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Hessenberg Matrices and the Generalized Fibonacci-Narayana Sequence

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Abstract. In this note, we define the generalized Fibonacci-Narayana sequence $\{G_n(a,b,c)\}_{n\in\mathbb{N}}$. After that, we derive some relations between these sequences, and permanents and determinants of one type of upper Hessenberg matrix.

1. Introduction

There are so many studies in the literature that concern about the generalized Fibonacci sequences (*cf.* [5–9, 18, 19]). In particular, in this paper we introduce the sequence $\{G_n(a,b,c)\}_{n\in\mathbb{N}}$, which is defined as follows:

$$G_n(a,b,c) = aG_{n-1}(a,b,c) + bG_{n-c}(a,b,c), \quad 2 \leqslant c \leqslant n, \tag{1}$$

with initial conditions $G_0(a,b,c) = 0$, $G_i(a,b,c) = 1$, for $i = 1,2,\ldots,c-1$. The constants a and b are nonzero real numbers. We call this sequence *generalized Fibonacci-Narayana sequence*. Note that, if $a = b = 1$ and $c = 2$, the Fibonacci sequence is obtained, and if $a = 1 = b$ and $c = 3$, the Narayana sequence is obtained [1, 12].

Other particular cases of the sequence $\{G_n(a,b,c)\}_{n\in\mathbb{N}}$ are

- If $a = b = 1$, the generalized Fibonacci sequence is obtained [2].

$$G_n = G_{n-1} + G_{n-c}.$$

- If $c = 2$, the generalized Fibonacci sequence is obtained.

$$G_n = aG_{n-1} + bG_{n-2}.$$

- If $a = k, b = 1$ and $c = 2$, the k-Fibonacci sequence is obtained [10].

$$F_{k,n} = kF_{k,n-1} + F_{k,n-2}.$$

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- If $a = 1$ and $b = c = 2$, the Jacobsthal sequence is obtained [13].

$$J_n = J_{n-1} + 2J_{n-2}.$$

On the other hand, an upper Hessenberg matrix, A_n, is an $n \times n$ matrix, where $a_{i,j} = 0$ whenever $i > j+1$ and $a_{j+1,j} \neq 0$ for some j. That is, all entries bellow the superdiagonal are 0 but the matrix is not upper triangular.

$$A_n = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n-1} & a_{1,n} \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,n-1} & a_{2,n} \\ 0 & a_{3,2} & a_{3,3} & \cdots & a_{3,n-1} & a_{3,n} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1,n-1} & a_{n-1,n} \\ 0 & 0 & 0 & \cdots & a_{n,n-1} & a_{n,n} \end{bmatrix}. \tag{2}$$

In this paper, we consider a type of upper Hessenberg matrix whose permanent is the generalized Fibonacci-Narayana sequence. The permanent of a matrix is similar to the determinant but all the sign used in the Laplace expansion of minors are positive [21].

There are a lot of relations between determinants or permanents of matrices and number sequences. For example, Yilmax and Bozkurt [27] defined the matrix

$$H_n = \begin{bmatrix} 1 & 1 & -1 & & & & & 0 \\ 1 & 1 & 1 & 1 & & & & \\ & 1 & 1 & 1 & 1 & & & \\ & & \ddots & \ddots & \ddots & \ddots & & \\ & & 1 & 1 & 1 & 1 & & \\ & & & 1 & 1 & 1 & -1 & \\ & & & & 1 & 1 & 1 & \\ 0 & & & & & 1 & 1 & \end{bmatrix}, \tag{3}$$

and showed that

$$\operatorname{per}(H_n) = P_n,$$

where P_n is the n-th Pell number, i.e., $P_n = 2P_{n-1} + P_{n-2}, (n > 2)$, where $P_1 = 1, P_2 = 2$. In [28], the authors obtained some relations between Padovan sequence and permanents of one type of Hessenberg matrix. Kiliç [16] obtained some relations between the Tribonacci sequence and permanents of one type of Hessenberg matrix. Öcal et al. [22] studied some determinantal and permanental representations of k-generalized Fibonacci and Lucas numbers. Janjić [14] considered a particular upper Hessenberg matrix and showed its relations with a generalization of the Fibonacci numbers. In [20], Li obtained three new Fibonacci-Hessenberg matrices and studied its relations with Pell and Perrin sequence. More examples can be found in [4, 11, 15, 17, 24–26].

2. The Main Theorem

Definition 2.1. *The permanent of an n-square matrix is defined by*

$$\operatorname{per}A = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)},$$

where the summation extends over all permutations σ of the symmetric group S_n.

