Filomat 29:9 (2015), 2069–2077
DOI 10.2298/FIL1509069B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On Matrix Methods of Convergence of Order α in (ℓ)-Groups

Antonio Boccuto[a], **Pratulananda Das**[b]

[a] *Department of Mathematics and Computer Sciences, University of Perugia, via Vanvitelli,1, I-06123 Perugia (Italy)*
[b] *Department of Mathematics, Jadavpur University, Kolkata-700032, West Bengal, India*

Abstract. We introduce a concept of convergence of order α, with $0 < \alpha \leq 1$, with respect to a summability matrix method A for sequences (which generalizes the notion of statistical convergence of order α), taking values in (ℓ)-groups. Some main properties and differences with the classical A-convergence are investigated. A Cauchy-type criterion and a closedness result for the space of convergent sequences according our notion is proved.

1. Introduction

The notion of statistical convergence was introduced in [16, 24]. In the literature there are several extensions of this concept: indeed, statistical convergence can be viewed as a particular case of convergence with respect to a summability matrix method (see for example [21]) as well as ideal and filter convergence (see for example [3, 19, 20]).

Several properties of summability matrices, matrix convergence and their various applications to approximation theory can be seen from [2] where many important references can be found.

For different works on the statistical and ideal convergences one can see, for example, [1, 10, 12, 13] where many more references can be found. In [8] the notion of ideal convergence in (ℓ)-groups was introduced and the main properties were examined, while in [6–8] there are some versions of basic matrix theorems and limit theorems for ideal pointwise convergent measures taking values in an (ℓ)-group R.

In [4] (also independently in [11]) a natural extension of statistical convergence is presented, by replacing n with a non linear term n^α, $0 < \alpha < 1$, in the definition of asymptotic density. This is motivated by the investigation of different kinds of densities, and by the problem of comparing them with the natural density.

Since the notion of A-density (density with respect to regular summability matrix A) is a natural extension of asymptotic density, it seems natural to investigate different kinds of A-densities and associated convergence in line of [4]. In this paper we precisely do that and extend the statistical convergence of order α to convergence of order α with respect to matrix methods, and we deal with the (ℓ)-group setting. One must note that though our definition is very similar to the notion of A-statistical convergence with the rate $o(a_n)$ for a given sequence of non-increasing positive reals (a_n) [14], (from where also our inspiration came) but it is not similar to that notion. We prove a Cauchy-type criterion, some main properties and some fundamental differences of the behavior of this kind of convergence between the cases $\alpha = 1$ and $0 < \alpha < 1$. Furthermore we present a property of closedness for the space of sequences, converging according our definition.

2010 *Mathematics Subject Classification*. Primary 28A20, 28B15, 54A20.
Keywords. (ℓ)-group, summability matrix, A^α-convergence.
Received: 07 February 2014; Accepted: 05 April 2014
Communicated by Eberhard Malkowsky
Email addresses: antonio.boccuto@unipg.it (Antonio Boccuto), pratulananda@yahoo.co.in (Pratulananda Das)

2. Preliminaries

We begin with recalling some notions about densities and summability matrix methods, which will be useful in the sequel.

Definitions 2.1. (a) Let $\mathbb{N}$ be the set of all natural numbers and $B \subset \mathbb{N}$. If $m, n \in \mathbb{N}$, $m < n$, we denote by $B(m, n)$ the cardinality of the set $B \cap \{m, m+1, \ldots, n\}$.

(b) Let $0 < \alpha \leq 1$ be a real number. The *lower* and *upper asymptotic density of order* α of the set B are defined by

$$\underline{d}^{\alpha}(B) = \liminf_n \frac{B(1,n)}{n^{\alpha}}, \quad \overline{d}^{\alpha}(B) = \limsup_n \frac{B(1,n)}{n^{\alpha}}. \tag{1}$$

If the limit $\lim_n \frac{B(1,n)}{n^{\alpha}}$ exists in $\mathbb{R}$, then the common value in (1) is said to be the *asymptotic density of the set B of order α* and is denoted by $d^{\alpha}(B)$ ([11], [4]).

(c) If $(x_k)_k$ is a sequence of real numbers, we say that $(x_k)_k$ *converges statistically of order α to $x_0 \in \mathbb{R}$* (shortly, $S^{\alpha} \lim_k x_k = x_0$) iff for each $\varepsilon > 0$ we have $d^{\alpha}(A(\varepsilon)) = 0$, where $A(\varepsilon) := \{k \in \mathbb{N} : |x_k - x_0| > \varepsilon\}$ ([11], [4]).

