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Improvement of Grüss and Ostrowski Type Inequalities

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Abstract. Several inequalities of Ostrowski-Grüss-type available in the literature are generalized considering the weighted case of them. The inequality of Grüss type proved by P. Cerone and S.S. Dragomir [3] is extended for the weighted case.

1. Introduction

In 1935 Grüss [8] proved a integral inequality that establishes a connection between the integral of the product of two functions and the product of the integrals. In 1938, Ostrowski [13] established an interesting integral inequality which gives an upper bound for the approximation of the integral average by the value of mapping in a certain point of the interval. In 1997, Dragomir and Wang [6] combined the Ostrowski inequality with Grüss inequality and obtained a new result for bounded differentiable mappings which is well known in the literature as Ostrowski-Grüss inequality. In 2000, B. Gavrea and I.Gavrea [7] obtained some generalizations of these inequalities using the least concave majorant of the modulus of continuity and the second order modulus of smoothness. During the last few years, many researchers focused their attention on the study and generalizations of these inequalities ([3], [4], [9], [10], [11], [14], [15]). In this paper we generalize these type of inequalities considering the weighted case of them and we improve some Ostrowski-Grüss type inequalities available in the literature.

The functional given by

$$T(f,g) := \frac{1}{b-a}\int_a^b f(t)g(t)dt - \frac{1}{b-a}\int_a^b f(t)dt \cdot \frac{1}{b-a}\int_a^b g(t)dt, \tag{1}$$

where $f, g : [a,b] \to \mathbb{R}$ are integrable functions, is well known in the literature as the Chebyshev functional (see [2]). In 1935, G. Grüss [8] obtained the following result.

Theorem 1.1. *Let f and g be two functions defined and integrable on $[a,b]$. If $m \le f(x) \le M$ and $p \le g(x) \le P$ for all $x \in [a,b]$, then we have*

$$\left|T(f,g)\right| \le \frac{1}{4}(M-m)(P-p). \tag{2}$$

The constant 1/4 is the best possible.

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Another celebrated classical inequality was proved by A. Ostrowski [13] in 1938, which we cite below in the form given by G.A. Anastassiou in 1995 (see [1]).

Theorem 1.2. *Let f be in $C^1[a,b]$, $x \in [a,b]$. Then*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \le \frac{(x-a)^2+(b-x)^2}{2(b-a)}\|f'\|_\infty. \tag{3}$$

In 1997, S.S. Dragomir and S. Wang [6] applied Theorem 1.1 to the mappings $f'(t)$ and $p(x,t) = \begin{cases} t-a, & t\in[a,x] \\ t-b, & t\in(x,b] \end{cases}$ obtaining a new result for bounded differentiable mappings, as shown in the relation (6), which is known as the Ostrowski-Grüss-type inequality. This inequality has been improved by M. Matić et al. ([12]), and we recall their result in (5). An improvement of this result is given by X.L. Cheng in [5], as shown in relation (4). He also proved that the constant $1/8$ is sharp.

Theorem 1.3. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping. If $\gamma \le f'(x) \le \Gamma$, $x \in [a,b]$ for some constants $\gamma, \Gamma \in \mathbb{R}$, then*

$$\begin{aligned} \left|\mathcal{L}(f)(x)\right| &\le \frac{1}{8}(b-a)(\Gamma-\gamma) && (4)\\ &\le \frac{1}{4\sqrt{3}}(b-a)(\Gamma-\gamma) && (5)\\ &\le \frac{1}{4}(b-a)(\Gamma-\gamma), && (6)\end{aligned}$$

where $\mathcal{L}(f)(x) := f(x) - \frac{1}{b-a}\int_a^b f(t)dt - \frac{f(b)-f(a)}{b-a}\left(x - \frac{a+b}{2}\right)$.

In [17], N. Ujević proved the following result, involving the second derivative of the mapping f.

