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Estimates of $(1+x)^{1/x}$ Involved in Carleman's Inequality and Keller's Limit

Cristinel Mortici[a], **X.-J. Jang**[b]

[a] *Valahia University of Târgovişte, Bd. Unirii 18, 130082, Târgovişte, Romania and Academy of Romanian Scientists, Splaiul Independenţei 54, 050094 Bucharest Romania*
[b] *Department of Mathematics, Tsinghua University, Beijing 100084, China*

Abstract. The aim of this work is to extend the results obtained by Yang Bicheng and L. Debnath in [*Some inequalities involving the constant e and an application to Carleman's inequality, J. Math. Anal. Appl., 223 (1998), 347-353*]. We present a simple proof of our new result which can be also used as a direct proof for Yang Bicheng and L. Debnath results. Finally some applications to generalized Keller's limit and further directions are provided.

1. Introduction and Motivation

Yang Bicheng and L. Debnath [4, Lemma 2.1] presented the following double inequality for every x in $0 < x \leq \frac{1}{5}$:

$$e - \frac{e}{2}x + \frac{11e}{24}x^2 - \frac{21e}{48}x^3 < (1+x)^{1/x} < e - \frac{e}{2}x + \frac{11e}{24}x^2. \quad (1)$$

Such inequalities were proven to be of great interest through the researchers, especially in the recent past, due to many practical problems where they can be applied. As example, we refer to inequality (1) which is the main tool for improving Carleman's inequality in [4]. As in [4] it is provided a long, difficult proof, we propose in this paper a simple, direct proof of (1). The proof we provide shows us that (1) is true for every real number $x \in (0, 1]$. We show our method in case of the following improvement of (1).

Theorem 1. *For every real number* $x \in (0, 1]$, *we have*

$$a(x) < (1+x)^{1/x} < b(x), \quad (2)$$

where

$$a(x) = e - \frac{e}{2}x + \frac{11e}{24}x^2 - \frac{21e}{48}x^3 + \frac{2447e}{5760}x^4 - \frac{959e}{2304}x^5$$

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Email addresses: cristinel.mortici@hotmail.com (Cristinel Mortici), yangxj@mail.tsinghua.edu.cn (X.-J. Jang)

and

$$b(x) = a(x) + \frac{959e}{2304}x^5.$$

2. The Proof of the Theorem

Proof of Theorem 1. Inequalities (2) are equivalent to $f < 0$ and $g > 0$ on $[1, \infty)$, where

$$f(x) = x\ln\left(1 + \frac{1}{x}\right) - \ln\left(1 - \frac{1}{2x} + \frac{11}{24x^2} - \frac{21}{48x^3} + \frac{2447}{5760x^4}\right) - 1$$

and

$$g(x) = x\ln\left(1 + \frac{1}{x}\right) - \ln\left(1 - \frac{1}{2x} + \frac{11}{24x^2} - \frac{21}{48x^3} + \frac{2447}{5760x^4} - \frac{959}{2304x^5}\right) - 1.$$

We have

$$f''(x) = -\frac{A(x-1)}{x^2(x+1)^2 P^2(x-1)} < 0$$

and

$$g''(x) = \frac{B(x-1)}{x^2(x+1)^2 Q^2(x-1)} > 0,$$

where

$$\begin{aligned} A(x) &= 2597\,148\,593x + 4933\,395\,316x^2 + 4673\,488\,800x^3 \\ &\quad +2204\,426\,400x^4 + 414\,288\,000x^5 + 571\,054\,913 \end{aligned}$$

$$P(x) = 17\,160x + 28\,560x^2 + 20\,160x^3 + 5760x^4 + 5447$$

$$\begin{aligned} B(x) &= 17\,596\,034\,319x + 42\,104\,671\,375x^2 + 53\,575\,389\,600x^3 \\ &\quad +38\,240\,425\,136x^4 + 14\,508\,374\,400x^5 + 2285\,212\,800x^6 + 2934\,528\,883 \end{aligned}$$

$$Q(x) = 45\,214x + 91\,440x^2 + 97\,440x^3 + 51\,840x^4 + 11\,520x^5 + 6099.$$

As f is strictly concave and g is strictly convex on $(1, \infty)$ with $f(\infty) = g(\infty) = 0$, we deduce that $f < 0$ and $g > 0$ on $(1, \infty)$. As

$$f(1) = \ln 2 - \ln\frac{5447}{5760} - 1 = -0.250\,98... < 0$$

and

$$g(1) = \ln 2 - \ln\frac{2033}{3840} - 1 = 0.329\,11... > 0,$$

the conclusion follows.□

3. Keller's limit and Generalization

As an application, a new proof of the limit

$$\lim_{n\to\infty}\left(\frac{(n+1)^{n+1}}{n^n}-\frac{n^n}{(n-1)^{n-1}}\right)=e \tag{3}$$