(d) Let $A := (a_{j,k})_{j,k}$ be an infinite summability matrix. For a given sequence $\underline{x} = (x_k)_k$ in $\mathbb{R}$, the *A-transform of $\underline{x}$*, denoted by $(A\underline{x})_j$, is given by

$$(A\underline{x})_j := \sum_{k=1}^{\infty} a_{j,k}\, x_k, \tag{2}$$

provided that the series in (2) converges for each $j \in \mathbb{N}$ (see [2]).

We say that A is *regular* iff $\lim_j (A\underline{x})_j = L$ whenever $\lim_k x_k = L$ with $L \in \mathbb{R}$.

The following characterization of regularity of a matrix A is known in the literature as the *Silverman-Toeplitz conditions* (see also [2, Theorem 1.6]).

Theorem 2.2. *An infinite summability matrix $A = (a_{j,k})_{j,k}$ is regular if and only if the following conditions are satisfied:*

(i) $\sup_j \Big(\sum_{k=1}^{\infty} |a_{j,k}| \Big) < +\infty$,

(ii) $\lim_j a_{j,k} = 0$ *for each* $k \in \mathbb{N}$,

(iii) $\lim_j \Big(\sum_{k=1}^{\infty} a_{j,k} \Big) = 1$.

Using regular matrices, Freedman and Sember ([17]) extended the idea of the statistical convergence to A-statistical convergence as follows.

Definitions 2.3. (a) Let $A = (a_{j,k})_{j,k}$ be a non-negative regular summability matrix. The *A-density* of a subset $K \subset \mathbb{N}$ is defined by

$$\delta_A(K) := \lim_j \sum_{k \in K} a_{j,k}, \tag{3}$$

provided that the limit in (3) exists in $\mathbb{R}$. Similarly it is possible to define the concepts of *A-upper density* and *A-lower density* of $K \subset \mathbb{N}$ as follows:

$$\overline{\delta}_A(K) := \limsup_j \sum_{k \in K} a_{j,k}, \quad \underline{\delta}_A(K) := \liminf_j \sum_{k \in K} a_{j,k}$$

respectively.

(b) A sequence $(x_k)_k$ of real numbers is said to be *A-convergent* to $x_0 \in \mathbb{R}$ if $\delta_A(A(\varepsilon)) = 0$ or equivalently

$$\lim_j \sum_{k \in A(\varepsilon)} a_{j,k} = 0,$$

where $A(\varepsilon)$ is as in Definitions 2.1 (c). Note that, when $A := C_1 = (c_{j,k})_{j,k}$ is the Cesàro matrix, defined by

$$c_{j,k} := \begin{cases} \frac{1}{j} & \text{if } 1 \leq k \leq j, \\ 0 & \text{otherwise,} \end{cases}$$

the A-density and A-convergence become the usual asymptotic density and statistical convergence respectively.

In [14] the following notion was introduced.

Definition 2.4. *Let $A = (a_{j,k})_{j,k}$ be a non-negative regular summability matrix and let $a = (a_n)$ be a non-increasing sequence of positive real numbers. We say that $x = (x_k)$ is A-statistically convergent to L with the rate of $o(a_n)$ if for each $\epsilon > 0$,*

$$\lim_n \frac{1}{a_n} \sum_{k; |x_k - L| \geq \epsilon} a_{nk} = 0$$

We now recall some concepts about (ℓ)-groups (see for example [22]).

An (ℓ)-group R is said to be *Dedekind complete* if every nonempty subset $A \subset R$, bounded from above, has a supremum in R. A Dedekind complete (ℓ)-group R is said to be *super Dedekind complete* if for any nonempty set $A \subset R$, bounded from above, there exists a countable subset $A^* \subset A$, such that $\sup A = \sup A^*$.

From now on, we always suppose that R is a Dedekind complete (ℓ)-group.

An (O)-*sequence* $(\sigma_p)_p$ in R is a monotone decreasing sequence, such that $\bigwedge_p \sigma_p = 0$.

A (D)-*sequence* or *regulator* in R is a bounded double sequence $(a_{i,j})_{i,j}$, such that for every $i \in \mathbb{N}$ the sequence $(a_{i,j})_j$ is an (O)-sequence.

