Improved Chen Inequality of Sasakian Space Forms
with the Tanaka-Webster Connection

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Abstract. In this paper, we establish an improved Chen inequality between the pseudo-Ricci curvature and the square of pseudo mean curvature with respect to the Tanaka-Webster connection in Sasakian space forms, and also we study an improved Chen inequality for anti-invariant submanifolds. The equality case is considered.

1. Introduction

One of the basic interests in the submanifold theory is to establish simple relationship between intrinsic invariants and extrinsic invariants of a submanifold. Gauss-Bonnet theorem, isoperimetric inequality and Chern-Lashof theorem are those such kind of study.

B.-Y. Chen ([2]) established a nice basic inequality related the Ricci curvature and the squared mean curvature $\|H\|^2$ of submanifolds in a real space form. The inequality drew attention of several authors and they established similar inequalities for different kind of submanifolds in ambient manifolds possessing different kind of structures. The submanifolds included mainly invariant, anti-invariant and slant submanifolds.
In 2005, T. Oprea [6] proved Chen inequality by using optimization techniques applied in the setup of Riemannian geometry. He also improved Chen inequality in terms of the Ricci curvature and the squared mean curvature for Lagranian submanifolds of complex space formes. On the other hand, for the above mentioned content, S. Deng [4] proved the improved Chen inequality for Lagrangian submanifolds of complex space forms just by using some crucial algebraic inequalities and also discussed the equality case, which is not discussed in Oprea's paper. Furthermore, M. M. Tripathi [9] proved Chen inequality and improved Chen inequality for curvature like tensors. He also applied these inequalities to Lagrangian and Kaehlerian slant submanifolds of complex space forms, and C-totally real submanifolds of Sasakian space forms.
Recently, D. H. Jin and the second author [5] investigated a basic inequality in Sasakian space forms with the Tanaka-Webster connection.

In this paper, we define a pesudo-Ricci curvature for the Tanaka-Webster connection in a Sasakian space form. After then, we study the relationship of inequalities for submanifolds of a Sasakian space form in

terms of constant pseudo-sectional curvature and a pseudo-Ricci curvature. We also investigate the equality case of the inequality.

2. Half Lightlike Submanifolds

Let $\widetilde{M}$ be an odd-dimensional Riemannian manifold with a Riemannian metric $\tilde{g}$ satisfying

$$\eta(\xi) = 1, \quad \varphi^2 = -I + \eta \otimes \xi, \quad \eta(X) = \tilde{g}(X, \xi)$$

$$\tilde{g}(\varphi X, \varphi Y) = \tilde{g}(X, Y) - \eta(X)\eta(Y). \tag{2.1}$$

Then $(\varphi, \xi, \eta, \tilde{g})$ is called the almost contact metric structure on $\widetilde{M}$. Let Φ denote the fundamental 2-form in $\widetilde{M}$ given by $\Phi(X, Y) = \tilde{g}(X, \varphi Y)$ for all $X, Y \in T\widetilde{M}$. If $\Phi = d\eta$, then $\widetilde{M}$ is said to be a contact metric manifold. Moreover, if ξ is a Killing vector field with respect to $\tilde{g}$, the contact metric structure is called a $\mathcal{K}$-contact structure. Recall that a contact metric manifold is $\mathcal{K}$-contact if and only if

$$\widetilde{\nabla}_X \xi = -\varphi X \tag{2.2}$$

for any $X \in T\widetilde{M}$, where $\widetilde{\nabla}$ is the Levi-Civita connection of $\widetilde{M}$. The structure of $\widetilde{M}$ is said to be normal if $[\varphi, \varphi] + 2d\eta \otimes \xi = 0$, where $[\varphi, \varphi]$ is the Nijenhuis torsion of φ. A Sasakian manifold is a normal contact metric manifold. In fact, an almost contact metric structure is Sasakian if and only if

$$(\widetilde{\nabla}_X \varphi)Y = \tilde{g}(X, Y)\xi - \eta(Y)X \tag{2.3}$$

for all vector fields X and Y. Every Sasakian manifold is a $\mathcal{K}$-contact manifold.

