Upward and Downward Statistical Continuities

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Abstract. A real valued function f defined on a subset E of $\mathbb{R}$, the set of real numbers, is statistically upward (resp. downward) continuous if it preserves statistically upward (resp. downward) half quasi-Cauchy sequences; A subset E of $\mathbb{R}$, is statistically upward (resp. downward) compact if any sequence of points in E has a statistically upward (resp. downward) half quasi-Cauchy subsequence, where a sequence (x_n) of points in $\mathbb{R}$ is called statistically upward half quasi-Cauchy if

$$\lim_{n \to \infty} \frac{1}{n} |\{k \le n : x_k - x_{k+1} \ge \varepsilon\}| = 0,$$

and statistically downward half quasi-Cauchy if

$$\lim_{n \to \infty} \frac{1}{n} |\{k \le n : x_{k+1} - x_k \ge \varepsilon\}| = 0$$

for every $\varepsilon > 0$. We investigate statistically upward and downward continuity, statistically upward and downward half compactness and prove interesting theorems. It turns out that any statistically upward continuous function on a below bounded subset of $\mathbb{R}$ is uniformly continuous, and any statistically downward continuous function on an above bounded subset of $\mathbb{R}$ is uniformly continuous.

1. Introduction

A subset E of $\mathbb{R}$ is bounded if and only if any sequence of points in E has a Cauchy subsequence. Boundedness coincides not only with ward compactness ([13]), but also either of the following kinds of compactness: slowly oscillating compactness ([7]), statistical ward compactness ([11, Lemma 2]), lacunary statistical ward compactness ([12, Theorem 3]), N_θ-ward compactness ([3, Theorem 3.3]). Two of our results in this paper state necessary and sufficient conditions for below boundedness and above boundedness of a subset of $\mathbb{R}$. Using the idea of continuity of a real function in terms of sequences, many kinds of continuities were introduced and investigated, not all but some of them we recall in the following: slowly oscillating continuity ([7]), quasi-slowly oscillating continuity ([23]), ward continuity ([13],[2]), δ-ward continuity ([9]), statistical ward continuity ([11]), and N_θ-ward continuity ([16]) which enabled some authors to obtain conditions on the domain of a function for some characterizations of uniform continuity in terms of sequences in the sense that a function preserves a certain kind of sequences (see [36],[2],[23]).

The purpose of this paper is to introduce the concepts of statistically upward and statistically downward continuities, and prove interesting theorems.

2. Preliminaries

Throughout of this paper, $\mathbb{N}$, and $\mathbb{R}$ will denote the set of all positive integers, and the set of all real numbers, respectively. A sequence (x_n) of points in $\mathbb{R}$ is quasi-Cauchy if (Δx_n) is a null sequence, where $\Delta x_n = x_n - x_{n+1}$. These sequences were named as *quasi-Cauchy* by Burton and Coleman [2, page 328], while they were called forward convergent to 0 sequences in [13, page 226]. A subset E of $\mathbb{R}$ is *bounded* if and only if any sequence of points in E has a quasi-Cauchy subsequence. The concept of statistical convergence is a generalization of the usual notion of convergence that, for real-valued sequences, parallels the usual theory of convergence. A sequence (x_k) of points in $\mathbb{R}$ is called statistically convergent to an element ℓ of $\mathbb{R}$ if for each $\varepsilon > 0$ $\lim_{n\to\infty} \frac{1}{n}|\{k \leq n : |x_k - \ell| \geq \varepsilon\}| = 0$, and this is denoted by $st - \lim_{k\to\infty} x_k = \ell$ (see [28], [5], [27], [8], [24], [25], [17], and [29]). A subset E of $\mathbb{R}$ is called *statistically ward compact* if any sequence of points in E has a statistically quasi-Cauchy subsequence, where a sequence (x_n) of points in $\mathbb{R}$ is statistically quasi-Cauchy if $st - \lim_{n\to\infty} \Delta x_n = 0$ ([12]).

