Filomat 29:7 (2015), 1663–1669
DOI 10.2298/FIL1507663A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Absolute Purity in The Category of Quasi Coherent Sheaves

Mansour Aghasi[a], Hamidreza Nemati[a]

[a]*Department of Mathematical Sciences, Isfahan University of Technology, Isfahan, 84156-83111, Iran*

Abstract. In this paper we consider the class of absolutely pure and Fp-injective quasi-coherent sheaves. We show that these two classes of quasi-coherent sheaves over a locally coherent scheme are equivalent. As a corollary we will show that the class of absolutely pure quasi-coherent sheaves over such a scheme is an enveloping and a covering class. It is proved that over a locally coherent scheme, the pair $({}^{\perp}\big(\mathrm{Abs}(X)\,,\ \mathrm{Abs}(X)\big)$ is a cotorsion theory. The existence of a duality between absolutely pure envelopes and flat covers is proved as expected.

1. Introduction

Through all of this paper X stands for a semi separated quasi compact scheme and $\mathfrak{U} = \{\mathrm{Spec}A_i\}_{i=1}^m$ denotes a semi-separating cover of X.

We recall the definition of a finitely presented quasi coherent sheaf from [13]. A quasi coherent sheaf $\mathcal{P}$ is finitely presented if $\mathrm{Hom}(\mathcal{P}, -) : \mathfrak{Qco}(X) \to \mathbf{A}b$ preserves direct limits. It is known that the category of quasi coherent sheaves over a semi separated quasi compact scheme X is a locally finitely presented category. Thus there is an standard notion of purity in $\mathfrak{Qco}X$. A sequence $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ of sheaves is called fp-pure if the induced morphism

$$0 \to \mathrm{Hom}(\mathcal{P}, \mathcal{F}) \to \mathrm{Hom}(\mathcal{P}, \mathcal{G}) \to \mathrm{Hom}(\mathcal{P}, \mathcal{H}) \to 0$$

is an exact sequence of abelian groups for every finitely presented sheaf $\mathcal{P}$. An exact sequence

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

of quasi coherent sheaves is said to be tensor-pure if

$$0 \to \mathcal{T} \otimes \mathcal{F} \to \mathcal{T} \otimes \mathcal{G} \to \mathcal{T} \otimes \mathcal{H} \to 0$$

is exact for every $\mathcal{T} \in \mathfrak{Qco}(X)$. We denote by $\mathbf{Pure}_{\otimes}$ and by $\mathbf{Pure}_{fp}$ the classes of fp-pure and tensor-pure (pure) short exact sequences in $\mathfrak{Qco}(X)$. A quasi coherent sheaf $\mathcal{H}$ is called flat if $\mathcal{H}_x$ is a flat $O_{X,x}$-module for each point $x \in X$ or equivalently every exact sequence

$$\varepsilon : 0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

2010 *Mathematics Subject Classification*. Primary 14F05; Secondary 18E20, 18E15
Keywords. quasi-coherent sheaf, absolutely pure, Fp-injective, absolutely pure cover
Received: 23 September 2013; Accepted: 07 July 2015
Communicated by Dragan S. Djordjević
Email addresses: m.aghasi@cc.iut.ac.ir (Mansour Aghasi), hr.nemati@math.iut.ac.ir (Hamidreza Nemati)

is pure (exactness of the functor $\mathcal{H} \otimes_{O_X} -$ is another equivalent definition of flatness). $\mathcal{H}$ is called fp-flat if every exact sequence of the form ε is fp-pure (this is the standard definition of flatness in any locally finitely presented category). The class of flat and fp-flat quasi coherent sheaves are denoted by $\mathbf{Flat}_{\otimes}$ and $\mathbf{Flat}_{fp}$ respectively. It is proved in [5, section 3] and in [7] that $\mathbf{Pure}_{fp} \subseteq \mathbf{Pure}_{\otimes}$ and $\mathrm{Flat}_{fp} \subseteq \mathrm{Flat}_{\otimes}$.