An (ℓ)-group R is *weakly σ-distributive* if

$$\bigwedge_{\varphi \in \mathbb{N}^{\mathbb{N}}} \left(\bigvee_{i=1}^{\infty} a_{i,\varphi(i)} \right) = 0 \tag{4}$$

for every (D)-sequence $(a_{i,j})_{i,j}$.

A sequence $(x_k)_k$ in R is (O)-*convergent to* $x \in R$ (and we write $(O) \lim_k x_k = x$) if there is an (O)-sequence $(\sigma_p)_p$ such that for each $p \in \mathbb{N}$ there exists $\overline{k} \in \mathbb{N}$ with

$$|x_k - x| \leq \sigma_p \quad \text{for all } k \geq \overline{k}.$$

A sequence $(x_k)_k$ in R is (O)-*Cauchy* if there exists an (O)-sequence $(\tau_p)_p$ with the property that for every $p \in \mathbb{N}$ there is $\overline{k} \in \mathbb{N}$ such that

$$|x_h - x_k| \leq \tau_p \quad \text{whenever } h, k \geq \overline{k}.$$

An (ℓ)-group R is (O)-*complete* if every (O)-Cauchy sequence in R is (O)-convergent in R. Note that every Dedekind complete (ℓ)-group is (O)-complete.

We now fix an exponent $0 < \alpha \leq 1$, and introduce the concepts of A-density and A-convergence of order α for a summability matrix $A = (a_{j,k})_{j,k}$.

Definition 2.5. The *A-density of order* α of a subset $K \subset \mathbb{N}$ is defined by

$$\delta_A^\alpha(K) := \lim_j \frac{1}{j^{\alpha-1}} \sum_{k \in K} a_{j,k}, \tag{5}$$

provided that the limit in (5) exists in $\mathbb{R}$. Analogously we define the *upper A-density of order* α and *lower A-density of order* α as follows:

$$\overline{\delta}_A^\alpha(K) := \limsup_j \frac{1}{j^{\alpha-1}} \sum_{k \in K} a_{j,k}, \quad \underline{\delta}_A^\alpha(K) := \liminf_j \frac{1}{j^{\alpha-1}} \sum_{k \in K} a_{j,k},$$

respectively. Note that for $A = C_1$, A-density of order α reduces to density of order α (see Definition 2.1 (b)).

Remark 2.6. At a first glance though it appears that Definition 2.5 is a special case of Definition 2.4 but that is not the case. Because in our definition we have actually considered the sequence $a = (a_n)$ where $a_n = n^{\alpha-1}$ but it is a strictly increasing sequence whereas in Definition 2.4 the sequence $a = (a_n)$ must be non-increasing.

Remark 2.7. Observe that, if $0 < \alpha < 1$ and $E \subset \mathbb{N}$ is such that $\delta_A^\alpha(E) = 0$, then $\delta_A^\alpha(\mathbb{N} \setminus E) = +\infty$. Indeed we get

$$\begin{aligned} \lim_j \frac{1}{j^{\alpha-1}} \sum_{k=1}^{\infty} a_{j,k} &= \lim_j \frac{1}{j^{\alpha-1}} \sum_{k \in E} a_{j,k} + \\ + \quad \lim_j \frac{1}{j^{\alpha-1}} \sum_{k \in \mathbb{N}\setminus E} a_{j,k} &= \delta_A^\alpha(E) + \delta_A^\alpha(\mathbb{N} \setminus E), \end{aligned} \tag{6}$$

provided that the limits involved exist in $[0, +\infty]$. Since $\delta_A^\alpha(E) = 0$, from (6) and regularity of A we obtain

$$\delta_A^\alpha(\mathbb{N} \setminus E) = \lim_j \frac{1}{j^{\alpha-1}} \sum_{k=1}^{\infty} a_{j,k} = +\infty. \tag{7}$$

Definitions 2.8. (a) A sequence $(x_k)_k$ in an (ℓ)-group R is *A-convergent of order* α or A^α*-convergent* to $x_0 \in R$ (shortly, $A^\alpha \lim_k x_k = x_0$) if there exists an (O)-sequence $(\sigma_p)_p$ such that, for every $p \in \mathbb{N}$, $\delta_A^\alpha(B_p) = 0$, where

$$B_p := \{k \in \mathbb{N} : |x_k - x_0| \not\leq \sigma_p\}, \tag{8}$$

or equivalently

$$\lim_j \frac{1}{j^{\alpha-1}} \sum_{k \in B_p} a_{j,k} = 0.$$

In this case we write $A^\alpha \lim_n x_n = x_0$.