Theorem 1.4. *(see Theorem 4 in [17]) Let $f : I \to \mathbb{R}$, where $I \subset \mathbb{R}$ an interval, be a twice continuously differentiable mapping in the interior Int(I) of I with $f'' \in L_2(a,b)$, and let $a,b \in Int(I)$, $a < b$. Then we have, for all $x \in [a,b]$,*

$$\left|\mathcal{L}(f)(x)\right| \le \frac{(b-a)^{3/2}}{2\pi\sqrt{3}}\|f''\|_2. \tag{7}$$

In this paper we will consider the weighted variant of the functional $\mathcal{L}$ and some new inequalities which involve second derivative of mapping are proved. The weight function is assumed to be non-negative and integrable over its entire domain. In order to formulate the main results we will define the following:

Definition 1.5. *Let $w : (a,b) \to (0,\infty)$ be integrable, i.e., $\int_a^b w(t)dt < \infty$, then $m(\alpha,\beta) := \int_\alpha^\beta w(t)dt$, $M(\alpha,\beta) := \int_\alpha^\beta tw(t)dt$ and $M_2(\alpha,\beta) := \int_\alpha^\beta t^2w(t)dt$ are the first moments, for $[\alpha,\beta] \subseteq [a,b]$. Define the mean of the interval $[\alpha,\beta]$ with respect to the weight function w as $\sigma(\alpha,\beta) := \frac{M(\alpha,\beta)}{m(\alpha,\beta)}$.*

The weighted variant of the functional $\mathcal{L}$ can be written in the following way

$$\mathcal{L}_w(f)(x) := f(x) - \frac{1}{m(a,b)}\int_a^b f(t)w(t)dt - \frac{f(b)-f(a)}{b-a}\left(x-\sigma(a,b)\right).$$

The structure of this paper is as follows: in Section 2 we give new bounds for the functional $\mathcal{L}_w$ and improve some inequalities available in literature. In Section 3 involving the least concave majorant of the modulus of continuity we provide new estimations of the functional $\mathcal{L}_w$. Finally, in Section 4 we extend for the weighted case a Grüss-type inequality proved by P. Cerone and S.S. Dragomir [3].

2. Ostrowski-Grüss-Type Inequalities

The aim of this section is to give new inequalities for the functional $\mathcal{L}_w$ involving the second derivative of mapping.

Denote

$$\mathcal{K}(x,t) := \begin{cases} -\dfrac{1}{m(a,b)}\displaystyle\int_a^t (u-t)w(u)du - \dfrac{t-a}{b-a}\left(x-\sigma(a,b)\right),\ t\in[a,x),\\ \dfrac{1}{m(a,b)}\displaystyle\int_t^b (u-t)w(u)du - \dfrac{t-b}{b-a}\left(x-\sigma(a,b)\right),\ t\in[x,b]. \end{cases} \tag{8}$$

Theorem 2.1. *Let $f:[a,b]\to\mathbb{R}$ be twice differentiable on the interval (a,b), with the second derivative bounded on (a,b), i.e., $\|f''\|_\infty := \sup\limits_{t\in(a,b)} |f''(t)| < \infty$. Then, for all $x\in[a,b]$, we get*

$$\left|\mathcal{L}_w(f)(x)\right| \le u(x;a,b)\|f''\|_\infty, \quad where\ \ u(x;a,b) := \int_a^b |\mathcal{K}(x,t)|dt. \tag{9}$$

Proof. Integrating by parts, we have $\displaystyle\int_a^b \mathcal{K}(x,t)f''(t)dt = -\mathcal{L}_w(f)(x)$. Therefore, we get

$$|\mathcal{L}_w(f)(x)| = \left|\int_a^b \mathcal{K}(x,t)f''(t)dt\right| \le \|f''\|_\infty\cdot\int_a^b |\mathcal{K}(x,t)|dt = \|f''\|_\infty\cdot u(x;a,b).$$