Let $A = [a_{ij}]$ be an $m \times n$ real matrix with row vectors $r_1, r_2, \ldots, r_m$. We say A is contractible on column k if column k contains exactly two nonzero entries. Suppose A is contractible on column k with $a_{ik} \neq 0 \neq a_{jk}$ and $i \neq j$. Then the $(m-1) \times (n-1)$ matrix $A_{ij:k}$ obtained from A replacing row i with $a_{jk} r_i + a_{ik} r_j$, and deleting row j and column k is called the contraction of A on column k relative to rows i and j. If A is contractible on row k with $a_{ki} \neq 0 \neq a_{kj}$ and $i \neq j$, then the matrix $A_{k:ij} = [A^T_{ij:k}]^T$ is called the contraction of A on row k relative to columns i and j.

Bruouldi and Gibson [3] proved the following result about the permanent of a matrix.

Lemma 2.2. *Let A be a nonnegative integral matrix of order $n > 1$ and let B be a contraction of A. Then*

$$\mathrm{per}A = \mathrm{per}B.$$

Define the n-square Hessenberg matrix $D_n(a, b, c)$ as $d_{s+1,s} = 1$, for $s = 1, \ldots, n-1$, $d_{i,i} = a$ for $i = 1, \ldots, n$, $d_{1,2} = d_{1,3} = \cdots = d_{1,c} = b$, $d_{t,c+t-1} = b$ for $t = 2, \ldots, n-c+1$, and otherwise 0, i.e.,

$$D_n(a,b,c) = \begin{bmatrix} a & b & b & \cdots & & b & & 0 \\ 1 & a & 0 & 0 & \cdots & & b & \\ & \ddots & \ddots & \ddots & \ddots & & \ddots & \\ & & 1 & a & 0 & 0 & \cdots & b \\ & & & \ddots & \ddots & \ddots & \ddots & \\ & & & & 1 & a & 0 & 0 \\ & & & & & 1 & a & 0 \\ 0 & & & & & & 1 & a \end{bmatrix}. \tag{4}$$

Theorem 2.3. *Let $D_n(a, b, c)$ be an n-square matrix as in (4), then*

$$\mathrm{per}D_n(a,b,c) = G_{n+c-1}(a,b,c), \tag{5}$$

for all integer $n \geq c \geqslant 2$.

Proof. Let $D_n(a, b, c)^{(r)}$ be the r-th contraction of $D_n(a, b, c)$. By definition of the matrix $D_n(a, b, c)$, it can be contracted on column 1, then

$$D_n(a,b,c)^{(1)} = \begin{bmatrix} a^2+b & b & b & \cdots & b & ab & & 0 \\ 1 & a & 0 & 0 & \cdots & & b & \\ & \ddots & \ddots & \ddots & \ddots & & \ddots & \\ & & 1 & a & 0 & 0 & \cdots & b \\ & & & \ddots & \ddots & \ddots & \ddots & \\ & & & & 1 & a & 0 & 0 \\ & & & & & 1 & a & 0 \\ 0 & & & & & & 1 & a \end{bmatrix}.$$

$D_n(a, b, c)^{(1)}$ also can be contracted on first column:

$$D_n(a,b,c)^{(2)} = \begin{bmatrix} a^3+ab+b & b & b & \cdots & b & ab & a^2b+b^2 & & & 0 \\ 1 & a & 0 & 0 & \cdots & & & b & & \\ & \ddots & \ddots & \ddots & \ddots & & & & \ddots & \\ & & 1 & a & 0 & 0 & & & & b \\ & & & \ddots & \ddots & \ddots & \ddots & & & \\ & & & & & & 1 & a & 0 & 0 \\ & & & & & & & 1 & a & 0 \\ 0 & & & & & & & & 1 & a \end{bmatrix}.$$

