considerable time after Einstein's original paper, the above mentioned Schwarzschild solution (and its "charged" equivalent) together with some cosmological models to be discussed later and a few other metrics (which included metrics representing wave-like solutions to (3)) were essentially the only exact solutions of Einstein's equations known. This has changed dramatically in the last few decades and a large compendium of known solutions can be found in the remarkable book [19].

Some final remarks can now be given on Einstein's formulation of general relativity theory. First, regarding the Einstein principle of equivalence, it can be shown that in many cases (and certainly in the vacuum case) the timelike geodesics themselves determine the metric uniquely (up to a constant conformal factor, that is, up to units of measurement). Thus the paths of particles falling freely in the important case of a vacuum gravitational field, according to Einstein's postulate, fix the metric in this sense (see [16–18]). Second, regarding Einstein's principle of covariance, a little more discussion is required. Suppose one starts with a study of Newtonian mechanics in a Cartesian coordinate system using the usual Lagrangian formulation and then rewrites Newton's equations in standard (configuration space) generalised coordinates. The resulting Euler-Lagrange equations are essentially just Newton's equations in a "covariant" form. Has one succeeded in making Newton's theory generally covariant? This example, together with others (which include making Maxwell's equations generally covariant in special relativity by replacing certain partial derivatives with covariant derivatives) suggest that many theories can be made generally covariant and that such a principle is not so powerful (see [8]). However, in these examples, an extra field variable is introduced; the (configuration space) metric in the generalised coordinates of the first example and the (flat) metric in Minkowski space-time in the second. This extra variable does not occur in the field equations of the theory in discussion and is often referred to as an *absolute* variable, being imposed from the outset (as opposed to the other occurring variables which do take part in the field equations and are thus *dynamical* [5, 6]). The covariance referred to in general relativity theory is quite different; there the only field variable is the metric and it is dynamical. No absolute variables are required for such covariance. It is in this stronger sense that general relativity is generally covariant.

5. Cosmology

In 1917, Einstein [11] published what was to be the first paper on relativistic cosmology. (It is remarked here that a cosmology can be based on Newtonian theory but contains several interpretative problems–see [13]). When Einstein wrote his paper the universe was understood to be, on the whole, static. Later, during the 1920s with the advent of the big telescopes, Hubble and others discovered that the universe was, in some sense, expanding and hence not static. This led to the "Hubble Law" of galactic expansion and later to the description "big bang" being applied to this model of the universe. The mathematicians were quick to respond to the question of modifying Einstein's original static universe so as to include this expansion. Their achievements can be summarised in the assumptions which follow. A cosmological model is an all-embracing theory of the universe and hence complicated but which can be approximated by a model which satisfies the conditions that (i) at each event, there exists an observer (called a *fundamental observer*) who "sees" no large-scale difference in one space direction from another (isotropy) and (ii) there exists a *cosmic time* for the universe such that any two fundamental observers have indistinguishable surroundings at the same cosmic time (homogeneity). Such assumptions lead to an energy-momentum tensor which is necessarily of the "perfect fluid" type and for which the tangent ($4-$velocity) to the fundamental observer at any point is a timelike eigenvector of it with respect to the metric at that point. They also reveal that the universe is a "conformally flat" manifold. [A more precise statement is [20] that a cosmological model is a pair (M, g), as mentioned above, whose Killing algebra $K(M)$ of global Killing vector fields on M is such that the isotropy subalgebra $I_p \equiv \{X \in K(M) : X(p) = 0\}$ at $p \in M$ acts transitively, through its local flows, on the set of all null directions at m. This is the isotropy assumption applied to all information in the incoming light beams at m. When allied to the mathematical structure of Killing orbits on M [21] it then follows that I_m is either Lie isomorphic to $so(3)$ *at each* $m \in M$ or to the full Lorentz algebra *at each* $m \in M$. The associated structure of the Killing orbits then leads to a natural cosmic time, the homogeneity condition and the usual mathematical cosmological models.] The general result of such assumptions (and some reasonable physics) is that the metric solution of Einstein's field equations is of the type discovered collectively by Friedmann, Robertson, Walker and Lemaitre and known as the FRWL models (for a history see [12]). If one rules out (the Einstein) static and constant curvature models, to arrive at generic FRWL models, the world lines of the fundamental observers constitute a unique family of geodesics (in the sense that through any $p \in M$ passes a unique such geodesic world line). Any observer other than one of these (that is, an observer who is in non-trivial relative motion at p to the fundamental observer at p) will not observe isotropy. Thus in one sense, one achieves a form of "cosmological absolute space" (the rest space of the fundamental observers) but one which, unlike Newton's, has been determined by the physics of the universe. Only the fundamental observers will observe, for example, the isotropy of the microwave background radiation (usually taken to be the "leftovers" of the big bang, discovered in 1965 and which, for many, confirmed this theory of cosmology).

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References

[1] Hilbert D., The Foundations of Geometry, Open Court, Chicago, (1902).
[2] Meschkowski H., Noneuclidean Geometry, Academic Press, (1964).
[3] Bonola R., Non-Euclidean Geometry, (Trans H. S. Carslaw) Dover, New York, (1955).
[4] Riemann G. F. B., On the Hypotheses which lie at the Foundation of Geometry. Translated in; Ewald William., From Kant to Hilbert, Vol 2, Clarendon Press, Oxford, 1996.
[5] Trautman A., The General Theory of Relativity. (Report from the Conference on Relativity Theory, London, 1965).
[6] Anderson J. L., *Gen. Rel. Grav.* **2**, (1971), 161-172.
[7] Petkov V. (ed) Minkowski Spacetime: A Hundred Years Later. Springer, 2010.
[8] Kretchsmann E., *Ann. der. Physik.*, **53**, (1917), 575.
[9] Eddington Sir A., The Mathematical Theory of Relativity. C.U.P. (1965).
[10] Lovelock D., Mathematical Aspects of Variational Principles in the General Theory of Relativity. D.Sc Thesis, University of Waterloo, Canada, (1973).
[11] Einstein A., *Sitz. Pruss. Akad. Wiss.* (1917) (Translated in The Principle of Relativity, Dover, 177-188, (1923).
[12] d'Inverno R., Introducing Einstein's Relativity. Oxford (1992).
[13] Bondi H., Cosmology. C.U.P. (1960).
[14] Einstein A., *Ann. der. Physik.* **17**, (1905). (Translated in The Principle of Relativity. Dover, (1923), 37-71.)
[15] Einstein A., *Ann. der. Physik.* **49**, (1916). (Translated in The Principle of Relativity. Dover, (1923), 111-164.)
[16] Petrov A. Z., Einstein Spaces, Pergamon, (1969).
[17] Kiosak V. and Matveev V., *Comm. Math. Phys.,* **289**, (2009), 383.
[18] Hall G.S. and Lonie D.P., *Class Quant Grav* **24**, (2007), 3617. (For an improved proof see Hall G.S. and Lonie D.P., *Class Quant Grav* **26**, (2009), 125009-1-125009-10
[19] Stephani H., Kramer D., MacCallum M.A.H., Hoenselears C. and Held E., Exact Solutions to Einstein's Equations. 2nd edition, CUP (2003).
[20] Hall G. S., *J. Appl. Computat. Math.* (2013), e130.doi:10.4172/2168-9679.1000e130 and for more details, Hall G. S., The Geometry of Newton's and Einstein's Theories, in ed. Ashtekar A. and Petkov V., "Springer Handbook of Spacetime", Springer, (2014), chapter 5, 71 – 89.
[21] Hall G. S. Symmetries and Curvature Structure in General Relativity. World Scientific, (2004).