Given a Sasakian manifold $\widetilde{M}$, a plane section π in $T_p\widetilde{M}$ is called a φ-section if it is spanned by X and φX, where X is a unit tangent vector field orthogonal to ξ. The sectional curvature $\widetilde{K}(\pi)$ of a φ-section π is called φ-sectional curvature. If a Sasakian manifold $\widetilde{M}$ has constant φ-sectional curvature c, $\widetilde{M}$ is called a Sasakian space-form, denoted by $\widetilde{M}(c)$ (For more details, see [1]).

Now let M be a submanifold immersed in $(\widetilde{M}, \varphi, \xi, \eta, \tilde{g})$. We denote by g the induced metric on M. Let TM be the Lie algebra of vector fields in M and $T^\perp M$. We denote by h the second fundamental form of M and by A_v the Weingarten endomorphism associated with any $v \in T^\perp M$. We put $h_{ij}^r = \tilde{g}(h(e_i, e_j), e_r)$ for any orthonormal vector $e_i, e_j \in TM$ and $e_r \in T^\perp M$. The mean curvature vector field H is defined by $H = \frac{1}{dimM} trace(h)$. M is said to be totally geodesic if the second fundamental form vanishes identically.

From now on, we assume that the dimension of M is $n + 1$ and that of the ambient manifold $\widetilde{M}$ is $2n + 1(n \geq 2)$. We also assume that the structure vector field ξ is tangent to M. Hence, if we denote by D the orthogonal distribution to ξ in TM, we have the orthogonal direct decomposition of TM by $TM = D \oplus span\{\xi\}$. For any $X \in TM$, we write $\varphi X = TX + NX$, where TX (NX, resp.) is the tangential (normal, resp.) component of φX. If M is a $\mathcal{K}$-contact manifold, (2.2) gives

$$h(X, \xi) = -NX, \tag{2.4}$$

for any X in TM. The submanifold M is said to be invariant if N is identically zero, that is, $\varphi X \in TM$ for any $X \in TM$. On the other hand, M is said to be an anti-invariant submanifold if T is identically zero, that is, $\varphi X \in T^\perp M$ for any $X \in TM$.

3. The Tanaka-Webster Connection for Sasakian Space Form

The Tanaka-Webster connection [7, 10] is the canonical affine connection defined on a non-degenerate pseudo-Hermitian CR-manifold. Tanno [8] defined the Tanaka-Webester connection for contact metric manifolds by the canonical connection which coincides with the Tanaka-Webester connection if the associated

CR-structure is integrable. We define the Tanaka-Webster connection for submanifolds of Sasakian manifolds by the naturally extended affine connection of Tanno's Tanaka-Webster connection. Now we recall the Tanaka-Webster connection $\widehat{\nabla}$ for contact metric manifolds:

$$\widehat{\nabla}_X Y = \widetilde{\nabla}_X Y + \eta(X)\varphi Y + (\widetilde{\nabla}_X \eta)(Y)\xi - \eta(Y)\widetilde{\nabla}_X \xi,$$

for all vector fields $X, Y \in T\widetilde{M}$ [3]. Together with (2.1), $\widehat{\nabla}$ is written by

$$\widehat{\nabla}_X Y = \widetilde{\nabla}_X Y + \eta(X)\varphi Y + \eta(Y)\varphi X + \tilde{g}(X, \varphi(Y))\xi. \tag{3.1}$$

Also, by using (2.1)~(2.3), we can see that

$$\widehat{\nabla}\eta = 0, \quad \widehat{\nabla}\xi = 0, \quad \widehat{\nabla}\varphi = 0, \quad \widehat{\nabla}\tilde{g} = 0. \tag{3.2}$$

We define the Tanaka-Webster curvature tensor of $\widehat{R}$ (in terms of $\widehat{\nabla}$) by

$$\widehat{R}(X, Y)Z = \widehat{\nabla}_X \widehat{\nabla}_Y Z - \widehat{\nabla}_Y \widehat{\nabla}_Y Z - \widehat{\nabla}_{[X,Y]} Z,$$

for all vector fields X, Y, Z in $T\widetilde{M}$.