(b) A sequence $(x_k)_k$ in R is A^α*-Cauchy* if there is an (O)-sequence $(\tau_p)_p$ such that for every $p \in \mathbb{N}$ there exists $n \in \mathbb{N}$ with

$$\delta_A^\alpha(\{k \in \mathbb{N} : |x_k - x_n| \not\leq \tau_p\}) = 0.$$

Remark 2.9. For $A = C_1$, the Cesàro matrix, A^α-convergence reduces to statistical convergence of order α (see [4]). Furthermore, if $(x_k)_k$ is A-convergent of order α when $\alpha = 1$, we see simply that $(x_k)_k$ is *A-convergent*. The collection of all sequences in an (ℓ)-group R which are A-convergent and A^α-convergent are denoted by Am_0 and Am_0^α respectively.

3. The Main Results

We begin with a Cauchy-type criterion, which extends (see also [18]) [8, Proposition 2.13] to A^α-convergence in the (ℓ)-group setting.

Proposition 3.1. *A sequence $(x_k)_k$ in R is A^α-convergent if and only if it is A^α-Cauchy.*

Proof. We begin with the sufficient part. Let $(\sigma_p)_p$ be an (O)-sequence, related with the Cauchy condition. There is a sequence $(n_p)_p$ in $\mathbb{N}$ with $\delta_A^\alpha(\mathbb{N} \setminus F_p) = 0$ for all $p \in \mathbb{N}$, where

$$F_p := \{k \in \mathbb{N} : |x_k - x_{n_p}| \leq \sigma_p\}. \tag{9}$$

We now claim that $F_p \cap F_q \neq \emptyset$ whenever $p \neq q$. Otherwise we have $F_p \subset \mathbb{N} \setminus F_q$. But $\delta_A^\alpha(\mathbb{N} \setminus F_q) = 0$, while $\delta_A^\alpha(F_p) = +\infty$ or 1 according as $0 < \alpha < 1$ or $\alpha = 1$ (see also (7)). This is a contradiction, which proves the claim.

Let now $p \neq q$. There exists $k_{p,q} \in \mathbb{N}$ with $|x_{k_{p,q}} - x_{n_p}| \leq \sigma_p$ and $|x_{k_{p,q}} - x_{n_q}| \leq \sigma_q$. So we get $|x_{n_p} - x_{n_q}| \leq \sigma_p + \sigma_q$, and hence $(x_{n_p})_p$ is an (O)-Cauchy sequence. As every Dedekind complete (ℓ)-group is (O)-complete (see [9]), there is $y \in R$ with $(O)\lim_p x_{n_p} = y$. So for each $p \in \mathbb{N}$ we get

$$\begin{aligned} & \{k \in \mathbb{N} : |x_k - y| \not\leq 2\,\sigma_p\} \subset \{k \in \mathbb{N} : |x_k - x_{n_p}| + |x_{n_p} - y| \not\leq 2\,\sigma_p\} \\ \subset \quad & \{k \in \mathbb{N} : |x_{n_p} - y| \not\leq \sigma_p\} \cup \{k \in \mathbb{N} : |x_k - x_{n_p}| \not\leq \sigma_p\}, \end{aligned}$$

and hence

$$\delta_A^\alpha(\{k \in \mathbb{N} : |x_k - y| \not\leq 2\,\sigma_p\}) = 0.$$

So, $A^\alpha \lim_k x_k = y$, which proves the sufficient part.

We now turn to the necessary part. We know that there exist an (O)-sequence $(\sigma_p)_p$ and $y \in R$ with $d_A^\alpha(B_p) = 0$ for every $p \in \mathbb{N}$, where

$$B_p := \{k \in \mathbb{N} : |x_k - y| \not\leq \sigma_p\}.$$

Observe that $\mathbb{N} \setminus B_p \neq \emptyset$ for all $p \in \mathbb{N}$, because $d_A^\alpha(B_p) = +\infty$ or 1 according as $0 < \alpha < 1$ or $\alpha = 1$ (see also (7)). If $k, n \in \mathbb{N} \setminus B_p$, then

$$|x_k - x_n| \leq |x_k - y| + |x_n - y| \leq 2\,\sigma_p. \tag{10}$$

Let $V_p := \{k \in \mathbb{N} : |x_k - x_n| \not\leq 2\sigma_p\}$, $p \in \mathbb{N}$. From (10), for any $p \in \mathbb{N}$ we get $V_p \subset B_p$, and thus $d_A^\alpha(V_p) = 0$, since $d_A^\alpha(B_p) = 0$. Thus the assertion of the necessary part follows. This concludes the proof. □

We now prove uniqueness of the A^α-limit in the (ℓ)-group setting.