Going with this process, we obtain the $n-c$ contraction. We have two cases. If $n \geq 2c$,

$$D_n(a,b,c)^{(n-c)} = \begin{bmatrix} G_n(a,b,c) & g_{n-2c+4}(a,b,c) & g_{n-2c+5}(a,b,c) & \cdots & g_{n-c+2}(a,b,c) \\ 1 & a & 0 & \cdots & 0 \\ & \ddots & \ddots & & \vdots \\ & 1 & a & 0 & 0 \\ & & 1 & a & 0 \\ 0 & & & 1 & a \end{bmatrix},$$

or if $n < 2c$,

$$D_n(a,b,c)^{(n-c)} = \begin{bmatrix} G_n(a,b,c) & g_2(a,b,c) & \cdots & g_2(a,b,c) & g_3(a,b,c) & \cdots & g_{n-c+2}(a,b,c) \\ 1 & a & 0 & \cdots & 0 & \cdots & 0 \\ & \ddots & \ddots & & & \ddots & \\ & & & 1 & a & 0 & 0 \\ & & & & 1 & a & 0 \\ 0 & & & & & 1 & a \end{bmatrix},$$

where $\{g_n(a,b,c)\}_{n\in\mathbb{N}}$ is the sequence defined by

$$g_0(a,b,c) = 0, \;\; g_1(a,b,c) = b, \;\; g_2(a,b,c) = b, \;\; \text{and} \;\; g_n(a,b,c) = bG_{c+n-3}(a,b,c), \; n \geq 3.$$

It is clear that

$$bG_i(a,b,c) = g_1(a,b,c) = g_2(a,b,c) = b, \text{ for } i = 1,2,\ldots,c-1. \tag{6}$$

Suppose that $n \geq 2c$. Then from definition of the sequence $\{g_n(a,b,c)\}_{n\in\mathbb{N}}$ we have

$$D_n(a,b,c)^{(n-c)} = \begin{bmatrix} G_n(a,b,c) & bG_{n-c+1}(a,b,c) & bG_{n-c+2}(a,b,c) & \cdots & bG_{n-1}(a,b,c) \\ 1 & a & 0 & \cdots & 0 \\ & \ddots & \ddots & & \vdots \\ & 1 & a & 0 & 0 \\ & & 1 & a & 0 \\ 0 & & & 1 & a \end{bmatrix}.$$

$D_n(a,b,c)^{(n-c)}$ also can be contracted on first column:

$$D_n(a,b,c)^{(n-c+1)} = \begin{bmatrix} G_{n+1}(a,b,c) & bG_{n-c+2}(a,b,c) & bG_{n-c+3}(a,b,c) & \cdots & bG_{n-1}(a,b,c) \\ 1 & a & 0 & \cdots & 0 \\ & \ddots & \ddots & & \vdots \\ & 1 & a & 0 & 0 \\ & & 1 & a & 0 \\ 0 & & & 1 & a \end{bmatrix}.$$

Hence, the $(n-3)$th contraction is

$$D_n(a,b,c)^{(n-3)} = \begin{bmatrix} G_{n+c-3}(a,b,c) & bG_{n-2}(a,b,c) & bG_{n-1}(a,b,c) \\ 1 & a & 0 \\ 0 & 1 & a \end{bmatrix}$$

which, by contraction of $D_n(a,b,c)^{(n-3)}$ on column 1,

$$D_n(a,b,c)^{(n-2)} = \begin{bmatrix} G_{n+c-2}(a,b,c) & bG_{n-1}(a,b,c) \\ 1 & a \end{bmatrix}.$$

Then from Lemma 2.2

$$\operatorname{per} D_n(a,b,c) = \operatorname{per} D_n(a,b,c)^{(n-2)} = aG_{n+c-2}(a,b,c) + bG_{n-1}(a,b,c) = G_{n+c-1}(a,b,c).$$

If $n < 2c$ the proof runs like. □

Particular cases of the previous theorem are

- If $a = b = 1$ and $c = 2$ we have the matrix

$$D_n(1,1,2) = \begin{bmatrix} 1 & 1 & & & & & & 0 \\ 1 & 1 & 1 & & & & & \\ & \ddots & \ddots & \ddots & & & & \\ & & 1 & 1 & 1 & & & \\ & & & \ddots & \ddots & \ddots & & \\ & & & & 1 & 1 & 1 & \\ & & & & & 1 & 1 & 1 \\ 0 & & & & & & 1 & 1 \end{bmatrix}, \tag{7}$$

  such that $\operatorname{per} D_n(1,1,2) = F_{n+1}$ for all $n \geqslant 2$.