(also known as Keller's limit, see e.g. [3]) can be constructed. Indeed, using Theorem 1, we get

$$\begin{aligned} & (n+1)\,a\left(\frac{1}{n}\right)-nb\left(\frac{1}{n-1}\right)\\ < \quad & \frac{(n+1)^{n+1}}{n^n}-\frac{n^n}{(n-1)^{n-1}}\\ < \quad & (n+1)\,b\left(\frac{1}{n}\right)-na\left(\frac{1}{n-1}\right). \end{aligned}$$

Extreme-side expressions are rational functions in n having e as common limit, so (3) is true. Moreover,

$$\begin{aligned} & n^2\left(\left((n+1)\,a\left(\frac{1}{n}\right)-nb\left(\frac{1}{n-1}\right)\right)-e\right) \qquad (4)\\ < \quad & n^2\left(\left(\frac{(n+1)^{n+1}}{n^n}-\frac{n^n}{(n-1)^{n-1}}\right)-e\right)\\ < \quad & n^2\left(\left((n+1)\,b\left(\frac{1}{n}\right)-na\left(\frac{1}{n-1}\right)\right)-e\right). \end{aligned}$$

With some patient, or better using a computer software such as Maple, we obtain

$$n^2\left(\left((n+1)\,a\left(\frac{1}{n}\right)-nb\left(\frac{1}{n-1}\right)\right)-e\right)=\frac{U(n)}{11520n^3(n-1)^4}e \tag{5}$$

and

$$n^2\left(\left((n+1)\,b\left(\frac{1}{n}\right)-na\left(\frac{1}{n-1}\right)\right)-e\right)=\frac{V(n)}{11520n^2(n-1)^5}e \tag{6}$$

where

$$\begin{aligned} U(n) &= 19\,279n-29\,312n^2+20\,598n^3-7507n^4\\ &\quad -1717n^5-1920n^6+480n^7-4795, \end{aligned}$$

$$\begin{aligned} V(n) &= 24\,616n-49\,910n^2+52\,080n^3-24\,970n^4\\ &\quad +9793n^5-2400n^6+480n^7-4894. \end{aligned}$$

As expressions in (5)-(6) tends to $e/24$, (4) reads as

$$\lim_{n\to\infty} n^2\left(\left(\frac{(n+1)^{n+1}}{n^n}-\frac{n^n}{(n-1)^{n-1}}\right)-e\right)=\frac{e}{24}. \tag{7}$$

Using the same method we discovered and present now the following new results.

Theorem 2. *For every $c\in\mathbb{R}$, we have*

$$\lim_{n\to\infty}\left((n+1)\left(1+\frac{1}{n+c}\right)^{n+c}-n\left(1+\frac{1}{n+c-1}\right)^{n+c-1}\right)=e \tag{8}$$

and

$$\lim_{n\to\infty} n^2\left(\left((n+1)\left(1+\frac{1}{n+c}\right)^{n+c} - n\left(1+\frac{1}{n+c-1}\right)^{n+c-1}\right) - e\right) = \frac{e}{24}(1-12c). \tag{9}$$

In $c = 1/12$ *case, we have*

$$\lim_{n\to\infty} n^3\left(\left((n+1)\left(1+\frac{1}{n+\frac{1}{12}}\right)^{n+\frac{1}{12}} - n\left(1+\frac{1}{n-\frac{11}{12}}\right)^{n-\frac{11}{12}}\right) - e\right) = \frac{5e}{144}. \tag{10}$$

Limits (3) and (7) are case $c = 0$ in (8), respective (9).