Let $\widetilde{M}(c)$ be a Sasakian space form of constant sectional curvature c. Then we have the following Gauss' equation:

$$\begin{aligned}
\widehat{R}(X, Y)Z &= \frac{c+3}{4}[\{g(Y, Z) - \eta(Y)\eta(Z)\}X - \{g(X, Z) - \eta(X)\eta(Z)\}Y \\
&\quad + \{g(X, Z)\eta(Y) - g(Y, Z)\eta(X)\} + g(\varphi Y, Z)\varphi X\xi \\
&\quad - g(\varphi X, Z)\varphi Y - 2g(\varphi X, Y)\varphi Z]
\end{aligned} \tag{3.3}$$

for any tangent vector fields X, Y, Z tangent to $\widetilde{M}(c)$.

Let (M, g) be a submanifold of $\widetilde{M}(c)$ with the induced metric g and define the connection $\mathring{\nabla}$ on M induced from the Tanaka-Webster connection $\widehat{\nabla}$ on $\widetilde{M}$ given by

$$\widehat{\nabla}_X Y = \mathring{\nabla}_X Y + \hat{h}(X, Y), \quad \widehat{\nabla}_X V = -\mathring{A}_V X + \hat{D}_X V \tag{3.4}$$

for any $X, Y \in \Gamma(TM)$, where $\hat{h}$ is called the lightlike second fundamental form of M with respect to the induced connection $\mathring{\nabla}$ and $\hat{D}$ is the normal connection. Then the lightlike second fundamental form $\hat{h}$ is related to $\mathring{A}_V$ by

$$\tilde{g}(\hat{h}(X, Y), V) = \tilde{g}(\mathring{A}_V X, Y).$$

The pseudo-mean curvature vector field $\widehat{H}$ is defined by

$$\widehat{H} = \frac{1}{dimM} trace(\hat{h}).$$

M is said to be totally pseudo-geodesic if the second fundamental form $\hat{h}$ vanishes identically. In the view of (3.1) and (3.4),

$$\mathring{\nabla}_X Y + \hat{h}(X, Y) = \nabla_X Y + h(X, Y) + \eta(X)\varphi Y + \eta(Y)\varphi X - \bar{g}(Y, \varphi X)\xi. \tag{3.5}$$

From (3.5), we obtain

$$\mathring{\nabla}_X Y = \nabla_X Y + \eta(X)TY + \eta(Y)TX - \bar{g}(Y, \varphi X)\xi, \tag{3.6}$$

$$\hat{h}(X, Y) = h(X, Y) + \eta(X)NY + \eta(Y)NX, \tag{3.7}$$

where $\varphi X = TX + NX$.

From (3.2), (3.6) and (3.7), it is easy to verify the following:

$$\mathring{\nabla}\eta = 0, \quad \mathring{\nabla}\xi = 0, \quad \mathring{\nabla}\varphi = 0, \quad \mathring{\nabla}g = 0. \tag{3.8}$$

From the definition of $\widehat{R}$, together with (3.3), we have

$$
\begin{aligned}
g(\mathring{R}(X,Y)Z,W) \;=\; & \frac{c+3}{4}[\{g(Y,Z) - \eta(Y)\eta(Z)\}g(X,W) \\
& -\{g(X,Z) - \eta(X)\eta(Z)\}g(Y,W) \\
& +\{g(X,Z)\eta(Y) - g(Y,Z)\eta(X)\}\eta(W) \\
& +g(\varphi Y,Z)g(\varphi X,W) - g(\varphi X,Z)g(\varphi Y,W) \\
& -2g(\varphi X,Y)g(\varphi Z,W)] \\
& +\bar{g}(\widehat{h}(X,W),\widehat{h}(Y,Z)) - \bar{g}(\widehat{h}(X,Z),\widehat{h}(Y,W)),
\end{aligned}
\tag{3.9}
$$

for any $X, Y, Z, W \in TM$.