It is known that quasi-coherent sheaves play the role of modules in algebraic geometry. They behave similar to modules in homological point of view. In relative homological algebra usually the study of some special classes of quasi coherent sheaves is considered. For example in [4] the class of flat quasi coherent sheaves has been studied. It is proved that the category of quasi coherent sheaves, $\mathfrak{Qco}(X)$, admits flat covers. Flat quasi coherent sheaves on a scheme X has been also studied during the recent years as a nice tool for studying the derived category of (quasi coherent) sheaves (for example see [8]). In the current paper we extend the notion of absolutely pure modules to that of absolutely pure quasi coherent sheaves and study the class of absolutely pure quasi coherent sheaves, Abs($\mathfrak{Qco}$)X, as a continuation of [5]. The study of the class of all absolutely pure modules were first considered in [1] and continued in several papers such as [12], [11], [15], [14], etc. A quasi coherent sheaf $\mathcal{F}$ is called absolutely pure if every exact sequence $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is pure. $\mathcal{F}$ is called locally absolutely pure if $\mathcal{F}_x$ is an absolutely pure $O_{X,x}$-module for each point $x \in X$.

In [5], the authors have proved that every locally absolutely pure is absolutely pure. They mentioned that the converse is not clear in general. We prove that the converse is true provided X is a locally coherent scheme. It is a generalization of a classical result which states that over a locally noetherian scheme X, every injective quasi-coherent sheaf $\mathcal{I}$ is locally injective. As a corollary we prove that any absolutely pure quasi coherent sheaf is actually an absolutely pure O_X-modules which is quasi coherent sheaf, that is $\mathrm{Abs}(\mathfrak{Qco}X) = \mathfrak{Qco}X \cap \mathrm{Abs}(\mathfrak{Sh}X)$.

A quasi coherent sheaf $\mathcal{J}$ is said to be Fp-injective if it is an fp-pure quasi coherent subsheaf in every sheaf of any quasi coherent sheaf $\mathcal{G}$ containing it. It is equivalent to saying that $\mathcal{J}$ has an injective property with respect to exact sequences of finitely presented sheaves. It is known that any Fp-injective quasi coherent sheaf $\mathcal{F}$ is absolutely pure since generally $\mathbf{Pure}_{\mathrm{fp}} \subseteq \mathbf{Pure}_{\otimes}$ [5]. As an important corollary of the previous result we prove that over a locally coherent scheme X, any quasi coherent sheaf $\mathcal{F}$ is absolutely pure if and only if it is Fp-injective.

One of our main purposes is to relate absolute purity to flatness as a generalization of classical results in the category of R-modules. The results can be used to relate absolutely pure dimension to flat dimension of quasi coherent sheaves. It is proved that over a locally coherent scheme the following statements will happens as it is expected.

1. A quasi coherent sheaf $\mathcal{F}$ is absolutely pure if and only if $\mathcal{F}^*$ is flat.
2. A quasi coherent sheaf $\mathcal{F}$ is absolutely pure if and only if $\mathcal{F}^{**}$ is injective.
3. A quasi coherent sheaf $\mathcal{F}$ is flat if and only if $\mathcal{F}^{**}$ is flat.

One can use theses results to prove that the Fp-injective dimension of any quasi-coherent sheaf, $\mathrm{Fp}-\mathrm{injd}\mathcal{F}$ is equal to the flat dimension of its character quasi coherent sheaf, $\mathrm{fd}\mathcal{F}^*$.

2. Preliminaries

In this paper all rings will be commutative with identity.

In a category C with direct limits, an object P is called finitely presented if the functor $\mathrm{Hom}_C(P,-)$ preserves direct limits. A category C with direct limits is called locally finitely presented if every object in C is a direct limit of finitely presented objects. For example the category of R-modules over a ring R and the category of quasi-coherent sheaves over a semi-separated quasi-compact scheme X are locally finitely presented categories.

The following results are all selected from [5].