□

Theorem 2.2. *Let $f:[a,b]\to\mathbb{R}$ be twice differentiable on the interval (a,b), with $f''\in L_2[a,b]$. Then, for all $x\in[a,b]$, we have*

$$|\mathcal{L}_w(f)(x)| \le \mu(x;a,b)\|f''\|_2,\ where\ \ \mu(x;a,b) = \left[\int_a^b \left(\mathcal{K}(x,t)\right)^2 dt\right]^{1/2}. \tag{10}$$

Proof. If we consider the function $\mathcal{K}(x,t)$ defined in (8), it follows

$$|\mathcal{L}_w(f)(x)| = \left|\int_a^b \mathcal{K}(x,t)f''(t)dt\right| \le \|f''\|_2\left[\int_a^b \left(\mathcal{K}(x,t)\right)^2 dt\right]^{1/2} = \mu(x;a,b)\|f''\|_2.$$ □

Theorem 2.3. *Let $f:[a,b]\to\mathbb{R}$ be twice differentiable on the interval (a,b), with $f''\in L_1[a,b]$. Then, for all $x\in[a,b]$, we have*

$$|\mathcal{L}_w(f)(x)| \le \nu(x;a,b)\cdot\|f''\|_1,\ where\ \nu(x;a,b) := \sup_{t\in[a,b]} |\mathcal{K}(x,t)|. \tag{11}$$

Proof. If we consider the function $\mathcal{K}(x,t)$ defined by (8), it follows

$$|\mathcal{L}_x[f]| = \left|\int_a^b \mathcal{K}(x,t)f''(t)dt\right| \le \|f''\|_1\cdot\sup_{t\in[a,b]}|\mathcal{K}(x,t)| = \nu(x;a,b)\cdot\|f''\|_1.$$

□

Remark 2.4. *For $a=0$, $b=1$ and $w(x)=1$ the inequalities (9)-(11) and (7) become*

$$|\mathcal{L}(f)(x)| \le C_1(x)\cdot\|f''\|_\infty \le \frac{1}{12}\|f''\|_\infty,\ where \tag{12}$$

$$C_1(x) = \begin{cases} \dfrac{1}{12}\left[-16x^3 + 18x^2 - 6x + 1\right], & x \in [0, 1/2] \\ \dfrac{1}{12}\left[16x^3 - 30x^2 + 18x - 3\right], & x \in (1/2, 1], \end{cases}$$

$$|\mathcal{L}(f)(x)| \leq C_2(x) \cdot \|f''\|_2 \leq \frac{\sqrt{30}}{60}\|f''\|_2 \approx 0.0913\|f''\|_2, \text{ where} \tag{13}$$

$$C_2(x) = \sqrt{\frac{x^4}{4} - \frac{x^3}{2} + \frac{x^2}{3} - \frac{x}{12} + \frac{1}{120}},$$

$$|\mathcal{L}(f)(x)| \leq C_3(x) \cdot \|f''\|_1 \leq \frac{1}{16}\|f''\|_1 \text{ where} \tag{14}$$

$$C_3(x) = \begin{cases} \dfrac{1}{2}\left(x - \dfrac{1}{2}\right)^2, & x \in \left[\dfrac{2-\sqrt{2}}{4}, \dfrac{2+\sqrt{2}}{4}\right] \\ \dfrac{1}{2}x(1-x), & x \in [0,1] \setminus \left[\dfrac{2-\sqrt{2}}{4}, \dfrac{2+\sqrt{2}}{4}\right], \end{cases}$$

$$|\mathcal{L}(f)(x)| \leq \frac{1}{2\pi\sqrt{3}}\|f''\|_2 \approx 0.0919\|f''\|_2. \tag{15}$$

In this particular case, our estimates (13) is better than N. Ujević's result (15).