Let M^{n+1} be an anti-invariant submanifold of a $(2n+1)$-dimensioinal manifold $\tilde{M}(c)$ whose characteristic vector field ξ is tangent to M. We choose a local orthonormal frame field in $\tilde{M}(c)$:

$$\{e_1,\cdots,e_n,e_{n+1}=\xi;\phi e_1,\cdots,\phi e_n\},$$

where $e_1,\cdots,e_n,e_{n+1}=\xi$ are tangent to M. Then for any r we have

$$\hat{h}_{ij}^r = \hat{h}_{rj}^i, \quad i,j \in \{1,\cdots,n\}, \tag{3.10}$$

$$\hat{h}_{in+1}^r = 0, \quad i \in \{1,\cdots,n+1\}, \tag{3.11}$$

where $\hat{h}_{ij}^r$ is the ϕe_r component of the vector $\hat{h}(e_i,e_j)$.

Let M^{n+1} be a Riemannian $(n+1)$-manifold and X be a unit vector. We choose an orthonormal frame $\{e_1,\cdots,e_{n+1}\}$ in T_xM such that $e_1 = X$. We denote the pseudo-Ricci curvature at X by

$$\mathring{Ric}(X) = \mathring{K}_{12} + \cdots + \mathring{K}_{1n+1},$$

where $\mathring{K}_{ij}$ denotes the pseudo-sectional curvature of the 2-plane section spanned by e_i, e_j.

We re-format the following lemmas from [4].

Lemma 3.1. *Let $(x_1,\cdots,x_n)$ be a point in $\mathbb{R}^n$. If $x_1 + \cdots + x_n = (n+1)a$, we have*

$$x_1^2 + \cdots + x_n^2 \geq \frac{(n+1)^2}{n}a^2.$$

The equality sign holds if and only if $x_1 = \cdots = x_n = \frac{n+1}{n}a$.

Proof. $\mathbb{R}^n$. If $x_1 + \cdots + x_n = (n+1)a$ is a plane tangent to the sphere $x_1^2 + \cdots + x_n^2 = \frac{(n+1)^2}{n}a^2$ at the point $\frac{n+1}{n}(a,a,\cdots,a)$. The proof is complete from the face that the distance between any point in the plane and the origin is bigger than or equal to the radius of the sphere and the minimum occurs at the point $\frac{n+1}{n}(a,a,\cdots,a)$. $\square$

Lemma 3.2. *Let $f_1(x_1,\cdots,x_n)$ be a function in $\mathbb{R}^n$ defined by*

$$f_1(x_1,\cdots,x_n) = x_1 \sum_{j=2}^{n} x_j - \sum_{j=2}^{n} x_j^2.$$

If $x_1 + x_2 + \cdots + x_n = 2(n + 1)a$, we have

$$f_1(x_1, \cdots, x_n) \leq \frac{n-1}{4n} \left(\sum_{j=1}^{n} x_j \right)^2.$$

The equality holds if and only if $\frac{1}{n+1}x_1 = x_2 = \cdots = x_n = \frac{n+1}{n}a$.

Proof. From $x_1 + x_2 + \cdots + x_n = 2(n + 1)a$, we have

$$[x_1 - (n + 1)a] + x_2 + \cdots + x_n = (n + 1)a.$$

From Lemma 3.1, we have

$$[x_1 - (n + 1)a]^2 + x_2^2 + \cdots + x_n^2 \geq \frac{(n+1)^2}{n}a^2,$$

with the equality holds if and only if $\frac{1}{n+1}x_1 = x_2 = \cdots = x_n = \frac{n+1}{n}a$. Therefore, we have

$$(n+1)^2 a^2 - \frac{(n+1)^2}{n}a^2 \geq x_1 \left[2(n+1)a - x_1 \right] - \sum_{j=2}^{n} x_j.$$