Proposition 2.1. *If $\mathfrak{Qco}(X)$ is a locally finitely presented category, then categorically pure short exact sequences are pure exact, that is, $\mathbf{Pure}_{fp} \subseteq \mathbf{Pure}_{\otimes}$.*

Definition 2.2. *Let $(X, \mathcal{O}_X)$ be a scheme. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. $\mathcal{F}$ is called absolutely pure in $\mathfrak{Qco}(X)$ if every exact sequence $0 \to \mathcal{F} \to \mathcal{G}$ in $\mathfrak{Qco}(X)$ is pure exact.*

Lemma 2.3. *Every locally absolutely pure quasi-coherent sheaf is absolutely pure in $\mathfrak{Qco}(X)$.*

Proposition 2.4. *A closed subscheme $X \subseteq \mathbb{P}^n(R)$ which is locally coherent is locally noetherian if and only if locally absolutely pure quasi-coherent sheaves are locally injective.*

Theorem 2.5. *Let X be a locally coherent scheme. The class of locally absolutely pure quasi-coherent sheaves is a covering class.*

Proposition 2.6. *If E_P is absolutely pure over R_P for every prime ideal P, then E is absolutely pure over R. The converse is true when R is a coherent ring R.*

Proposition 2.7. *Let $(X, \mathcal{O}_X)$ be a locally coherent scheme. Then the following conditions are equivalent for a quasi-coherent sheaf $\mathcal{F}$.*

1. *$\mathcal{F}(U)$ is absolutely pure for every affine U.*
2. *$\mathcal{F}(U_i)$ is absolutely pure for all $i \in I$ of some cover $\{U_i\}_{i \in I}$ of affine open subsets.*
3. *$\mathcal{F}_x$ is absolutely pure for all $x \in X$.*

Lemma 2.8. *Let $\mathcal{F}$ be a quasi-coherent sheaf. Then $\mathcal{F}$ is locally absolutely pure if and only if $\mathcal{F}$ is absolutely pure in $\mathcal{O}_X$-Mod.*

Theorem 2.9. *[2, Corllary 3.5 and Corollary 4.2] The class of Fp-injective objects in a locally finitely presented additive category and a locally coherent category is a preenveloping and a covering class respectively.*

Let $\mathbb{F}$ be any class of quasi coherent sheaves closed under isomorphisms and extensions. For a given quasi coherent sheaf $\mathcal{G}$, the morphism $\phi : \mathcal{F} \to \mathcal{G}$ with $\mathcal{F} \in \mathbb{F}$ is called an $\mathbb{F}$-precover of $\mathcal{G}$ if $\mathrm{Hom}(\mathcal{F}', \mathcal{F}) \to \mathrm{Hom}(\mathcal{F}, \mathcal{G})$ is an epimorphism of abelian groups for every $\mathcal{F}' \in \mathbb{F}$. $\phi : \mathcal{F} \to \mathcal{G}$ is called an $\mathbb{F}$-cover if $\phi \circ \psi = \phi$ implies that ψ is an isomorphism where $\psi \in \mathrm{Hom}(\mathcal{F}, \mathcal{F})$. $\mathbb{F}$-(pre)envelopes can be defined dually.

3. Absolutely Pure and Fp-Injective Quasi-Coherent Sheaves

The class of locally absolutely pure quasi-coherent sheaves is considered in [5]. It is proved that over a locally coherent scheme any locally absolutely pure quasi-coherent sheaf is absolutely pure. The existence of locally absolutely pure covering of quasi coherent sheaves over a locally coherent scheme is also proved. In this section we use a contra-variant functor $(-)^* : \mathfrak{Q}\mathfrak{o}X \to \mathfrak{Qco}X$ introduced in [9] to get an strong equivalent definition of purity to study absolutely pure quasi coherent sheaves as a continuation of [5].

Let $\mathcal{F}$ be a quasi-coherent sheaf. Set $\mathcal{F}^* = \bigoplus_{i=1}^m f_{i*}\widetilde{F_i^*}$, $\mathcal{F}^{**} = \bigoplus_{i=1}^m f_{i*}\widetilde{F_i^{**}}$ whenever $F_i = \mathcal{F}(U_i)$, $F_i^* = \mathrm{Hom}_{\mathbb{Z}}(F_i, \mathbb{Q}/\mathbb{Z})$ and $f_i : U_i \to X$ is the inclusion for each $1 \le i \le m$. Then $\mathcal{F}^*$ and $\mathcal{F}^{**}$ are both pure injective and $\mathcal{F} \to \mathcal{F}^{**}$ is a pure monomorphism.