3. Ostrowski-Grüss-Type Inequalities in Terms of the Least Concave Majorant

In order to formulate the next result we need the following

Definition 3.1. *Let $f \in C[a,b]$. If, for $t \in [0,\infty)$, the quantity $\omega(f;t) = \sup\left\{\left|f(x) - f(y)\right|, \left|x - y\right| \leq t\right\}$ is the usual modulus of continuity, its least concave majorant is given by*

$$\tilde{\omega}(f;t) = \sup\left\{\frac{(t-x)\omega(f;y) + (y-t)\omega(f;x)}{y-x};\ 0 \leq x \leq t \leq y \leq b-a, x \neq y\right\}.$$

Let $I = [a,b]$ be a compact interval of the real axis and $f \in C(I)$. In [16], the following result for the least concave majorant is proved:

$$K\left(\frac{t}{2}, f; C[a,b], C^1[a,b]\right) := \inf_{g \in C^1(I)}\left(\left\|f - g\right\|_\infty + \frac{t}{2}\left\|g'\right\|_\infty\right) = \frac{1}{2}\tilde{\omega}(f;t),\ t \geq 0.$$

Before we proceed to the result of this section we recall the following lemma established by J. Roumeliotis in [15].

Lemma 3.2. *[15] There exists $t^* \in [a,b]$ uniquely satisfying*

$$\frac{m(a,b)}{b-a}|x - \sigma(a,b)| = \begin{cases} m(t^*, b),\ a \leq x \leq \sigma(a,b), \\ m(a, t^*),\ \sigma(a,b) < x \leq b. \end{cases} \tag{16}$$

Denote

$$\mathcal{P}(x,t) = \begin{cases} \dfrac{1}{m(a,b)}\displaystyle\int_a^t w(u)du - \dfrac{1}{b-a}(x - \sigma(a,b)),\ t \in [a,x), \\ \dfrac{1}{m(a,b)}\displaystyle\int_b^t w(u)du - \dfrac{1}{b-a}(x - \sigma(a,b)),\ t \in [x,b]. \end{cases}$$

Lemma 3.3. *If* $\tilde{u}(x;a,b) := \int_a^b |P(x,t)|\,dt$, *then*

$$\tilde{u}(x;a,b) = \frac{2}{m(a,b)} \int_x^{t^*} (t-x)w(t)dt, \text{ for all } x \in [a,b] \text{ and } t^* \text{ as defined in Lemma 3.2.} \tag{17}$$

Proof. If $a \le x \le \sigma(a,b)$, then $\mathcal{P}(x,t) \ge 0,$ for $t \in [a,x] \cup [t^*,b]$ and $\mathcal{P}(x,t) \le 0,$ for $t \in [x,t^*]$.
Also, if $\sigma(a,b) \le x \le b$ we have $\mathcal{P}(x,t) \le 0,$ for $t \in [a,t^*] \cup [x,b]$ and $\mathcal{P}(x,t) \ge 0,$ for $t \in [t^*,x]$.
From proof of Theorem 3 ([15]) it follows

$$\tilde{u}(x;a,b) = \int_a^x \mathcal{P}(x,t)dt + \int_{t^*}^b \mathcal{P}(x,t)dt - \int_x^{t^*} \mathcal{P}(x,t)dt = \frac{2}{m(a,b)} \int_x^{t^*} (t-x)w(t)dt, \text{ for all } x \in [a,\sigma(a,b)].$$

In a similar way the identity (17) can be proved for $x \in (\sigma(a,b),b]$. □

Lemma 3.4. *There exists* $t^{**} \in [a,b]$ *uniquely satisfying*

$$-\frac{1}{m(a,b)} \int_b^{t^{**}} (u-t^{**})w(u)du = \frac{t^{**}-b}{b-a}(x-\sigma(a,b)), \text{ for } a \le x \le \sigma(a,b), \text{and} \tag{18}$$

$$-\frac{1}{m(a,b)} \int_a^{t^{**}} (u-t^{**})w(u)du = \frac{t^{**}-a}{b-a}(x-\sigma(a,b)), \text{ for } \sigma(a,b) < x \le b. \tag{19}$$

Proof. Let us consider $a \le x \le \sigma(a,b)$ and $f(t) = \frac{1}{m(a,b)} \int_t^b (u-t)w(u)du - \frac{t-b}{b-a}(x-\sigma(a,b))$, $t \in [x,b]$.
We have

$$f'(t) = -\frac{1}{m(a,b)} \int_t^b w(u)du - \frac{1}{b-a}(x-\sigma(a,b)).$$

Since $f'(t) \le 0$ on $t \in [x,t^*]$ and $f'(t) \ge 0$ on $t \in [t^*,b]$, where t^* is defined in Lemma 3.2, then to show that $t^{**} \in [x,t^*]$ exists such the identity (18) holds it will suffice to establish that $f(x) \ge 0$.