Recently, Palladino ([32]) introduced a concept of upward half Cauchyness, and a concept of downward half Cauchyness in the following way: a sequence (x_n) of points in $\mathbb{R}$ is called *upward half Cauchy* if for every $\varepsilon > 0$ there exists an $n_0 \in \mathbb{N}$ so that $x_n - x_m < \varepsilon$ for $m \geq n \geq n_0$, and *downward half Cauchy* if for every $\varepsilon > 0$ there exists an $n_0 \in \mathbb{N}$ so that $x_m - x_n < \varepsilon$ for $m \geq n \geq n_0$. Using the idea of the definition of an upward half Cauchy sequence, a concept of upward half quasi-Cauchy sequence is introduced in ([4]). A sequence (x_n) of points in $\mathbb{R}$ is called *upward half quasi-Cauchy* if for every $\varepsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that $x_n - x_{n+1} < \varepsilon$ for $n \geq n_0$. A subset E of $\mathbb{R}$ is called *upward compact* if any sequence of points in E has an upward half quasi-Cauchy subsequence. A sequence (x_n) of points in $\mathbb{R}$ is called *downward half quasi-Cauchy* if for every $\varepsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that $x_{n+1} - x_n < \varepsilon$ for $n \geq n_0$ ([4]). A subset E of $\mathbb{R}$ is called *downward compact* if any sequence of points in E has a downward half quasi-Cauchy subsequence. A subset E of $\mathbb{R}$ is called *up half compact* if any sequence of points in E has an up half Cauchy subsequence, and is called *down half compact* if any sequence of points in E has a down half Cauchy subsequence ([4]).

3. Statistically upward and downward quasi-Cauchy sequences

Weakening the condition on the definition of a statistically quasi-Cauchy sequence, omitting the absolute value symbol, i.e. replacing $"|x_k - x_{k+1}| \geq \varepsilon"$ with $"x_k - x_{k+1} \geq \varepsilon"$ in the definition of a statistically quasi-Cauchy sequence, we introduce the following definition.

Definition 3.1. *A sequence (x_n) of points in $\mathbb{R}$ is called statistically upward half quasi-Cauchy if*

$$\lim_{n\to\infty} \frac{1}{n}|\{k \leq n : x_k - x_{k+1} \geq \varepsilon\}| = 0$$

for every $\varepsilon > 0$.

Any statistically convergent sequence is statistically upward half quasi-Cauchy. Any statistically quasi-Cauchy sequence is statistically upward half quasi-Cauchy, but the converse is not always true. As a counterexample, the sequence (n) is a statistically upward half quasi-Cauchy sequence, but not statistically quasi-Cauchy. Any slowly oscillating sequence is statistically upward half quasi-Cauchy, so any Cauchy sequence is, so any convergent sequence is. Any upward half Cauchy sequence is statistically upward half quasi-Cauchy. The sum of two statistically upward half quasi-Cauchy sequences is statistically upward half quasi-Cauchy.
Now we introduce a definition of statistically upward compactness of a subset of $\mathbb{R}$, by using the main idea in the definition of sequential compactness.

Definition 3.2. *A subset E of $\mathbb{R}$ is called statistically upward compact if any sequence of points in E has a statistically upward half quasi-Cauchy subsequence.*

First, we note that any finite subset of $\mathbb{R}$ is statistically upward compact, the union of two statistically upward compact subsets of $\mathbb{R}$ is statistically upward compact and the intersection of any family of statistically upward compact subsets of $\mathbb{R}$ is statistically upward compact. Furthermore any subset of a statistically upward compact set is statistically upward compact, any compact subset of $\mathbb{R}$ is statistically upward compact, any bounded subset of $\mathbb{R}$ is statistically upward compact, and any slowly oscillating compact subset of $\mathbb{R}$ is statistically upward compact (see [7] for the definition of slowly oscillating compactness). These observations suggest to us the following.