Proposition 3.1. *Let X be any scheme and $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ be an exact sequence of quasi-coherent sheaves. It is pure exact if and only if the sequence $0 \to \mathcal{H}^* \to \mathcal{G}^* \to \mathcal{F}^* \to 0$ splits.*

Proof. It is a direct consequence of a similar result in the category of R-modules. □

Lemma 3.2. *Let X be a locally coherent scheme. Then every quasi-coherent sheaf can be embedded in a locally absolutely pure quasi coherent sheaf.*

Proof. Let $0 \to F_i \to E_i$ be the absolutely pure envelope for each $1 \le i \le m$. Then $\widetilde{E_i}_P = E_{iP}$ is absolutely pure by Proposition 2.6 and hence $\widetilde{E_i}$ is absolutely pure by Proposition 2.7. For each $0 \le i \le m$, the stalk of the quasi-coherent sheaf $F_*(\widetilde{E_i})$ at each point $x \in X$ is zero or E_{ix} depended in x is in U_i or not. So by Proposition 2.7 it is a locally absolutely pure quasi-coherent sheaf over X. Then the composition of $0 \to \mathcal{F} \to \bigoplus_{i=1}^m f_{i*}\widetilde{F_i} \to \bigoplus_{i=1}^m f_{i*}\widetilde{E_i}$ completes the proof. □

Proposition 3.3. *Let X be a locally coherent scheme. Then any quasi coherent sheaf $\mathcal{F}$ is absolutely pure if and only if $\mathcal{F}$ is locally absolutely pure. Moreover, over such scheme* $\mathrm{Abs}(\mathfrak{Qco}X) = \mathfrak{Qco}X \cap \mathrm{Abs}(\mathfrak{Sh}X)$.

Proof. Consider $0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{T} \to 0$ whenever $\mathcal{E}$ is locally absolutely pure. Then $0 \to \mathcal{T}^* \to \mathcal{E}^* \to \mathcal{F}^* \to 0$ and hence for every $1 \leq i \leq m$,

$$0 \to T_i^* \to E_i^* \to F_i^* \to 0$$

splits. Since E_i^* is flat, we conclude that F_i^* is flat and hence F_i is absolutely pure. The other part follows by combining Lemma 2.8 and the previous part of the theorem. □

The previous theorem generalizes the following known result which we recover it as follows.

Corollary 3.4. *Let X be a locally noetherian scheme. Then any injective quasi coherent sheaf is in fact an injective $\mathcal{O}_X$-module which is quasi coherent sheaf.*

Proof. It is a direct consequence of the Proposition 3.3 and Proposition 2.4. □

Theorem 3.5. *Let X be a locally coherent scheme. A quasi coherent sheaf $\mathcal{F}$ is absolutely pure if and only if it is Fp-injective.*

Proof. A quasi-coherent sheaf $\mathcal{F}$ is absolutely pure if and only if it is locally absolutely pure by Proposition 3.3. Since in the category of R-modules (with R a commutative ring with identity) the notions of Fp-injectivity and absolute purity are equivalent, a locally absolutely pure quasi-coherent sheaf is locally Fp-injective and wise versa. Notice that any locally Fp-injective quasi-coherent sheaf is Fp-injective since

$$\mathcal{E}\mathrm{xt}^1(\mathcal{P}, \mathcal{F})(U) = \mathrm{Ext}^1(\mathcal{P}|_U, \mathcal{F}|_U) = 0$$

for every finitely presented quasi coherent sheaf $\mathcal{P}$. Finally every Fp-injective quasi-coherent sheaf is absolutely pure by Proposition 2.1. □

Pinzon has devoted [14] to show that every R-module over a coherent ring R has an absolutely pure cover. The following corollary is a simple proof of one of the main results in [5].