Theorem 3.2. *Let $(x_k)_k$ be an A^α-convergent sequence. Then, its A^α-limit is unique.*

Proof. Let $(x_k)_k$ be a sequence in R, with $A^\alpha \lim_k x_k = x_0$ and $A^\alpha \lim_k x_k = y_0$. Then there are two (O)-sequences $(\sigma_p)_p$, $(\tau_p)_p$ such that

$$\delta_A^\alpha(\{k \in \mathbb{N} : |x_k - x_0| \not\leq \sigma_p\}) = \delta_A^\alpha(\{k \in \mathbb{N} : |x_k - y_0| \not\leq \tau_p\}) = 0 \text{ for all } p \in \mathbb{N}.$$

Fix now $p \in \mathbb{N}$, and let

$$D_1 := \{k \in \mathbb{N} : |x_k - x_0| \leq \sigma_p\}, \quad D_2 := \{k \in \mathbb{N} : |x_k - x_0| \leq \tau_p\}.$$

If $D_1 \cap D_2 = \emptyset$, then $D_1 \subset \mathbb{N} \setminus D_2$, and hence $\overline{\delta}_A^\alpha(D_1) \leq \overline{\delta}_A^\alpha(\mathbb{N} \setminus D_2)$, but $\overline{\delta}_A^\alpha(\mathbb{N} \setminus D_2) = 0$, while $\overline{\delta}_A^\alpha(D_1) = +\infty$ or 1 according as $0 < \alpha < 1$ or $\alpha = 1$ (see also (7)), a contradiction. Hence, $D_1 \cap D_2 \neq \emptyset$. Let $k \in D_1 \cap D_2$, then

$$|x_0 - y_0| \leq |x_0 - x_k| + |x_k - y_0| \leq \sigma_p + \tau_p.$$

Since $(\sigma_p)_p$, $(\tau_p)_p$ are (O)-sequences, by arbitrariness of p we get $x_0 = y_0$. □

The next result is a fundamental property of A^α-convergence in the case $0 < \alpha < 1$, and extends [4, Theorem 1] to the (ℓ)-group context.

Theorem 3.3. *Let A be a regular and lower triangular summability matrix. If a sequence $(x_k)_k$ in R is A^α-convergent to x_0 (with $0 < \alpha < 1$), then there is a set $C := \{k_1 < k_2 < \ldots < k_n < \ldots\} \subset \mathbb{N}$ with $\overline{\delta}^\alpha_A(C) = +\infty$ and $(O)\lim_n x_{k_n} = x_0$.*

Proof. *Suppose that $(x_n)_n$ is A^α-convergent to x_0. Then there exists an (O)-sequence $(\sigma_p)_p$ with $\delta^\alpha_A(B_p) = 0$ for each $p \in \mathbb{N}$, where B_p is as in (8). Set $C_p = \mathbb{N} \setminus B_p$. Since $\delta^\alpha_A(B_p) = 0$, then $\delta^\alpha_A(C_p) = +\infty$ (see also 7)). Moreover, it is easy to see that the sequence $(C_p)_p$ is decreasing.*

Let now $(G_i)_i$ be a strictly increasing sequence of positive real numbers. Choose arbitrarily $v_1 \in C_1$. Since $\delta^\alpha_A(C_2) = +\infty$, there is $v_2 \in C_2$, $v_2 > v_1$, with

$$\sum_{k\in C_2} a_{j,k} > G_2 \cdot j^{\alpha-1} \quad \text{for each } j \ge v_2$$