Theorem 3.3. *A subset of $\mathbb{R}$ is statistically upward compact if and only if it is bounded below.*

Proof. Let E be a bounded below subset of $\mathbb{R}$. If E is also bounded above, then it follows from [11, Lemma 2] and [12, Theorem 3] that any sequence of points in E has a quasi Cauchy subsequence which is also statistically upward half quasi-Cauchy. If E is unbounded above, and (x_n) is an unbounded above sequence of points in E, then for $k = 1$ we can find an x_{n_1} greater than 0. For k=2 we can pick an x_{n_2} such that $x_{n_2} > 1 + x_{n_1}$. We can successively find for each $k \in \mathbb{N}$ an x_{n_k} such that $x_{n_{k+1}} > k + x_{n_k}$. Then $x_{n_k} - x_{n_{k+1}} < -k$ for each $k \in \mathbb{N}$. Therefore we see that

$$\lim_{n \to \infty} \frac{1}{n} |\{k \le n : x_{n_k} - x_{n_{k+1}} \ge \varepsilon\}| = 0$$

for every $\varepsilon > 0$.
Conversely, suppose that E is not bounded below. Pick an element x_1 of E. Then we can choose an element x_2 of E such that $x_2 < -2 + x_1$. Similarly we can choose an element x_3 of E such that $x_3 < -3 + x_2$. We can inductively choose x_k satisfying $x_{k+1} < -k + x_k$ for each positive integer k. Then the sequence (x_k) does not have any statistically upward half quasi-Cauchy subsequence. Thus E is not statistically upward compact. This contradiction completes the proof. $\square$

Reversing the places of the terms "x_n" and "x_{n+1}" in the definition of a statistically upward half quasi-Cauchy sequence, we introduce a concept of a statistically downward half quasi-Cauchy sequence in the following.

Definition 3.4. *A sequence (x_n) of points in $\mathbb{R}$ is called statistically downward half quasi-Cauchy if*

$$\lim_{n \to \infty} \frac{1}{n} |\{k \le n : x_{k+1} - x_k \ge \varepsilon\}| = 0$$

for every $\varepsilon > 0$.

We note that a sequence (x_n) of points in $\mathbb{R}$ is statistically quasi-Cauchy if and only if it is both statistically upward half quasi-Cauchy and statistically downward half quasi-Cauchy. Any statistically convergent sequence is statistically downward half quasi-Cauchy. Any statistically quasi-Cauchy sequence is statistically downward half quasi-Cauchy, but there are statistically downward half quasi-Cauchy sequences which are not statistically quasi-Cauchy. A counterexample is the sequence defined by $x_n = -n$ for each positive integer n, i.e. the sequence $(-n)$ is a statistically downward half quasi-Cauchy sequence, which is not statistically quasi-Cauchy. Any slowly oscillating sequence is statistically downward half quasi-Cauchy, so any Cauchy sequence is, so any convergent sequence is. Any downward half Cauchy sequence is statistically downward half quasi-Cauchy. Throughout the paper ΔS, ΔS^+ and ΔS^- will denote the set of statistically quasi-Cauchy sequences, the set of statistically upward half quasi-Cauchy sequences, and the set of statistically downward half quasi-Cauchy sequences of points in $\mathbb{R}$. The sum of two statistically downward half quasi-Cauchy sequences is statistically downward half quasi-Cauchy.
Now we introduce a definition of statistically downward compactness of a subset of $\mathbb{R}$.

Definition 3.5. *A subset E of $\mathbb{R}$ is called statistically downward compact if any sequence of points in E has a statistically downward half quasi-Cauchy subsequence.*

First, we note that any finite subset of $\mathbb{R}$ is statistically downward compact, the union of two statistically downward compact subsets of $\mathbb{R}$ is statistically downward compact and the intersection of any collection of statistically downward compact subsets of $\mathbb{R}$ is statistically downward compact. Furthermore, any subset of a statistically downward compact set is statistically downward compact, any compact subset of $\mathbb{R}$ is statistically downward compact, any bounded subset of $\mathbb{R}$ is statistically downward compact, and any slowly oscillating compact subset of $\mathbb{R}$ is statistically downward compact. These observations lead us to state the following result.

Theorem 3.6. *A subset of $\mathbb{R}$ is statistically downward compact if and only if it is bounded above.*

Proof. The proof of this theorem is similar to the proof of Theorem 3.3, so is omitted. □

Recalling that a subset E of $\mathbb{R}$ is down half compact if any sequence of points in E has a down half Cauchy subsequence ([4]), we have the following result.

Corollary 3.7. *A subset of $\mathbb{R}$ is down half compact if and only if it is statistically downward compact.*

Corollary 3.8. *A subset of $\mathbb{R}$ is statistically ward compact if and only if it is both statistically upward and statistically downward compact.*

Now we give the following results that easily follow from Theorem 3.3 and Theorem 3.6.