We have

$$f(x) = -\frac{1}{m(a,b)} \int_b^x (u-x)w(u)du - \frac{x-b}{b-a}(x-\sigma(a,b)) \ge \frac{1}{m(a,b)} \int_a^b (u-x)w(u)du - \frac{x-b}{b-a}(x-\sigma(a,b))$$
$$= (\sigma(a,b)-x) - \frac{x-b}{b-a}(x-\sigma(a,b)) = (\sigma(a,b)-x)\frac{x-a}{b-a} \ge 0.$$

Similarly, we can show (19) to be true for $\sigma(a,b) < x \le b$. □

Lemma 3.5. *If* $u(x;a,b) := \int_a^b |K(x,t)|dt$, *then for all* $x \in [a,b]$ *and* t^{**} *as defined in Lemma 3.4, it follows*

$$\begin{aligned} u(x;a,b) = &\frac{x-\sigma(a,b)}{b-a}\left\{t^{**2} - (b-a)x - \frac{a^2+b^2}{2}\right\} - x\sigma(a,b) + \frac{x^2}{2} \\ &- \frac{1}{m(a,b)}\left\{M_2(t^{**},b) - \frac{1}{2}M_2(a,b) - t^{**2}m(t^{**},b)\right\}, \text{ for } a \le x \le \sigma(a,b) \end{aligned} \tag{20}$$

and

$$\begin{aligned} u(x;a,b) = &\frac{x-\sigma(a,b)}{b-a}\left\{-t^{**2} - (b-a)x + \frac{a^2+b^2}{2}\right\} - x\sigma(a,b) + \frac{x^2}{2} \\ &- \frac{1}{m(a,b)}\left\{M_2(a,t^{**}) - \frac{1}{2}M_2(a,b) - t^{**2}m(a,t^{**})\right\}, \text{ for } \sigma(a,b) < x \le b. \end{aligned} \tag{21}$$

Proof. If $a \le x \le \sigma(a,b)$, then

$$K(x,t) \ge 0, \text{ for } t \in [a,x] \cup (x,t^{**}] \text{ and } K(x,t) < 0, \text{ for } t \in (t^{**}, b]. \tag{22}$$

Also, if $\sigma(a,b) \le x \le b$ we have

$$K(x,t) \ge 0, \text{ for } t \in [t^{**},x] \cup (x,b] \text{ and } K(x,t) < 0, \text{ for } t \in [a, t^{**}). \tag{23}$$

From (22), (23) and Lemma 3.4 the relations (20) and (21) are proved. □

Theorem 3.6. *If $f \in C^1[a,b]$, then*

$$|\mathcal{L}_w(f)(x)| \le \frac{\tilde{u}(x;a,b)}{2}\tilde{\omega}\left(f'; \frac{2u(x;a,b)}{\tilde{u}(x;a,b)}\right), \tag{24}$$

where $\tilde{u}$ and u are defined in Lemma 3.3 and Lemma 3.5.