- If $c = 2$ we have the matrix

$$D_n(a,b,2) = \begin{bmatrix} a & b & & & & & & 0 \\ 1 & a & b & & & & & \\ & \ddots & \ddots & \ddots & & & & \\ & & 1 & a & b & & & \\ & & & \ddots & \ddots & \ddots & & \\ & & & & 1 & a & b & \\ & & & & & 1 & a & b \\ 0 & & & & & & 1 & a \end{bmatrix}, \tag{8}$$

  such that $\operatorname{per} D_n(a,b,2) = G_{n+1}$ for all $n \geqslant 2$, where G_n is the n-th generalized Fibonacci sequence. For example, if $a = 3$ and $b = 2$, we obtain that

$$\left\{ \operatorname{per}\begin{bmatrix} 3 & 2 \\ 1 & 3 \end{bmatrix}, \operatorname{per}\begin{bmatrix} 3 & 2 & 0 \\ 1 & 3 & 2 \\ 0 & 1 & 3 \end{bmatrix}, \operatorname{per}\begin{bmatrix} 3 & 2 & 0 & 0 \\ 1 & 3 & 2 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}, \operatorname{per}\begin{bmatrix} 3 & 2 & 0 & 0 & 0 \\ 1 & 3 & 2 & 0 & 0 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix}, \dots \right\}$$
$$= \{11, 39, 139, 495, \dots\}$$

- If $a = b = 1$ and $c = 3$ we have the matrix

$$D_n(1,1,3) = \begin{bmatrix} 1 & 1 & 1 & & & & & 0 \\ 1 & 1 & 0 & 1 & & & & \\ & \ddots & \ddots & \ddots & \ddots & & & \\ & & 1 & 1 & 0 & 1 & & \\ & & & \ddots & \ddots & \ddots & \ddots & \\ & & & & 1 & 1 & 0 & 1 \\ & & & & & 1 & 1 & 0 \\ 0 & & & & & & 1 & 1 \end{bmatrix}, \tag{9}$$

  such that $\operatorname{per} D_n(1,1,3) = H_{n+2}$ for all $n \geqslant 3$, where H_n is the n-th Narayana number, i.e., $\{N_n\}_{n\in\mathbb{N}} = \{0, 1, 1, 1, 2, 3, 4, 6, 9, 13, 19, 28, \dots\}$.

- If $a = 1$ and $b = c = 2$ we have the matrix

$$D_n(1,2,2) = \begin{bmatrix} 1 & 2 & & & & & & 0 \\ 1 & 1 & 2 & & & & & \\ & \ddots & \ddots & \ddots & \ddots & & & \\ & & 1 & 1 & 2 & & & \\ & & & \ddots & \ddots & \ddots & \ddots & \\ & & & & 1 & 1 & 2 & \\ & & & & & 1 & 1 & 2 \\ 0 & & & & & & 1 & 1 \end{bmatrix}, \tag{10}$$

such that $\mathrm{per} D_n(1,2,2) = J_{n+1}$ for all $n \geqslant 2$, where J_n is the n-th Jacobsthal number.

Finally, we find a matrix $B_n(a,b,c)$ such that $\det B_n(a,b,c) = G_{n+c-1}(a,b,c)$. We use the ideas of Killiç and Taşçi [17]. These authors introduced an n-square $(1,-1)$ matrix S, such that $\mathrm{per}A = \det(A \circ S)$, where $A \circ S$ denotes Hadamard product of A and S. The matrix S is defined as $s_{i,j} = -1$ if $i = j+1$ and otherwise 1, i.e.,

$$S = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ -1 & 1 & \cdots & 1 & 1 \\ 1 & -1 & \cdots & 1 & 1 \\ \vdots & & \cdots & & \vdots \\ 1 & 1 & \cdots & -1 & 1 \end{bmatrix}. \tag{11}$$

Let $B_n(a,b,c)$ be the matrix defined as $B_n(a,b,c) = D_n(a,b,c) \circ S$. Hence,

$$B_n(a,b,c) = \begin{bmatrix} a & b & b & \cdots & & b & & 0 \\ -1 & a & 0 & 0 & \cdots & & b & \\ & \ddots & \ddots & \ddots & \ddots & & \ddots & \\ & & -1 & a & 0 & 0 & \cdots & b \\ & & & \ddots & \ddots & \ddots & \ddots & \\ & & & & -1 & a & 0 & 0 \\ & & & & & -1 & a & 0 \\ 0 & & & & & & -1 & a \end{bmatrix}. \tag{12}$$

Then we obtain that $\det B_n(a,b,c) = \mathrm{per}(D_n(a,b,c)) = G_{n+c-1}(a,b,c)$ for all integer $n \geq c \geqslant 2$.

References

[1] J. P. Allouche, J. Johnson, Narayana's cows and delayed morphisms. In: Articles of 3rd Computer Music Conference JIM96, France, (1996).

