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On a Property of Moduli of Smoothness and K-Functionals

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Abstract. A new property of moduli of smoothness associated to functions belonging to some certain spaces is revealed. In terms of statistical convergence, we determine the behavior of these special functions at the point $\delta = 0$. In this respect, Peetre's K-functional is also investigated.

1. Introduction

Moduli of smoothness represent important tools in obtaining quantitative estimates of the error of approximation for positive processes. There are many such special functions associated with wide classes of function spaces. Investigate their properties are useful in evaluating the rate of convergence induced by linear operators used in approximation. Among the essential properties is the right continuity in origin of a given modulus, say $\omega^*(f;\cdot)$, where f belongs to a certain space $\mathcal{S}$. More precisely, let $(\delta_n)_{n\geq 1}$ be a positive real sequence. Over time, a justified concern was to find answer to the following question

$$\text{if } \lim_n \delta_n = 0, \text{ then will result } \lim_n \omega^*(f;\delta_n) = 0? \tag{1.1}$$

For example, we refer at k-th modulus of smoothness $\omega_k(f;\cdot)$ of f ($k \in \mathbb{N}$) defined by

$$\omega_k(f;\delta) = \sup_{\substack{|h|\leq\delta \\ x,x+kh\in I}} |\Delta_h^k f(x)|, \quad \delta > 0, \tag{1.2}$$

where $f : I \to \mathbb{R}$ is a bounded real function and $\Delta_h^k f(x)$ represents the k-th difference of f with step h ($h \neq 0$) at the point $x \in I$. It satisfies $\lim_{\delta\to 0^+} \omega_k(f;\delta) = 0$ provided f belongs to $UC_B(I)$, the space of all uniformly continuous and bounded real valued functions defined on the interval I, see, e.g., the monograph [1, *Lemma 5.1.1*].

Motivation to write this note comes from the following considerations. In Approximation Theory one recent topic is the analysis of linear and positive processes by using statistical convergence, the first step being done by Gadjiev and Orhan [5]. The main idea of statistical convergence of a sequence is that the majority (in a certain sense) of its elements converges and we are not concerned in what happens to the remaining elements. The advantage of replacing the ordinary convergence by statistical convergence

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consists in the fact that the second convergence is efficient in summing divergent sequences which may have unbounded subsequences. This way, the approach provides an optimization of the technique to approximate a signal by linear and positive operators.

In this context, we are interested to give a general answer to question (1.1) replacing the classical convergence by statistical convergence. Even if it seems like a basic problem, this has not been yet treated. The property which we will highlight is useful in the study of statistical convergence for sequences of linear approximation operators to the identity operator.

2. Results

At first we recall the concept of statistical convergence that was introduced by Fast [4] and Steinhauss [10]. It is based on the notion of the asymptotic density of subsets of $\mathbb{N}$. The density of $S \subseteq \mathbb{N}$ denoted by $\delta(S)$ is given by

$$\delta(S) = \lim_n n^{-1} \sum_{j=1}^{n} \chi_S(j),$$

where χ_S stands for characteristic function of the set S. A sequence $(x_n)_{n\geq 1}$ of real numbers is said to be statistically convergent to a real number l if, for every $\varepsilon > 0$,

$$\delta(\{n \in \mathbb{N} : \, |x_n - l| \geq \varepsilon\}) = 0,$$

this limit being denoted by $st - \lim_n x_n = l$.

We answer to question (1.1) reformulated in terms of statistical convergence by using a general modulus introduced by Nishishiraho [7]. This modulus is useful to determine quantitative estimates of the rates of convergence for positive approximation processes on spaces of real continuous functions defined on a convex compact set.