Proof. Let $A : C[a,b] \to \mathbb{R}$ be defined by $A(f)(x) = \int_a^b \mathcal{P}(x,t)f(t)dt$. We have

$$|A(f)(x)| \le \|f\|_\infty \cdot \tilde{u}(x;a,b). \tag{25}$$

Let $g \in C^1[a,b]$ and $\mathcal{K}(x,t)$ be the mapping defined by (8). We obtain $\int_a^b \mathcal{K}(x,t)g'(t)dt = -A(g)(x)$, namely

$$|A(g)(x)| = \left|\int_a^b \mathcal{K}(x,t)g'(t)dt\right| \le \|g'\|_\infty \cdot \int_a^b |\mathcal{K}(x,t)|dt = \|g'\|_\infty \cdot u(x;a,b). \tag{26}$$

From relations (25) and (26), we have

$$\begin{aligned} |A(f)(x)| &= |A(f-g+g)(x)| \le |A(f-g)(x)| + |A(g)(x)| \le \|f-g\|_\infty \cdot \tilde{u}(x;a,b) + \|g'\|_\infty \cdot u(x;a,b) \\ &\le \tilde{u}(x;a,b) \inf_{g \in C^1[a,b]} \left\{ \|f-g\|_\infty + \frac{u(x;a,b)}{\tilde{u}(x;a,b)}\|g'\|_\infty \right\}. \end{aligned}$$

Therefore,

$$|A(f)(x)| \le \frac{\tilde{u}(x;a,b)}{2}\tilde{\omega}\left(f; \frac{2u(x;a,b)}{\tilde{u}(x;a,b)}\right). \tag{27}$$

If we write (27) for the function f', we obtain inequality (24). □

Theorem 3.7. *If $f \in C[a,b]$, then for all $x \in [a,b]$, we have*

$$|\mathcal{L}(f)(x)| \le 4K\left(\frac{u(x;a,b)}{4}; f; C[a,b], C^2[a,b]\right), \text{ where} \tag{28}$$

$K\left(t; f; C[a,b], C^2[a,b]\right) := \inf_{g \in C^2[a,b]} \{\|f-g\|_\infty + t\|g''\|_\infty\}$ and u is defined in Lemma 3.5.

Proof. For any $f \in C[a,b]$, $|\mathcal{L}_w(f)(x)| \le 4\|f\|_\infty$. For $g \in C^2[a,b]$, from Theorem 2.1 we get $|\mathcal{L}_w(g)(x)| \le u(x;a,b)\|g''\|_\infty$. So, for $f \in C[a,b]$ fixed and $g \in C^2[a,b]$ arbitrary, we have

$$|\mathcal{L}_w(f)(x)| \le |\mathcal{L}_w(f-g)(x)| + |\mathcal{L}_w(g)(x)| \le 4\|f-g\|_\infty + u(x;a,b)\|g''\|_\infty = 4\left\{\|f-g\|_\infty + \frac{u(x;a,b)}{4}\|g''\|_\infty\right\}.$$

Passing to the infimum over $g \in C^2[a,b]$ gives relation (28). □

4. The Weighted Grüss Type Inequalities

In [3], P. Cerone and S.S. Dragomir proved the following inequality for functional $T(f,g)$.

Theorem 4.1. *([3])Assume that $f:[a,b]\to\mathbb{R}$ is a measurable function on $[a,b]$ and such that $\tilde{f}:=f-\frac{1}{b-a}\int_a^b f(t)dt$, $e\tilde{f}\in L_2[a,b]$, where $e(t)=t$, $t\in[a,b]$. If $g:[a,b]\to\mathbb{R}$ is absolutely continuous and $g'\in L_2[a,b]$, then we have the inequality*

$$|T(f,g)|\leq\frac{2}{b-a}\|g'\|_2\cdot\left[\frac{\int_a^b\tilde{f}(t)^2dt\cdot\int_a^b t^2\tilde{f}(t)^2dt-\left(\int_a^b t\tilde{f}(t)^2dt\right)^2}{\int_a^b\tilde{f}(t)^2dt}\right]^{1/2}\leq\frac{2}{b-a}\|g'\|_2\cdot\|e\tilde{f}\|_2.$$

In this section we will prove a Grüss type inequality for the weighted case. Let $w:[a,b]\to[0,\infty)$ a weight function and we will consider that w is a bounded function, namely exists $K\in\mathbb{R}$ such that $w(x)\leq K$ for all $x\in[a,b]$.