In other words,

$$x_1 \sum_{j=2}^{n} x_j - \sum_{j=2}^{n} x_j \leq \frac{(n-1)(n+1)^2}{n}a^2 = \frac{n-1}{4n} \left(\sum_{j=1}^{n} x_j \right)^2.$$

$\square$

Lemma 3.3. *Let $f_2(x_1, \cdots, x_n)$ be a function in $\mathbb{R}^n$ defined by*

$$f_2(x_1, \cdots, x_n) = x_1 \sum_{j=2}^{n} x_j - x_1^2.$$

If $x_1 + x_2 + \cdots + x_n = 4a$, we obtain

$$f_2(x_1, \cdots, x_n) \leq \frac{1}{8} \left(\sum_{j=1}^{n} x_j \right)^2.$$

The equality holds if and only if $x_1 = a, x_2 = \cdots = x_n = 3a$.

Proof. Let $u = x_1$ and $v = x_2 + \cdots + x_n - 2a$. We get

$$u + v = x_1 + x_2 + \cdots + x_n - 2a = 2a = 3\left(\frac{2}{3}a \right).$$

By Lemma 3.1., we have

$$u^2 + v^2 = x_1^2 + (x_2 + x_3 + \cdots + x_n - 2a)^2 \geq \frac{9}{2}\left(\frac{2}{3}a \right)^2 = 2a^2,$$

where the equality holds if and only if $x_1 = (\frac{3}{2})(\frac{2}{3}a) = a, x_2 + \cdots + x_n = 3a$. Therefore we obtain

$$(x_2 + \cdots + x_n)^2 - 4a(x_2 + \cdots + x_n) + 4a^2 \geq 2a^2 - x_1^2,$$

i.e.,

$$(x_2 + \cdots + x_n)\left[(x_2 + \cdots + x_n) - 4a\right] + 2a^2 \geq -x_1^2.$$

Since $x_1 = 4a - (x_2 + \cdots + x_n)$, we finally have

$$x_1(x_2 + \cdots + x_n) - x_1^2 \leq 2a^2 = \frac{1}{8}(x_1 + x_2 + \cdots + x_n)^2.$$

□

4. An Improved inequality for Pseudo-Ricci Curvature with Respect to the Tanaka-Webster Connection

Theorem 4.1. *Let M be an $(n + 1)$-dimensional anti-invariant submanifold of a Sasakian space form $\tilde{M}^{2n+1}(c)$. Assume that the characteristic vector field ξ is tangent to M. Let x be a point in M and X a unit tangent vector field in $T_x M$. The we have*

$$\overset{\circ}{Ric}(X) \leq \frac{(n-1)(n+1)^2}{4n}\|\hat{H}\|^2 + \frac{(n-1)(c+3)}{4}. \tag{4.1}$$

where $\hat{H}$ is the pseudo mean curvature of M in $\tilde{M}(c)$ for a Tanaka-Webster connection and $\overset{\circ}{Ric}(X)$ is the pseudo-Ricci curvature of M at x in terms of the Tanaka-Webster connection. The equality holds for any unit tangent vector at x if and only if either
(i) x is a totally pseudo-geodesic point or
(ii) $n = 2$ and

$$\hat{h}(e_1, e_1) = 3\mu\phi e_1, \quad \hat{h}(e_2, e_2) = \mu\phi e_1, \quad \hat{h}(e_1, e_2) = \mu\phi e_2,$$

for some function μ with respect to some orthonormal local frame field.