Corollary 3.9. *A subset of $\mathbb{R}$ is bounded if and only if it is both statistically upward and statistically downward compact.*

Corollary 3.10. *A subset of $\mathbb{R}$ is N_θ-ward compact if and only if it is both statistically upward and statistically downward compact.*

Proof. The proof follows from Corollary 3.9, and [3, Theorem 3.3]. □

4. Statistically Upward and Downward continuities

A real valued function f defined on a subset of $\mathbb{R}$ is statistically continuous if and only if, for each point ℓ in the domain, $st - \lim_{n\to\infty} f(x_n) = f(\ell)$ whenever $st - \lim_{n\to\infty} x_n = \ell$. This is equivalent to the statement that $(f(x_n))$ is a convergent sequence whenever (x_n) is. This is also equivalent to the statement that $(f(x_n))$ is a Cauchy sequence whenever (x_n) is Cauchy provided that the domain of the function is closed. These well known results for statistical continuity and continuity for real functions in terms of sequences might suggest to us introducing a new type of continuity, namely, statistically upward continuity.

Definition 4.1. *A function $f : E \to \mathbb{R}$ is called statistically upward continuous on a subset E of $\mathbb{R}$ if it preserves statistically upward half quasi-Cauchy sequences, i.e. the sequence $(f(x_n))$ is statistically upward half quasi-Cauchy whenever $x = (x_n)$ is a statistically upward half quasi-Cauchy sequence of points in E.*

It should be noted that statistically upward continuity cannot be given by any A-continuity in the manner of [26].

We see that the sum of two statistically upward continuous functions is statistically upward continuous and the composition of two statistically upward continuous functions is statistically upward continuous.

In connection with statistically upward half quasi-Cauchy sequences, statistically quasi-Cauchy sequences, statistically convergent sequences, and convergent sequences the problem arises to investigate the following types of continuity of functions on $\mathbb{R}$:

(δS^+) $(x_n) \in \Delta S^+ \Rightarrow (f(x_n)) \in \Delta S^+$

(δS^+c) $(x_n) \in \Delta S^+ \Rightarrow (f(x_n)) \in c$

(c) $(x_n) \in c \Rightarrow (f(x_n)) \in c$

(cδS⁺) $(x_n) \in c \Rightarrow (f(x_n)) \in \Delta S^+$

(S) $(x_n) \in S \Rightarrow (f(x_n)) \in S$

(δS) $(x_n) \in \Delta S \Rightarrow (f(x_n)) \in \Delta S$.

We see that (δS^+) is statistically upward continuity of f, and (S) is the statistical continuity. It is easy to see that $(\delta S^+ c)$ implies (δS^+); (δS^+) does not imply $(\delta S^+ c)$; (δS^+) implies $(c\delta S^+)$; $(c\delta S^+)$ does not imply (δS^+); $(\delta S^+ c)$ implies (c), and (c) does not imply $(\delta S^+ c)$; and (c) implies $(c\delta S^+)$. We see that (c) can be replaced by not only statistical continuity, but also lacunary statistical continuity, S_λ-statistical continuity, N_θ-sequential continuity, I-sequential continuity, and more generally G-sequential continuity, i.e. not only $st - \lim_{n\to\infty} f(x_n) = f(\ell)$ whenever $\mathbf{x} = (x_n)$ is a statistically convergent sequence with $st - \lim_{n\to\infty} x_n = \ell$, but also:
$S_\theta - \lim_{n\to\infty} f(x_n) = f(\ell)$ whenever $\mathbf{x} = (x_n)$ is a lacunary statistically convergent sequence with $S_\theta - \lim_{n\to\infty} x_n = \ell$;
$S_\lambda - \lim_{n\to\infty} f(x_n) = f(\ell)$ whenever $\mathbf{x} = (x_n)$ is a λ-statistically convergent sequence with $S_\lambda - \lim_{n\to\infty} x_n = \ell$;
$N_\theta - \lim_{n\to\infty} f(x_n) = f(\ell)$ whenever $\mathbf{x} = (x_n)$ is an N_θ-convergent sequence with $N_\theta - \lim_{n\to\infty} x_n = \ell$;
$I - \lim_{n\to\infty} f(x_n) = f(\ell)$, whenever $\mathbf{x} = (x_n)$ is a ideal convergent sequence with $I - \lim_{n\to\infty} x_n = \ell$ (see [15]);
$G(f(\mathbf{x})) = f(G(\mathbf{x}))$,
where I is a non-trivial admissible ideal of $\mathbb{N}$, and G is a regular subsequential method (see [10, Lemma 1, Corollary 9]).