Let F be a locally convex Hausdorff space over the field of real numbers and F' the space consisting of all linear functionals on F. Let X be a compact convex subset of F. Set $B(X)$, $C(X)$ the Banach lattice of all real-valued bounded, respectively continuous functions on X endowed with the supremum norm $\|\cdot\|_\infty$. The set of all restrictions of functions in F' to X is denoted by F'_X. Following [7, *Definition 1*] and [1, *page 270*], let $\{g_1, \ldots, g_k\}$ be a finite subset of F'_X. We define on $\mathbb{R}_+$ the modulus of smoothness of $f \in B(X)$ with respect to $g_1, \ldots, g_k$ as

$$\omega(f; g_1, \ldots, g_k; \delta) = \sup\Big\{|f(x) - f(y)| : (x, y) \in X\times X \text{ and } \sum_{j=1}^{k} (g_j(x) - g_j(y))^2 \leq \delta^2\Big\}. \tag{2.1}$$

The total modulus of smoothness of $f \in B(X)$ is defined on $\mathbb{R}_+$ as follows

$$\Omega(f;\delta) = \inf\Big\{\omega(f; g_1, \ldots, g_k; \delta) : k \in \mathbb{N},\, g_1, \ldots, g_k \in F'_X,\, \Big\|\sum_{j=1}^{k} g_j^2\Big\|_\infty = 1\Big\}. \tag{2.2}$$

Theorem 2.1. *Let $f \in C(X)$ and let the modulus of smoothness of f with respect to $g_1, \ldots, g_k$ be defined by (2.1). If $(\delta_n)_{n\geq 1}$ is a positive real sequence such that $st - \lim_n \delta_n = 0$, then*

$$st - \lim_n \omega(f; g_1, \ldots, g_k; \delta_n) = 0. \tag{2.3}$$

Proof. We use the following characterization of statistical convergence established by Šalát [9, *Lemma 1.1*].

A sequence $x = (x_n)_{n\geq 1}$ converges statistically to $l \in \mathbb{R}$ if and only if there exists a set $S = \{n_j : n_j < n_{j+1}, j \in \mathbb{N}\} \subset \mathbb{N}$ with the properties

$$\delta(S) = 1 \quad \text{and} \quad \lim_j x_{n_j} = l.$$

According to this criterion, since $st - \lim_n \delta_n = 0$, a strictly increasing sequence $(n_j)_{j\geq 1}$ of positive integers exists such that $\lim_j \delta_{n_j} = 0$ and the set $S = \{n_1, n_2, \ldots\}$ has full density, i.e., $\delta(S) = 1$.

Since $f \in C(X)$ and $\omega(f; g_1, \ldots, g_k; \cdot)$ is right continuous at zero, the relation $\lim_j \delta_{n_j} = 0$ implies

$$\lim_j \omega(f; g_1, \ldots, g_k; \delta_{n_j}) = 0.$$

Applying again the Šalát's result, this time for the sequence $(\omega(f; g_1, \ldots, g_k; \delta_n))_{n\geq 1}$ and the same set of indices S, we arrive at (2.3) and the proof is completed. □

Starting from the above result, we can assert

Corollary 2.2. *Let $f \in C(X)$ and let the total modulus of smoothness of f be defined by (2.2). If $(\delta_n)_{n\geq 1}$ is a positive real sequence such that $st - \lim_n \delta_n = 0$, then*

$$st - \lim_n \Omega(f; \delta_n) = 0.$$

The result of Theorem 2.1 includes the following special case. Choosing $F = \mathbb{R}^k$, $X = I_1 \times \ldots \times I_k$ a k-dimensional interval, $g_i = pr_i : \mathbb{R}^k \to \mathbb{R}$ the i-th canonical projection map, $i = \overline{1,k}$, the modulus of smoothness of $f \in B(X)$ with respect to $pr_1, \ldots, pr_k$ is given by

$$\omega(f; pr_1, \ldots, pr_k; \delta) = \sup\{|f(x) - f(y)| : (x, y) \in X \times X, \|x - y\|_{\mathbb{R}^k} < \delta\},$$

in brief $\omega(f; \delta)$, where $\|\cdot\|_{\mathbb{R}^k}$ is the Euclidian distance in $\mathbb{R}^k$. For $k = 1$, the property established in this theorem refers to the one-dimensional classical modulus of continuity, see (1.2) with $k = 1$ and $I = I_1$. In one-dimensional case this property was often used. Best of our knowledge, it has appeared for the first time in [3].

So called Peetre's K-functional introduced in [8] represents another way for measuring the smoothness of a function in terms of how well it can be approximated by smoother functions.