Proof. Fixed the point x in M, let X be any unit tangent vector at x. We choose an orthonormal frame $\{e_1, \cdots, e_n, e_{n+1}\}$ in $T_x M$ such that $e_1 = X$, $e_{n+1} = \xi$ and $\{\phi e_1, \cdots, \phi e_n\}$ an orthonormal frame in $T_x^{\perp} M$. From (3.9), we have

$$\overset{\circ}{R}(e_1, e_j, e_j, e_1) = \frac{c+3}{4}\left[1 - \left(\delta_r^{n+1}\right)^2\right] + \sum_{r=1}^{n}\left[\hat{h}_{11}^r \hat{h}_{jj}^r - \left(\hat{h}_{1j}^r\right)^2\right].$$

Hence, from (3.11), we have

$$
\begin{aligned}
\overset{\circ}{Ric}(X) - \frac{(n-1)(c+3)}{4} &= \sum_{j=2}^{n+1}\sum_{r=1}^{n}\left[\hat{h}_{11}^r \hat{h}_{jj}^r - \left(\hat{h}_{1j}^r\right)^2\right] \\
&= \sum_{j=2}^{n}\sum_{r=1}^{n}\left[\hat{h}_{11}^r \hat{h}_{jj}^r - \left(\hat{h}_{1j}^r\right)^2\right] \\
&\leq \sum_{j=2}^{n}\sum_{r=1}^{n}\hat{h}_{11}^r \hat{h}_{jj}^r - \sum_{j=2}^{n}\left(\hat{h}_{1j}^1\right)^2 - \sum_{j=2}^{n}\left(\hat{h}_{1j}^j\right)^2
\end{aligned}
\tag{4.2}
$$

Using (3.10), we have

$$
\begin{aligned}
\overset{\circ}{Ric}(X) - \frac{(n-1)(c+3)}{4} \;\leq\;& \sum_{j=2}^{n}\sum_{r=1}^{n} \hat{h}_{11}^{r}\hat{h}_{jj}^{r} - \sum_{j=2}^{n}\left(\hat{h}_{1j}^{1}\right)^2 - \sum_{j=2}^{n}\left(\hat{h}_{1j}^{j}\right)^2 \\
=\;& \sum_{r=1}^{n}\sum_{j=2}^{n} \hat{h}_{11}^{r}\hat{h}_{jj}^{r} - \sum_{j=2}^{n}\left(\hat{h}_{1j}^{1}\right)^2 - \sum_{j=2}^{n}\left(\hat{h}_{jj}^{1}\right)^2 \\
=\;& \hat{h}_{11}^{1}\sum_{j=2}^{n}\hat{h}_{jj}^{1} - \sum_{j=2}^{n}\left(\hat{h}_{jj}^{1}\right)^2 \\
&+ \sum_{r=2}^{n}\left[\hat{h}_{11}^{r}\sum_{j=2}^{n}\hat{h}_{jj}^{r} - \left(\hat{h}_{11}^{r}\right)^2\right].
\end{aligned}
\tag{4.3}
$$

Now, we assume that

$$
f_1(\hat{h}_{11}^1,\hat{h}_{22}^1,\cdots,\hat{h}_{nn}^1) = \hat{h}_{11}^1\sum_{j=2}^{n}\hat{h}_{jj}^1 - \sum_{j=2}^{n}\left(\hat{h}_{jj}^1\right)^2,
$$

$$
f_r(\hat{h}_{11}^r,\hat{h}_{22}^r,\cdots,\hat{h}_{nn}^r) = \hat{h}_{11}^r\sum_{j=2}^{n}\hat{h}_{jj}^r - \left(\hat{h}_{11}^r\right)^2, \quad r\in\{2,\cdots,n\}
$$

Since $(n+1)\hat{H}^1 = \hat{h}_{11}^1 + \hat{h}_{22}^1 + \cdots + \hat{h}_{nn}^1$, by Lemma 3.2, we have

$$
\begin{aligned}
f_1(\hat{h}_{11}^1,\hat{h}_{22}^1,\cdots,\hat{h}_{nn}^1) \;\leq\;& \frac{n-1}{4n}\left((n+1)\hat{H}^1\right)^2 \\
=\;& \frac{(n-1)(n+1)^2}{4n}\left(\hat{H}^1\right)^2.
\end{aligned}
\tag{4.4}
$$