Now we give the implication (δS^+) implies (δS), i.e. any statistically upward continuous function is statistically ward continuous.

Theorem 4.2. *If f is statistically upward continuous on a subset E of $\mathbb{R}$, then it is statistically ward continuous on E.*

Proof. Let (x_n) be any statistically quasi-Cauchy sequence of points in E. Then

$$(x_1, x_2, x_1, x_2, x_3, x_2, x_3, ..., x_{n-1}, x_n, x_{n-1}, x_n, x_{n+1}, x_n, x_{n+1}, ...)$$

is also statistically quasi-Cauchy, so it is statistically upward half quasi-Cauchy. Hence the corresponding sequence

$$(f(x_1), f(x_2), f(x_1), f(x_2), f(x_3), f(x_2), f(x_3), ..., f(x_n), f(x_{n+1}), f(x_n), f(x_{n+1}), f(x_{n+2}), ...)$$

is statistically upward half quasi-Cauchy. It follows from this that

$$\lim_{n\to\infty} \frac{1}{n} |\{k \le n : |f(x_k) - f(x_{k+1})| \ge \varepsilon\}| = 0$$

for every $\varepsilon > 0$. This completes the proof of the theorem. $\square$

It should be noted that the converse of the preceding theorem is not always true, i.e. there are statistically ward continuous functions which are not statistically upward continuous. As a counter example we can consider the function f defined by $f(x) = -x$ for every $x \in \mathbb{R}$.
Now we prove that (δS^+) implies (S) in the following:

Theorem 4.3. *If f is statistically upward continuous on a subset E of $\mathbb{R}$, then it is statistically continuous on E.*

Proof. Although the proof follows from [11, Theorem 3] and the preceding theorem, and the fact that statistical continuity coincides with continuity, we give a direct proof for completeness. Let (x_n) be any statistically convergent sequence with $st - \lim_{k\to\infty} x_k = \ell$. Then

$$(x_1, \ell, x_1, \ell, x_2, \ell, x_2, \ell, ..., x_n, \ell, x_n, \ell, ...)$$

is also statistically convergent to ℓ. Thus it is statistically upward half quasi-Cauchy. Hence

$$(f(x_1), f(\ell), f(x_1), f(\ell), f(x_2), f(\ell), f(x_2), f(\ell), ..., f(x_n), f(\ell), f(x_n), f(\ell), ...)$$

is statistically upward half quasi-Cauchy. Now it follows that

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n : |f(x_k) - f(\ell)| \geq \varepsilon\}| = 0$$

for every $\varepsilon > 0$. This completes the proof of the theorem. $\square$

The converse of the preceding theorem is not always true; a counterexample is the function f defined by $f(x) = -x^2$ for every $x \in \mathbb{R}$.

Observing that statistical continuity implies ordinary continuity, we note that it follows from Theorem 4.3 that statistically upward continuity implies not only ordinary continuity, but also some other kinds of continuities, namely, lacunary statistically continuity, λ-statistical sequential continuity ([21]), N_θ-sequential continuity, I-continuity for any non-trivial admissible ideal I of $\mathbb{N}$ ([15, Theorem 4], [14]), and G-continuity for any regular subsequential method G (see [26], [6], and [10]).