In particular, we get

$$\sum_{k\in C_2} a_{v_2,k} > G_2 \cdot {v_2}^{\alpha-1}.$$

At the next step, since $\delta^\alpha_A(C_3) = +\infty$, we can find an element $v_3 \in C_3$, $v_3 > v_2$, with

$$\sum_{k\in C_3} a_{v_3,k} > G_3 \cdot {v_3}^{\alpha-1}.$$

Proceeding by induction, we construct a strictly increasing sequence $(v_j)_j$ of positive integers, with $v_j \in C_j$ and

$$\sum_{k\in C_j} a_{v_j,k} > G_j \cdot {v_j}^{\alpha-1} \quad \text{for all } j \in \mathbb{N}.$$

Put now

$$C := [1, v_1] \cup \left(\bigcup_{j=2}^{\infty} ([v_{j-1}, v_j] \cap C_j) \right).$$

Since A is lower triangular, we get

$$a_{v_j,k} = 0 \quad \text{for all } j \in \mathbb{N} \text{ and } k > v_j. \tag{11}$$

As $C \cap [1, v_j] \supset C_j \cap [1, v_j]$ for every $j \in \mathbb{N}$, from (11) we get

$$\sum_{k\in C} a_{v_j,k} \ge \sum_{k\in C_j} a_{v_j,k} > G_j \cdot {v_j}^{\alpha-1},$$

that is

$$\frac{1}{{v_j}^{\alpha-1}} \sum_{k\in C} a_{v_j,k} > G_j, \quad \text{for all } j \in \mathbb{N}. \tag{12}$$

From (12) it follows that

$$\limsup_j \frac{1}{{v_j}^{\alpha-1}} \sum_{k\in C} a_{v_j,k} = +\infty,$$

namely $\overline{\delta}^\alpha_A(C) = +\infty$.

Now, to show that the sequence $(x_k)_{k\in C}$ (O)-converges to x_0, it is sufficient to observe that for all $j \in \mathbb{N}$ and $k \ge v_j$ we have $|x_k - x_0| \le \sigma_j$, where $(\sigma_j)_j$ is an (O)-sequence. This completes the proof. □

Remarks 3.4. (a) Some examples of lower triangular non-negative regular matrices, whose C_1 is a particular case, are the matrices which generate the Erdős-Ulam ideals (see [15, Example 1.2.3 (d)]).

(b) Observe that, for $\alpha = 1$, A^α-convergence reduces to A-convergence, that is the convergence generated by the ideal of those subsets of $\mathbb{N}$ having A-density zero. From this (see [8, Proposition 2.8]) it follows that in a Dedekind complete (ℓ)-group R, a sequence $(x_k)_k$ is A-convergent to x_0 if and only if there is a subset $C \subset \mathbb{N}$, $C = \{k_1 < \ldots < k_n < \ldots\}$, with $\delta_A(C) = 1$ and $(O)\lim_n x_{k_n} = x_0$.

(c) For $0 < \alpha < 1$, the converse of Theorem 3.3 is in general not true, as can be seen by taking

$$x_k := \begin{cases} (4,4), & \text{if } k = n^2, n \in \mathbb{N}. \\ (0,0), & \text{otherwise.} \end{cases} \tag{13}$$

Indeed, for $\alpha = \dfrac{1}{4}$, $\overline{\delta}_A^\alpha(C) = +\infty$, where $C = \{n^2 : n \in \mathbb{N}\}$, but it is not true that $A^\alpha \lim_k x_k = (4,4)$.

From now on we do not require lower triangularity of the summability matrix involved, but only non-negativity and regularity.

We now check the following inclusion, extending [4, Theorem 2].

Theorem 3.5. *Let $0 < \alpha \leq \beta \leq 1$. Then $Am_0^\alpha \subset Am_0^\beta$.*

Proof. Let $(x_k)_k \in Am_0^\alpha$ and with $A^\alpha \lim_k x_k = x_0$. Then there is an (O)-sequence $(\sigma_p)_p$ such that for all $p \in \mathbb{N}$ we get $\delta_A^\alpha(B_p) = 0$, namely

$$\lim_j \frac{1}{j^{\alpha-1}} \sum_{k \in B_p} a_{j,k},$$

where $B_p := \{k \in \mathbb{N} : |x_k - x_0| \not\leq \sigma_p\}$. Since

$$0 \leq \limsup_j \frac{1}{j^{\beta-1}} \sum_{k \in B_p} a_{j,k} \leq \limsup_j \frac{1}{j^{\alpha-1}} \sum_{k \in B_p} a_{j,k} = 0,$$

then we get $\delta_A^\beta(B_p) = 0$, which shows that $(x_k)_k \in Am_0^\beta$. □

If we take $\beta = 1$ in Theorem 3.5, then we obtain the following result.