[2] M. Bicknell-Johnson, C. Spears, Classes of identities for the generalized Fibonacci numbers $G_n = G_{n-1} + G_{n-c}$ from matrices with constant valued determinants, Fibonacci Quart. 34(2) (1996) 121–128.

[3] R. Brualdi, P. Gibson, Convex polyhedra of doubly stochastic matrices I: applications of the permanent function, J. Comb. Theory 22(2) (1977) 194–230.

[4] N. D. Cahil, J. R. D'Errico, D. A. Narayan, J. Y. Narayan, Fibonacci determinants, College Math. J. 33(3) (2002) 221–225.

[5] G. B. Djordjević, Some generalizations of the Jacobsthal numbers, Filomat 24(2) (2010) 143–151.

[6] G. B. Djordjević, Generalizations of the Fibonacci and Lucas polynomials, Filomat 23(3) (2009) 291–301.

[7] G. B. Djordjević, H. M. Srivastava, Incomplete generalized Jacobsthal and Jacobsthal-Lucas numbers, Math. Comput. Modelling 42 (2005) 1049–1056.

[8] G. B. Djordjević, H. M. Srivastava, Some generalizations of certain sequences associated with the Fibonacci numbers, J. Indonesian Math. Soc. 12(2006) 99–112.

[9] G. B. Djordjević, H. M. Srivastava, Some generalizations of the incomplete Fibonacci and the incomplete Lucas polynomials, Adv. Stud. Contemp. Math. 11 (2005) 11–32.
[10] S. Falcón, A. Plaza, On the Fibonacci *k*-numbers, Chaos Solitons Fractals 32(5) (2007), 1615–1624.
[11] S. Falcón, On the generating matrices of the *k*-Fibonacci numbers, Proyecciones 32(4) (2013), 347–357.
[12] C. Flaut, V. Shpakivskyi, On generalized Fibonacci quaternions and Fibonacci-Narayana quaternions, Adv. Appl. Clifford Algebras 23(2013), 673–688.
[13] A. F. Horadam, Jacobsthal representation numbers, Fibonacci Quart. 34 (1)(1996), 40–49.
[14] M. Janjić, Hessenberg matrices and integer sequences, J. Integer Seq. 13, article 10.7.8 (2010) 1–10.
[15] J. Jína, P. Trojovský, On determinants of some tridiagonal matrices connected with Fibonacci numbers, Int. J. Pure Appl. Math. 88(4) (2013) 569–575.
[16] E. Kiliç, Tribonacci sequences with certain indices and their sums, Ars Comb. 86(2008) 13–22.
[17] E. Kiliç, D. Taşçi. Negatively subscripted Fibonacci and Lucas numbers and their complex factorizations, Ars Combin. 96(2010) 275–288.
[18] T. Koshy, Fibonacci and Lucas Numbers with Applications, A Wiley-Interscience Publication, 2001.
[19] P. Larcombe, O. Bagdasar, E. Fennessey, Horadam sequences: a survey, Bull. Inst. Comb. Appl. 67(2013) 49–72.
[20] H-C, Li, On Fibonacci-Hessenberg matrices and the Pell and Perrin numbers, Appl. Math. Comput. 218 (2012) 8353–8358.
[21] H. Minc, Permanents, Encyclopedia of Mathematics and its Applications, vol. 6, Addison-Wesley Publishing Company, London, 1978.
[22] A. A. Öcal, N. Tuglu, E. Altinisik, On the representation of *k*-generalized Fibonacci and Lucas numbers, Appl. Math. Comput. 170 (2005), 584–596.
[23] R. K. Raina, H. M. Srivastava, A class of numbers associated with the Lucas numbers, Math. Comput. Modelling 25(7) (1997), 15–22.
[24] J. Ramírez, Some properties of convolved *k*-Fibonacci numbers, ISRN Comb. vol 2013 (2013), 1–5.
[25] J. Ramírez, On Convolved Generalized Fibonacci and Lucas Polynomials, Appl. Math. Comput., 229(25) (2014), 208–213.
[26] Y. Yazlik, N. Taskara, On the negatively and positively subscripted generalized *k*-Horadam sequences and their matrix presentations, World Appl. Sci. J. 27(12) (2013), 1561–1565.
[27] F. Yilmaz, D. Bozkurt, Hessenberg matrices and the Pell and Perrin numbers, J. Number Theory 131(8) (2011), 1390–1396.
[28] F. Yilmaz, D. Bozkurt, Some properties of Padovan sequence by matrix methods, Ars Comb. 104 (2012), 49–160.