Theorem 4.4. *Statistically upward continuous image of any statistically upward compact subset E of $\mathbb{R}$ is statistically upward compact.*

Proof. Write $y_n = f(x_n)$, where $x_n \in E$ for each $n \in \mathbb{N}$, $\mathbf{x} = (x_n)$. Statistically upward compactness of E implies that there is a statistically upward half quasi-Cauchy subsequence $\mathbf{z}$ of the sequence of $\mathbf{x}$. Write $(t_k) = f(\mathbf{z}) = (f(z_k))$. (t_k) is a statistically upward half quasi-Cauchy subsequence of the sequence $f(\mathbf{x})$. This completes the proof of the theorem. $\square$

Corollary 4.5. *Statistically upward continuous image of any below bounded subset of $\mathbb{R}$ is bounded below.*

Proof. The proof follows from Theorem 4.3 and Theorem 3.3, so is omitted. $\square$

Corollary 4.6. *Statistically upward continuous image of any N_θ-sequentially compact subset of $\mathbb{R}$ is N_θ-sequentially compact.*

Proof. Let f be a statistically upward continuous function on a subset E of $\mathbb{R}$, and A be an N_θ-sequentially compact subset of E. It follows from Theorem 4.3 that f is statistically continuous on E. Lemma 1 and Corollary 9 of [10] ensure that f is N_θ-sequentially continuous. Now the proof follows from Theorem 7 of [6]. $\square$

Theorem 4.7. *Let E be a statistically upward compact subset of $\mathbb{R}$ and let $f : E \longrightarrow \mathbb{R}$ be a statistically upward continuous function on E. Then f is uniformly continuous on E.*

Proof. Suppose that f is not uniformly continuous on E so that there exists an $\varepsilon_0 > 0$ such that for any $\delta > 0$ there are $x, y \in E$ with $|x - y| < \delta$ but $|f(x) - f(y)| \geq \varepsilon_0$. For each positive integer n, choose x_n and y_n such that $|x_n - y_n| < \frac{1}{n}$, and $|f(x_n) - f(y_n)| \geq \varepsilon_0$. Since E is statistically upward compact, there exists a statistical upward half quasi-Cauchy subsequence (x_{n_k}) of the sequence (x_n). It is clear that the corresponding subsequence (y_{n_k}) of the sequence (y_n) is also statistically upward half quasi-Cauchy, since $(y_{n_k} - y_{n_{k+1}})$ is a sum of three statistically upward half quasi-Cauchy sequences, i.e.

$$y_{n_k} - y_{n_{k+1}} = (y_{n_k} - x_{n_k}) + (x_{n_k} - x_{n_{k+1}}) + (x_{n_{k+1}} - y_{n_{k+1}}).$$

Then the sequence

$$(x_{n_1}, y_{n_1}, x_{n_2}, y_{n_2}, x_{n_3}, y_{n_3}, ..., x_{n_k}, y_{n_k}, ...)$$

is statistically upward half quasi-Cauchy since the sequence $(x_{n_k} - y_{n_{k+1}})$ is statistically upward half quasi-Cauchy which follows from the equality

$$x_{n_k} - y_{n_{k+1}} = x_{n_k} - x_{n_{k+1}} + x_{n_{k+1}} - y_{n_{k+1}}.$$

But the sequence

$$(f(x_{n_1}), f(y_{n_1}), f(x_{n_2}), f(y_{n_2}), f(x_{n_3}), f(y_{n_3}), ..., f(x_{n_k}), f(y_{n_k}), ...)$$

is not statistically upward half quasi-Cauchy. Thus f does not preserve statistically upward half quasi-Cauchy sequences. This contradiction completes the proof of the theorem. $\square$

Now we give the definition of statistically downward continuity of a real function in the following:

Definition 4.8. *A function $f : E \to \mathbb{R}$ is called statistically downward continuous on $E \subset \mathbb{R}$ if it preserves statistically downward half quasi-Cauchy sequence, i.e. the sequence $(f(x_n))$ is statistically downward half quasi-Cauchy whenever $\mathbf{x} = (x_n)$ is a statistically downward half quasi-Cauchy sequence of points in E.*

It should be noted that statistically downward continuity cannot be given by any A-continuity in the manner of [26].

We note that the sum of two statistically downward continuous functions is statistically downward continuous and the composition of two statistically downward continuous functions is statistically downward continuous. Analogously to the cases of statistically upward continuity, we consider the following types of continuity of functions on $\mathbb{R}$:

(δS^-) $(x_n) \in \Delta S^- \Rightarrow (f(x_n)) \in \Delta S^-$,

$(\delta S^- c)$ $(x_n) \in \Delta S^- \Rightarrow (f(x_n)) \in c$,

$(c\delta S^-)$ $(x_n) \in c \Rightarrow (f(x_n)) \in \Delta S^-$.