Corollary 3.6. *The class of absolutely pure quasi coherent sheaves over a coherent scheme is a covering and an enveloping class.*

Proof. The class of Fp-injective objects is known by Theorem 2.9 to be an enveloping class in any locally finitely presented category and a covering class in any locally coherent category. When X is a coherent scheme, the category $\mathfrak{Qco}(X)$ is a locally coherent category. □

Theorem 3.7. *Over a coherent scheme X, the pair $({}^{\perp}\mathrm{A}bs(X), \mathrm{A}bs(X))$ is a complete cotorsion pair.*

Proof. We should only show that every quasi-coherent sheaf $\mathcal{H} \in ({}^{\perp}\mathrm{A}bs(X))^{\perp}$ is absolutely pure. Let $\mathcal{P}$ be a finitely presented quasi-coherent sheaf, then by Theorem 3.5, $\mathcal{P} \in ({}^{\perp}\mathrm{A}bs(X))$. So $\mathrm{E}xt^1(\mathcal{P}, \mathcal{H}) = 0$ and this implies that $\mathcal{H}$ is Fp-injective (and hence absolutely pure). □

It is known that over a locally noetherian scheme X, the restriction of an injective quasi coherent sheaf to any open subset is injective. The following result generalizes this result and by theorem Proposition 2.4 has this result as a corollary.

Proposition 3.8. *The following are equivalent.*

1. *X is locally coherent.*
2. *The restriction of any injective quasi coherent sheaf to any open subset is absolutely pure.*

Proof. 2 → 1: We show that any absolutely pure quasi coherent sheaf is locally absolutely pure. Let $\mathcal{F}$ be any absolutely pure quasi coherent sheaf. Consider

$$0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{G} \to 0$$

with $\mathcal{E}$ injective. Then

$$0 \to \mathcal{F}|_U \to \mathcal{E}|_U \to \mathcal{G}|_U \to 0$$

is pure. Hence $\mathcal{F}|_U$ is absolutely pure.
1 → 2: Theorem 3.3 □

The following theorem is an extract of this section.

Theorem 3.9. *Let X be a locally coherent scheme. The following happens.*

1. $\mathrm{Abs}(\mathfrak{Qco}X) = \mathrm{Abs}(\mathfrak{Sh}X) \cap \mathfrak{Qco}X$.
2. *A quasi coherent sheaf $\mathcal{F}$ is absolutely pure if and only if $\mathcal{F}$ is locally absolutely pure.*
3. *A quasi coherent sheaf $\mathcal{F}$ is absolutely pure if and only if $\mathcal{F}$ is Fp-injective.*
4. *the restriction of any injective sheaf to any open subset is absolutely pure.*
5. *the direct limit of any direct system of absolutely pure quasi coherent sheaves is absolutely pure.*
6. *Every quasi coherent sheaf admits an* $\mathrm{Abs}(\mathfrak{Qco}X)$*-cover.*

It is known that a ring R is a noetherian ring if and only if every injective module can be written as a direct sum of some indecomposable injective modules. A quasi-coherent version of this fact has been appeared in [5]. By comparing the treatment of injective quasi-coherent sheaves over noetherian schemes and absolutely pure quasi-coherent sheaves over locally coherent schemes, asking the following question is natural.

Question: Is it true to say that a scheme X is coherent if and only if every absolutely pure quasi-coherent sheaf is a direct sum of some indecomposable injective modules?

Huang has answered this question incompletely in [10, Corollary 3.8]. He has proved that over a coherent ring R, an absolutely pure module does not in general have such a decomposition. The following theorem answers to the above question.

Theorem 3.10. *Let X be a scheme. Then X is noetherian if and only if every absolutely pure quasi-coherent sheaf has a decomposition as a direct sum of indecomposable absolutely pure quasi-coherent subsheaves.*

Proof. The only if part is trivial. Conversely suppose that $\mathcal{E}$ is an injective quasi-coherent sheaf. Since $\mathcal{E}$ is absolutely pure, there is a family $\{\mathcal{A}_i\}_{i\in I}$ of indecomposable absolutely pure quasi-coherent sheaves such that $\mathcal{E} = \oplus_{i\in I}\mathcal{A}_i$. Each $\mathcal{A}_i$ is a direct summand of an injective quasi-coherent sheaf and hence is injective. This means that any injective module is a direct sum of some indecomposable injective modules. This implies that X is a noetherian scheme. □

4. Duality Between Flatness and Absolute Purity

In this setion we show that the functor $()^* : \mathfrak{Qco}(X) \to \mathfrak{Qco}(X)$ treats similar to the character functor for the category of R-modules and hence there is a duality between absolutely pure envelope and flat cover of quasi-coherent sheaves.