Similarly, by Lemma 2.3, we have for $2\leq r\leq n$,

$$
\begin{aligned}
f_r(\hat{h}_{11}^r,\hat{h}_{22}^r,\cdots,\hat{h}_{nn}^r) \;\leq\;& \frac{1}{8}\left((n+1)\hat{H}^r\right)^2 \\
=\;& \frac{(n+1)^2}{8}\left(\hat{H}^r\right)^2 \\
\leq\;& \frac{(n-1)(n+1)^2}{4n}\left(\hat{H}^r\right)^2.
\end{aligned}
\tag{4.5}
$$

From (4.3), (4.4) and (4.5), we have

$$
\overset{\circ}{Ric}(X) - \frac{(n-1)(c+3)}{4} \leq \frac{(n-1)(n+1)^2}{4n}\sum_{r=1}^{r}\left(\hat{H}^r\right)^2 = \frac{(n-1)(n+1)^2}{4n}\|\hat{H}\|^2.
$$

Now we assume that $n\geq 3$ and the equality of (4.1) holds for any unit tangent vector X at x. By (4.5), we have $\hat{H}^r = 0$ for $r\geq 2$ (or simply choose ϕe_1 parallel to $\hat{H}$). Combining this and Lemma 2.3, we have

$$
\hat{h}_{1j}^1 = \hat{h}_{11}^j = \frac{(n+1)}{4}\hat{H}^j = 0, \quad \forall j\geq 2.
$$

From (4.2), we have $\hat{h}_{jk}^1 = 0$, $\forall i,k\geq 2$, $j\neq k$. From the equality and Lemma 3.2, $(\hat{h}_{ij}^1)$ must be diagonal with $\hat{h}_{11}^1 = \frac{(n+1)^2}{2n}\hat{H}^1$ and $\hat{h}_{jj}^1 = \frac{n+1}{2n}\hat{H}^1$, $\forall j\geq 2$. Now if we compute $\overset{\circ}{Ric}(e_2)$ as we do for $\overset{\circ}{Ric}(X) = \overset{\circ}{Ric}(e_1)$ in (4.2), from the equality, we get $\hat{h}_{2j}^r = \hat{h}_{jr}^2 = 0$, $\forall r\neq 2$, $j\neq 2$, $r\neq j$. From the equality and Lemma 2.2, we get

$$
\frac{\hat{h}_{11}^2}{n+1} = \hat{h}_{22}^2 = \cdots = \hat{h}_{nn}^2 = \frac{n+1}{2n}\hat{H}^2 = 0.
$$

Since the equality holds for all unit tangent vector, the argument is also true for matrices $(\hat{h}^r_{ij})$. Now finally $\hat{h}^2_{2j} = \hat{h}^j_{22} = \frac{n+1}{2n}\hat{H}^2 = 0$, $\forall j \geq 3$. Therefore, matrix $(\hat{h}^2_{jk})$ has only two possible nonzero entries (i.e., $\hat{h}^2_{12} = \hat{h}^2_{21} = \hat{h}^1_{22} = \frac{n+1}{2n}\hat{H}^1$). Similarly, matrix $(\hat{h}^r_{jk})$ has only two possible nonzero entries

$$\hat{h}^r_{1r} = \hat{h}^r_{r1} = \hat{h}^1_{rr} = \frac{n+1}{2n}\hat{H}^1, \ \forall r \geq 3.$$

Now, we compute $\mathring{Ric}(e_2)$ as follows:
For $j = n + 1$, using (3.11),

$$\mathring{R}(e_2, e_{n+1}, e_{n+1}, e_2) = 0. \tag{4.6}$$

For all j such that $3 \leq j \leq n$,

$$
\begin{aligned}
\mathring{R}(e_2, e_j, e_j, e_2) &= \frac{c+3}{4} + \sum_{r=1}^{n}\left[\hat{h}^r_{22}\hat{h}^r_{jj} - \left(\hat{h}^r_{2j}\right)^2\right] \\
&= \frac{c+3}{4} + \left(\hat{h}^1_{22}\right)^2 \\
&= \frac{c+3}{4} + \left(\frac{n+1}{2n}\hat{H}^1\right)^2, \ \forall 3 \leq j \leq n.
\end{aligned}
\tag{4.7}
$$