Denote

$$T_w(f,g):=\frac{\int_a^b f(x)g(x)w(x)dx}{\int_a^b w(x)dx}-\frac{\int_a^b f(x)w(x)dx\int_a^b g(x)w(x)dx}{\left(\int_a^b w(x)dx\right)^2},$$

the Čebyšev functional for the weighted case.

Consider in Hilbert space $L_{2,w}[a,b]$ the inner product $\langle f,g\rangle_w=\int_a^b f(x)g(x)w(x)dx$ with the norm $\|f\|_{2,w}=\left(\int_a^b f(x)^2w(x)dx\right)^{1/2}$.

Lemma 4.2. *Assume that function $f:[a,b]\to\mathbb{R}$ is Lebesque integrable and $\int_a^b f(x)w(x)dx=0$. Define $F(x)=\int_a^x f(t)w(t)dt$, $e(x)=x$, $x\in[a,b]$ and assume that $F,f,ef\in L_{2,w}[a,b]$. Then we have*

$$\|F\|_{2,w}^2\leq 4K^2\cdot\frac{\|f\|_{2,w}^2\cdot\|fe\|_{2,w}^2-(\langle f,fe\rangle_w)^2}{\|f\|_{2,w}^2}\leq 4K^2\|fe\|_{2,w}^2. \tag{29}$$

Proof. We have

$$\begin{aligned}\|F\|_{2,w}^2 &= \int_a^b F(x)^2w(x)dx\leq K\int_a^b F(x)^2dx=K\int_a^b F(x)^2(x-\lambda)'dx\\ &= K\cdot F(x)^2(x-\lambda)\Big|_a^b-K\cdot\int_a^b 2F(x)f(x)w(x)(x-\lambda)dx=2K\cdot\int_a^b(\lambda-x)F(x)f(x)w(x)dx\\ &\leq 2K\cdot\left(\int_a^b(\lambda-x)^2f(x)^2w(x)dx\right)^{1/2}\cdot\left(\int_a^b F(x)^2w(x)dx\right)^{1/2}=2K\cdot\|F\|_{2,w}\cdot\left(\int_a^b(\lambda-x)^2f(x)^2w(x)dx\right)^{1/2}.\end{aligned}$$

From the above relation we have

$$\|F\|_{2,w}\leq 2K\left(\int_a^b(\lambda-x)^2f(x)^2w(x)dx\right)^{1/2},$$

namely

$$\|F\|_{2,w}^2\leq 4K^2\int_a^b(\lambda-x)^2f(x)^2w(x)dx.$$

Denote

$$g(\lambda) := \int_a^b (\lambda - x)^2 f(x)^2 w(x)dx = \lambda^2 \|f\|_{2,w}^2 - 2\lambda \langle f, ef \rangle_w + \|fe\|_{2,w}^2.$$

Since $\inf_{\lambda \in \mathbb{R}} g(\lambda) = \frac{\|f\|_{2,w}^2 \cdot \|fe\|_{2,w}^2 - (\langle f, fe \rangle_w)^2}{\|f\|_{2,w}^2}$, we obtain the relation (29).

Corollary 4.3. *Assume that f satisfies the assumption in Lemma 4.2 and $0 < a < b$. Then we have the inequality*

$$\|F\|_{2,w}^2 \leq K^2 \frac{(b-a)^2}{ab} \cdot \frac{(\langle f, ef \rangle_w)^2}{\|f\|_{2,w}^2} \leq K^2 \frac{(b-a)^2}{ab} \|ef\|_{2,w}^2.$$

Proof. We use the following integral version of Cassels' inequality

$$\frac{\int_a^b p(t) l(t)^2 dt \int_a^b p(t) h(t)^2 dt}{\left(\int_a^b p(t) l(t) h(t) dt\right)^2} \leq \frac{(M+m)^2}{4mM}, \tag{30}$$

provided $0 < m \leq \frac{h(t)}{l(t)} \leq M < \infty$ a.e.t $t \in [a,b]$ and $p \geq 0$ a.e.t on $[a,b]$.