We consider the space $L_p(I)$, $1 \leq p < \infty$, $I = (a, b)$, where $a = -\infty$ or $a = 0$ and $n = 0$ or $b = \infty$. Let φ be a compatible step-weight function, subject to certain restrictions. The K-functional is given by

$$K_{r,\varphi}(f; \delta)_p = \inf_g \{\|f - g\|_p + \delta \|\varphi^r g^{(r)}\|_p : g^{(r-1)} \in A.C._{loc}\}, \tag{2.4}$$

$\delta \geq 0$, where $g^{(r-1)} \in A.C._{loc}$ means that g is $(r-1)$−times differentiable and $g^{(r-1)}$ is absolutely continuous in every compact interval J such that $J \subset I$.

Theorem 2.3. *If $(\delta_n)_{n\geq 1}$ is a positive real sequence such that $st - \lim_n \delta_n = 0$, then the K-functional defined by (2.4) satisfies*

$$st - \lim_n K_{r,\varphi}(f; \delta_n)_p = 0.$$

Proof. In the proof of this limit we turn to the connection between this functional and Ditzian-Totik modulus of smoothness $\omega_\varphi^r(f; \cdot)_p$ given by

$$\omega_\varphi^r(f; \delta)_p = \sup_{0<h\leq\delta} \|\Delta_{h\varphi}^r f\|_p,$$

see [2, *Eq. (2.1.2)]*, where $\Delta^r_{h\varphi} f$ represents the r-th central difference of the function f, i.e.,

$$\Delta^r_{h\varphi(x)} f(x) = \sum_{k=0}^{r} (-1)^k \binom{r}{k} f\left(x + \left(\frac{r}{2} - k\right) h\varphi(x)\right).$$

If $x + rh\varphi(x)/2$ or $x - rh\varphi(x)/2$ do not belong to I, then we consider $\Delta^r_{h\varphi(x)} f(x) = 0$.

According to [2, *Theorem 2.1.1*], one has

$$M^{-1}\omega^r_\varphi(f;\delta)_p \le K_{r,\varphi}(f;\delta^r)_p \le M\omega^r_\varphi(f;\delta)_p, \quad 0 < t \le t_0, \tag{2.5}$$

for some constants $M > 0$ and t_0. Let $st - \lim_n \delta_n = 0$. Since

$$\lim_{\delta\to 0^+} \omega^r_\varphi(f;\delta)_p = 0 \text{ for all } f \in L_p(I) \text{ if } 1 \le p < \infty,$$

following the same technique as in the proof of Theorem 2.1, we get

$$st - \lim_n \omega^r_\varphi(f;\delta_n)_p = 0.$$

Relation (2.5) leads us to the desired result. □

Remark 2.4. *Statistical convergence implies statistical σ-convergence, a concept based on invariant mean and recently introduced [6]. Consequently the proven statements of this section are also valid for this new type of convergence.*

References

[1] F. Altomare, M. Campiti, Korovkin-type Approximation Theory and its Applications, de Gruyter Studies in Mathematics, Vol. 17, Walter de Gruyter, Berlin, 1994.

[2] Z. Ditzian, V. Totik, Moduli of Smoothness, Springer Series in Computational Mathematics, Vol. 9, Springer Verlag, New York Inc., 1987.

[3] O. Duman, M.K. Khan, C. Orhan, *A-Statistical convergence of approximating operators*, Math. Inequal. Appl., **6**(2003), 689-699.

[4] H. Fast, Sur le convergence statistique, Colloq. Math. 2 (1951) 241-244.

[5] A.D. Gadjiev, C. Orhan, Some approximation theorems via statistical convergence, Rocky Mountain J. Math. 32 (2002) 129-138.

[6] M. Mursaleen, O.H.H. Edely, On invariant mean and statistical convergence, Appl. Math. Lett. 22 (2009) 1700-1704.

[7] T. Nishishiraho, The degree of convergence of positive linear operators. Tôhoku Math. Journal 29 (1977) 81-89.

[8] J. Peetre, A Theory of Interpolation of Normed Spaces, Notas de Matemática, Instituto de Matemática Pura e Aplicada, Rio de Janeiro 39 (1968) 1-86.

[9] T. Šalát, On statistically convergent sequences of real numbers, Math. Slovaca 30 (1980) 139-150.

[10] H. Steinhauss, Sur la convergence ordinaire et la convergence asymptotique, Colloq. Math. 2 (1951) 73-74.