For the case of $j = 1$,

$$
\begin{aligned}
\mathring{Ric}(e_2, e_1, e_1, e_2) &= \frac{c+3}{4} + \sum_{r=1}^{n}\left[\hat{h}^r_{22}\hat{h}^r_{11} - \left(\hat{h}^r_{21}\right)^2\right] \\
&= \frac{c+3}{4} + \hat{h}^1_{22}\hat{h}^1_{11} - \left(\hat{h}^2_{21}\right)^2 \\
&= \frac{c+3}{4} + \left(\frac{n+1}{2n}\hat{H}^1\right)\left(\frac{(n+1)^2}{2n}\hat{H}^1\right) - \left(\frac{n+1}{2n}\hat{H}^1\right)^2.
\end{aligned}
\tag{4.8}
$$

By combining (4.6) ~ (4.8), we get

$$
\begin{aligned}
&\mathring{Ric}(e_2) - \frac{(n-1)(c+3)}{4} \\
&= \frac{(n+1)^3}{4n^2}\left(\hat{H}^1\right)^2 - \frac{(n+1)^2}{4n^2}\left(\hat{H}^1\right)^2 + \frac{(n-1)(n+1)^2}{4n^2}\left(\hat{H}^1\right)^2 \\
&= \frac{(2n-1)(n+1)^2}{4n^2}\left(\hat{H}^1\right)^2.
\end{aligned}
$$

On the other hand, from the assumption for the equality on the equation (4.1), we have

$$\mathring{Ric}(X) + \frac{(n-1)(c+3)}{4} = \frac{(n-1)(n+1)^2}{4n}\|\hat{H}\|^2 = \frac{(n-1)(n+1)^2}{4n}\left(\hat{H}^1\right)^2.$$

Therefore, we have

$$\frac{(2n-1)(n+1)^2}{4n^2}\left(\hat{H}^1\right)^2 = \frac{(n-1)(n+1)^2}{4n}\left(\hat{H}^1\right)^2.$$

Then we have $\hat{H}^1 = 0$, and hence $\left(\hat{h}^r_{ij}\right)$ are all zero and x is a totally pseudo-geodesic point.
For the case of $n = 2$, we have

$$\hat{h}^1_{11} = \frac{9}{4}\hat{H}^1, \ \hat{h}^1_{22} = \frac{3}{4}\hat{H}^1, \ \hat{h}^2_{11} = \hat{h}^2_{22} = 0.$$

Moreover, the lightlike second fundamental form takes the following form:

$$\hat{h}(e_1, e_1) = \frac{9}{4}\hat{H}^1 \phi e_1, \quad \hat{h}(e_2, e_2) = \frac{3}{4}\hat{H}^1 \phi e_1, \quad \hat{h}(e_1, e_2) = \frac{3}{4}\hat{H}^1 \phi e_2.$$

The converse can be proved by simple computation. □

Corollary 4.2. *Let M be an $(n + 1)$-dimensional anti-invariant submanifold of a Sasakian space form $\tilde{M}^{2n+1}(c)$. Assume that the characteristic vector field ξ is tangent to M. Let x be a point in M and X a unit tangent vector field in T_xM. If*

$$\mathring{Ric}(X) = \frac{(n-1)(n+1)^2}{4n}\|\hat{H}\|^2 - \frac{(n-1)(c+3)}{4}.$$

X in TM, where $\hat{H}$ is the pseudo mean curvature of M in $\tilde{M}(c)$ for a Tanaka-Webster connection and $\mathring{Ric}(X)$ is the pseudo Ricci curvature of M at x in terms of the Tanaka-Webster connection. Then either M is a totally pseudo-geodesic submanifold or $n = 2$ and

$$\hat{h}(e_1, e_1) = 3\mu\phi e_1, \quad \hat{h}(e_2, e_2) = \mu\phi e_1, \quad \hat{h}(e_1, e_2) = \mu\phi e_2,$$

for some function μ with respect to some orthonormal local frame field..

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