Lemma 4.1. *Let ε and ε' be short exact sequences with ε' a pure short exact sequence. If $0 \to \varepsilon \to \varepsilon'$ is pure exact then ε is pure.*

Proof. Let $\mathcal{A}$ be any quasi coherent sheaf and $0 \to \varepsilon \to \varepsilon'$ be as follows.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{F} \arrow[r, "f"] \arrow[d] & \mathcal{G} \arrow[r, "g"] \arrow[d] & \mathcal{H} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{M} \arrow[r, "f'"] & \mathcal{N} \arrow[r, "g'"] & \mathcal{P} \arrow[r] & 0
\end{tikzcd}$$

Then it is not difficult to show that $1_{\mathcal{A}} \otimes f$ is a monomorphism.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & \mathcal{F} \otimes_{O_X} \mathcal{A} \arrow[r, "1\otimes f"] \arrow[d] & \mathcal{G} \otimes_{O_X} \mathcal{A} \arrow[r, "1\otimes g"] \arrow[d] & \mathcal{H} \otimes_{O_X} \mathcal{A} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{M} \otimes_{O_X} \mathcal{A} \arrow[r, "1\otimes f'"] & \mathcal{N} \otimes_{O_X} \mathcal{A} \arrow[r, "1\otimes g'"] & \mathcal{P} \otimes_{O_X} \mathcal{A} \arrow[r] & 0
\end{tikzcd}$$

□

Proposition 4.2. *The following are equivalent for a quasi coherent sheaf $\mathcal{F}$.*

1. *$\mathcal{F}$ is flat.*
2. *$\mathcal{F}^*$ is absolutely pure.*
3. *$\mathcal{F}^*$ is injective.*

Proof. $1 \to 3$: Suppose that we are given any exact sequence

$$0 \to \mathcal{F}^* \xrightarrow{f} \mathcal{G} \to \mathcal{H} \to 0.$$

Taking pull back of $\mathcal{F} \to \mathcal{F}^{**}$ and $\mathcal{G}^* \to \mathcal{F}^{**}$ yields the following diagram.

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{H}^* \arrow[r] \arrow[d] & \mathcal{P} \arrow[r, "h"] \arrow[d, "g"] & \mathcal{F} \arrow[r] \arrow[d, "i"] & 0 \\
0 \arrow[r] & \mathcal{H}^* \arrow[r] & \mathcal{G}^* \arrow[r, "f^*"] & \mathcal{F}^{**} \arrow[r] & 0
\end{tikzcd}$$

The top row splits since $\mathcal{H}^*$ is pure injective and $\mathcal{F}$ is flat hence there exists $h' : \mathcal{F} \to \mathcal{P}$ such that $hh' = 1_{\mathcal{F}}$. Putting $g_1 := gh'$ we have $f^*gh' = f^*g_1 = i$ and $g_1^* f^{**} = i^*$. In the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}^* \arrow[r, "f"] \arrow[d, "j"] & \mathcal{G} \arrow[r] \arrow[d, "k"] & \mathcal{H} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{F}^{***} \arrow[r, "f^{**}"'] & \mathcal{G}^{**} \arrow[r] & \mathcal{H}^{**} \arrow[r] & 0
\end{tikzcd}$$

$g_1^* k$ splits the top row since $g_1^* kf = g_1^* f^{**} j = i^* j = 1_{\mathcal{F}^*}$. Thus $\mathcal{F}^*$ is injective.
$3 \to 2$ is a direct consequence of definitions.
$2 \to 1$: For a given

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{F} \to 0$$

consider

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{A} & \longrightarrow & \mathcal{B} & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{A}^{**} & \longrightarrow & \mathcal{B}^{**} & \longrightarrow & \mathcal{F}^{**} & \longrightarrow & 0 \end{array}$$

whenever the vertical arrows are all pure and the down row splits. By Lemma 4.1, the top row is pure and hence $\mathcal{F}$ is flat. □

Theorem 4.3. *Let X be a locally coherent scheme. Then*

1. *$\mathcal{F}$ is absolutely pure if and only if $\mathcal{F}^*$ is flat.*
2. *$\mathcal{F}$ is absolutely pure if and only if $\mathcal{F}^{**}$ is injective.*
3. *$\mathcal{F}$ is flat if and only if $\mathcal{F}^{**}$ is flat.*