We see that (δS^-) is statistically downward continuity of f. It is easy to see that $(\delta S^- c)$ implies (δS^-); (δS^-) does not imply $(\delta S^- c)$; (δS^-) implies $(c\delta S^-)$; $(c\delta S^-)$ does not imply (δS^-); $(\delta S^- c)$ implies (c); (c) does not imply $(\delta S^- c)$, and (c) implies $(c\delta S^-)$.

Now we give the implication (δS^-) implies (δS), i.e. any statistically downward continuous function is statistically ward continuous.

Theorem 4.9. *If f is statistically downward continuous on a subset E of $\mathbb{R}$, then it is statistically ward continuous on E.*

Proof. The proof can be obtained using similar techniques to that of Theorem 4.2, so is omitted. $\square$

We note that the converse of the preceding theorem is not always true, i.e. there are statistically ward continuous functions which are not statistically downward continuous. As a counter example we can consider the function f defined by $f(x) = -x$ for every $x \in \mathbb{R}$.

Now we give the implication (δS^-) implies (c), i.e. any statistically downward continuous function is statistically continuous.

Theorem 4.10. *If f is statistically downward continuous on a subset E of $\mathbb{R}$, then it is statistically continuous on E.*

Proof. The proof follows from [11, Theorem 3], the preceding theorem, and the fact that statistical continuity coincides with continuity, so is omitted. $\square$

We note that the converse of the preceding theorem is not always true. As a counter example we can consider the function f defined by $f(x) = x^2$ for every $x \in \mathbb{R}$.

Recalling that statistical continuity coincides with ordinary continuity, we note that it follows from Theorem 4.10 that statistically downward continuity implies not only ordinary continuity, but also some other kinds of continuities, namely, lacunary statistically continuity, N_θ-sequential continuity, I-continuity for any non-trivial admissible ideal I of $\mathbb{N}$ ([15, Theorem 4]), and G-continuity for any regular subsequential method G (see [26], [6], and [10]).

Theorem 4.11. *Statistically downward continuous image of any statistically downward half compact subset of* ℝ *is statistically downward half compact.*

Proof. The proof of the theorem can be obtained using a similar method to that of Theorem 4.4, so is omitted. □

Corollary 4.12. *Statistically downward continuous image of any compact subset of* ℝ *is compact.*

Theorem 4.13. *Let E be a statistically downward compact subset E of* ℝ *and let f : E ⟶* ℝ *be a statistically downward continuous function on E. Then f is uniformly continuous on E.*

Proof. The proof can be done by using similar techniques to those of Theorem 4.7, so is omitted. □

5. Conclusion

In this paper, we introduce and investigate not only statistically upward and statistically downward continuities of real functions, but also some other kinds of continuities defined via statistically upward and statistically downward half quasi-Cauchy sequences. We also investigate necessary and sufficient conditions for a subset of ℝ to be below bounded and necessary and sufficient conditions for a subset of ℝ to be above bounded. We prove results related to these newly defined continuities, newly defined compactness, and some other kinds of continuities and compactness; namely slowly oscillating continuity, statistical ward continuity, slowly oscillating compactness, statistical ward compactness, lacunary statistical ward compactness, ordinary compactness, ordinary continuity, and uniform continuity. It turns out that not only the set of statistically upward continuous functions on a below bounded subset of ℝ, but also the set of statistically downward continuous functions on an above bounded subset of ℝ is contained in the set of uniformly continuous functions. We suggest to investigate statistically upward and downward half quasi-Cauchy sequences of fuzzy points or soft points (see [19], for the definitions and related concepts in fuzzy setting, and see [1] related concepts in soft setting). We also suggest to investigate statistically upward and downward half quasi-Cauchy double sequences (see for example [30], [33], and [18] for the definitions and related concepts in the double sequences case). For another further study, we suggest to investigate statistically upward and downward half Cauchy sequences of points in an asymmetric cone metric space since in a cone metric space the notion of an statistically upward half Cauchy sequence coincides with the notion of a statistically downward half Cauchy sequence, and therefore statistically upward continuity coincides with statistically downward continuity (see [34], [22], [31], [20], and [35]).

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