Applying (30) for $p(t) = f(t)^2 w(t)$, $l(t) = 1$, $h(t) = t$, $t \in [a,b]$ we get

$$\frac{\|f\|_{2,w}^2 \cdot \|fe\|_{2,w}^2}{(\langle f, fe \rangle_w)^2} \leq \frac{(b+a)^2}{4ab}, \text{ namely}$$

$$\|f\|_{2,w}^2 \cdot \|fe\|_{2,w}^2 - (\langle f, fe \rangle_w)^2 \leq \frac{(b-a)^2}{4ab} (\langle f, fe \rangle_w)^2.$$

Using relation (29) we have

$$\|F\|_{2,w}^2 \leq K^2 \frac{(b-a)^2}{ab} \frac{(\langle f, fe \rangle_w)^2}{\|f\|_{2,w}^2} \leq K^2 \frac{(b-a)^2}{ab} \|ef\|_{2,w}^2.$$

Theorem 4.4. *Assume that $f : [a,b] \to \mathbb{R}$ is a measurable function on $[a,b]$ and such that $\tilde{f} := f - \frac{1}{\int_a^b w(t)dt} \int_a^b f(t) w(t) dt$, $e\tilde{f} \in L_{2,w}[a,b]$. If $g : [a,b] \to \mathbb{R}$ is absolutely continuous and $g' \in L_{2,w}[a,b]$, then we have the inequality*

$$|T_w(f,g)| \leq \frac{2K}{\int_a^b w(t)dt} \|g'\|_{2,w} \cdot \frac{\left[\|\tilde{f}\|_{2,w}^2 \cdot \|e\tilde{f}\|_{2,w}^2 - \langle \tilde{f}, e\tilde{f} \rangle_w^2\right]^{1/2}}{\|\tilde{f}\|_{2,w}} \leq \frac{2K}{\int_a^b w(t)dt} \|g'\|_{2,w} \cdot \|e\tilde{f}\|_{2,w}. \tag{31}$$

Proof. Denote

$$\tilde{F}(x) = \int_a^x \tilde{f}(t) w(t) dt, \;\; x \in [a,b]$$

We have

$$\frac{1}{\int_a^b w(t)dt} \int_a^b \tilde{F}(x) g'(x) dx = \frac{1}{\int_a^b w(t)dt} \tilde{F}(x) g(x) \Big|_a^b - \frac{1}{\int_a^b w(t)dt} \int_a^b \tilde{f}(x) w(x) g(x) dx.$$

But

$$\tilde{F}(b) = \int_a^b \left(f(t) - \frac{1}{\int_a^b w(u)du} \int_a^b f(u) w(u) du \right) w(t) dt = \int_a^b f(t) w(t) dt - \frac{1}{\int_a^b w(u)du} \int_a^b w(t) dt \cdot \int_a^b f(u) w(u) du = 0.$$

Therefore

$$\frac{1}{\int_a^b w(t)dt}\int_a^b \tilde{F}(x)g'(x)dx = -\frac{1}{\int_a^b w(t)dt}\int_a^b \left(f(x) - \frac{1}{\int_a^b w(t)dt}\int_a^b f(t)w(t)dt\right)w(x)g(x)dx$$
$$= -\frac{1}{\int_a^b w(t)dt}\cdot\left[\int_a^b f(x)g(x)w(x)dx - \frac{1}{\int_a^b w(t)dt}\int_a^b f(t)w(t)dt\cdot\int_a^b g(x)w(x)dx\right] = -T_w(f,g).$$

From the above relation we can write

$$|T_w(f,g)| \le \frac{1}{\int_a^b w(t)dt}\cdot ||\tilde{F}||_{2,w}\cdot ||g'||_{2,w}.$$

From Lemma 4.2 we obtain the inequality (31).

Remark 4.5. *For $w(x) = 1$ we have the results obtained by Cerone and Dragomir in [3].*

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