Proof. 1): Suppose that we are given an exact sequence

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{F}^* \to 0.$$

Then $0 \to \mathcal{F}^{**} \to \mathcal{B}^* \to \mathcal{A}^* \to 0$ splits since $\mathcal{F}^{**}$ is absolutely pure and pure injective. Conversely let $\mathcal{F}^*$ be flat. Then $\mathcal{F}^{**}$ is absolutely pure by Proposition 4.2. Hence $\mathcal{F}^{**}$ is absolutely pure since $\mathcal{F} \to \mathcal{F}^{**}$ is a pure monomorphism.

2): Let $\mathcal{F}$ be absolutely pure. Then $\mathcal{F}^{**} = \bigoplus_{i=1}^{m} f_{i*}\widetilde{F_i^{**}}$ is absolutely pure since X is locally coherent. On the other hand $\mathcal{F}^{**}$ is pure injective. Notice that a quasi coherent sheaf $\mathcal{F}$ is injective if and only if $\mathcal{F}$ is absolutely pure and pure injective.

Conversely suppose that $\mathcal{F}^{**}$ is injective. Then by the previous proposition $\mathcal{F}^*$ is flat and thus $\mathcal{F}$ is absolutely pure by part (1).

3): $\mathcal{F}$ is flat if and only if $\mathcal{F}^*$ is absolutely pure if and only if $(\mathcal{F}^*)^{**} = (\mathcal{F}^{**})^*$ is injective if and only if $\mathcal{F}^{**}$ is flat. □

Theorem 4.4. *Let X be a locally coherent scheme and $\mathcal{F}$ be a quasi-coherent sheaf. If $f : \mathcal{F} \to \mathcal{A}$ is an absolutely pure preenvelope, then $F^* : \mathcal{A}^* \to \mathcal{F}^*$ is a flat precover.*

Proof. The proof is similar to the proof of Theorem 3.1 in [6]. □

References

[1] Adams, D. D. *Absolutely Pure Modules*, Ph.D. Thesis, University of Kentucky, 1978.
[2] Crevei S., Prest M., and Torrecillas B. *Covers in Finitely Accessible Categories*, Alg. Rep. Theor. **9**, 423-430, 2006.
[3] Enochs E., *Shortening Filtration*, Sci. China Math. **55** (4), 687-693, 2012.
[4] Enochs E. and Estrada, S. *Relative homological algebra in the category of quasi-coherent sheaves*, Adv. in Math. **194** (3), 284-295, 2005.
[5] Enochs E., Estrada, S. and Odabasi, S. *Pure Injective and Absolutely Pure quasi-coherent sheaves*, available at http://arxiv-web3.library.cornell.edu/abs/1307.1729.
[6] Enochs E. and Huang, Z. *Injective Envelopes and (Gorenstein) Flat Covers*, Algebr. Represent. Theor. **15** (6), 1131-1145, 2012.
[7] Estrada S. and Saorín, M. *Locally finitely presented categories with no flat objects*, Forum Math. **0**, 1435-5337, 2012.
[8] Gilespie, J. *The Flat Model Structure on Chain Complexes of Sheaves*, Trans. Amer. MAth. Soc. **358** (7), 2855-2874, 2006.
[9] Hosseini, E. *Flat Quasi Coherent Sheaves of Finite Cotorsion Dimension*, available at http://arxiv.org/abs/1307.5505.
[10] Huang Z. *Duality of preenvelopes and pure injective modules*, Canad. Math. Bull. **57**(2), 318-325, 2014.
[11] Maddox, B. H. *Absolutely Pure Modules*, Proc. amer. Math. soc. **18** (1), 155-158, 1970.
[12] Megibben, C. *Absolutely Pure Modules*, Proc. amer. Math. soc. **26** (4), 561-566, 1970.
[13] Murfet, D. *Modules Over a Scheme*, Available at www.therisingsea.org
[14] Pinzon, K. *Absolutely Pure Covers*, Comm. Algebra, **36**, 2186-2194, 2008.
[15] Stenstrom, B. *Coherent Rings and Fp-injective Modules*, J. London Math. Soc. **2** (2), 323–329, 1970.