By analogue remarks, we have

$$\begin{aligned} & (n+1)\,a\left(\frac{1}{n+c}\right) - nb\left(\frac{1}{n-1+c}\right) \\ < \; & (n+1)\left(1+\frac{1}{n+c}\right)^{n+c} - n\left(1+\frac{1}{n+c-1}\right)^{n+c-1} \\ < \; & (n+1)\,b\left(\frac{1}{n+c}\right) - na\left(\frac{1}{n-1+c}\right). \end{aligned}$$

and

$$\begin{aligned} & \lim_{n\to\infty}\left((n+1)\,a\left(\frac{1}{n+c}\right) - nb\left(\frac{1}{n-1+c}\right)\right) \\ = \; & \lim_{n\to\infty}\left((n+1)\,b\left(\frac{1}{n+c}\right) - na\left(\frac{1}{n-1+c}\right)\right) \\ = \; & e, \end{aligned}$$

as the involved sequences are rational functions in n. The exact forms were obtained and can be verified by the reader using a computer software such as Maple.

Finally,

$$\begin{aligned} & n^2\left((n+1)\,a\left(\frac{1}{n+c}\right) - nb\left(\frac{1}{n-1+c}\right) - e\right) \\ < \; & n^2\left((n+1)\left(1+\frac{1}{n+c}\right)^{n+c} - n\left(1+\frac{1}{n+c-1}\right)^{n+c-1} - e\right) \\ < \; & n^2\left((n+1)\,b\left(\frac{1}{n+c}\right) - na\left(\frac{1}{n-1+c}\right) - e\right) \end{aligned}$$

and the extreme-side sequences converge to $\frac{e}{24}(1-12c)$.

The sequence (10) is bounded below and above by

$$u_n = n^3\left(\left((n+1)\,a\left(\frac{1}{n+\frac{1}{12}}\right) - nb\left(\frac{1}{n-\frac{11}{12}}\right)\right) - e\right),$$

respective

$$v_n = n^3\left(\left((n+1)\,b\left(\frac{1}{n+\frac{1}{12}}\right) - na\left(\frac{1}{n-\frac{11}{12}}\right)\right) - e\right),$$

where

$$u_n = \frac{6en^3 P(n)}{5\,(12n+1)^5\,(12n-11)^4}$$

and

$$v_n = \frac{6en^3 Q(n)}{5(12n+1)^4 (12n-11)^5},$$

with

$$\begin{aligned} P(n) &= 5011\,380\,414n - 7958\,707\,488n^2 + 5128\,087\,104n^3 \\ &\quad -1280\,800\,512n^4 - 2395\,132\,416n^5 + 149\,299\,200n^6 - 1164\,676\,909 \end{aligned}$$

and

$$\begin{aligned} Q(n) &= 5884\,739\,370n - 12\,534\,662\,304n^2 + 13\,665\,440\,448n^3 \\ &\quad -4838\,973\,696n^4 + 1035\,016\,704n^5 + 149\,299\,200n^6 - 1088\,865\,811. \end{aligned}$$

Now easy, $\lim_{n\to\infty} u_n = \lim_{n\to\infty} v_n = 5e/144$.

Finally, we are convinced that the inequalities presented in Theorem 1 can be succesfully used to obtain other new results, such as those presented in this paper, but also in the problem of improving inequalities of Carleman's type. See recent works [1], [3]-[6].

Remark 3.1. *Computations made in this paper were performed using Maple software. The latest changes of this paper were completed during the first author's visit to University of Jaen, Spain.*

References

[1] Yan Ping, Guozheng Sun, A strengthened Carleman's inequality, J. Math. Anal. Appl., 240 (1999), 290-293.
[2] G. Pólya, G. Szegö, Problems and Theorems in Analysis, vol. I, Springer Verlag, New York, 1972.
[3] J. Sandor, On certain inequalities involving the constant e and their applications, J. Math. Anal. Appl., 249 (2000), 569-582.
[4] Bicheng Yang, L. Debnath, Some inequalities involving the constant e and an application to Carleman's inequality, J. Math. Anal. Appl., 223 (1998), 347-353.
[5] Xiaojing Yang, On Carleman's inequality, J. Math. Anal. Appl., 253 (2001), 691-694.
[6] Hu Yue, A strengthened Carleman's inequality, Commun. Math. Anal., 1 (2006), no. 2, 115-119.