Corollary 3.6. *If a sequence $(x_k)_k$ in R is A^α-convergent to x_0 for some $0 < \alpha \leq 1$, then it is A-convergent to x_0, that is $Am_0^\alpha \subset Am_0$.*

We now prove a closedness property for the space of A^α-convergent sequences, extending [4, Theorem 3] to the (ℓ)-group setting.

Theorem 3.7. *Let $\alpha \in (0,1]$, $(\underline{x}^{(k)})_k$ be a sequence in Am_0^α, where $\underline{x}^{(k)} = (x_j^{(k)})_j$ and for every $k \in \mathbb{N}$ $(x_j^{(k)})_j$ is A^α-convergent with respect to a common (O)-sequence $(\sigma_p)_p$, independent of k. If $\underline{x} = (x_j)_j$ is such that*

$$(O)\lim_k \left(\bigvee_j |x_j^{(k)} - x_j| \right) = 0, \tag{14}$$

then $\underline{x} \in Am_0^\alpha$.

Proof. Let $(\underline{x}^{(k)})_k$ satisfy (14), where $\underline{x}^{(k)} \in Am_0$ for all $k \in \mathbb{N}$. Suppose that

$$A^\alpha \lim_j x_j^{(k)} = y_k \quad \text{for all } k \in \mathbb{N}. \tag{15}$$

By (15) there is an (O)-sequence $(\sigma_p)_p$ with

$$\delta_A^\alpha(\{n \in \mathbb{N} : |x_n^{(k)} - y_k| \not\leq \sigma_p\}) = 0 \quad \text{for all } k, p \in \mathbb{N}.$$

Fix now $p \in \mathbb{N}$, and put

$$E_k := \{n \in \mathbb{N} : |x_n^{(k)} - y_k| \le \sigma_p\} \tag{16}$$

for all $k \in \mathbb{N}$. If $E_k \cap E_r = \emptyset$, then $E_k \subset \mathbb{N} \setminus E_r$ and hence $\overline{\delta}_A^{\alpha}(E_k) \le \overline{\delta}_A^{\alpha}(\mathbb{N} \setminus E_r) = 0$. But $\overline{\delta}_A^{\alpha}(E_k) = +\infty$ or 1 according as $0 < \alpha < 1$ or $\alpha = 1$ (see also (7)), and so we obtain a contradiction. Thus for every $k, r \in \mathbb{N}$ there exists $\overline{n} \in E_k \cap E_r$.

By (14) there is an (O)-sequence $(\tau_p)_p$ such that for all $p \in \mathbb{N}$ there is $n_0 = n_0(p) \in \mathbb{N}$ with

$$\bigvee_{j=1}^{\infty} |x_j^{(k)} - x_j| \le \tau_p \quad \text{for all } k \ge n_0.$$

Then

$$\begin{aligned} & |x_j^{(k)} - x_j^{(r)}| \le |x_j^{(k)} - x_j| + |x_j^{(r)} - x_j| \\ \le \quad & \bigvee_{j=1}^{\infty} |x_j^{(k)} - x_j| + \bigvee_{j=1}^{\infty} |x_j^{(r)} - x_j| \le 2\,\tau_p \end{aligned}$$

for every $j \in \mathbb{N}$ and $k, r \ge n_0$. Thus we get

$$\begin{aligned} & |y_k - y_r| \le |y_k - x_{\overline{n}}^{(k)}| + |x_{\overline{n}}^{(k)} - x_{\overline{n}}^{(r)}| + |y_k - x_{\overline{n}}^{(k)}| \\ \le \quad & 2\,\sigma_p + \bigvee_{j=1}^{\infty} |x_j^{(k)} - x_j| + \bigvee_{j=1}^{\infty} |x_j^{(r)} - x_j| \le 2\,\sigma_p + 2\,\tau_p \end{aligned}$$

whenever $k, r \ge n_0$. This shows that $(y_k)_k$ is an (O)-Cauchy sequence in R. Since R is Dedekind complete, then R is (O)-complete too (see also [9]), and so the sequence $(y_k)_k$ is (O)-convergent to some element $y \in R$. Hence there is an (O)-net $(\eta_p)_p$ such that for each $p \in \mathbb{N}$ there exists $n^* = n^*(p)$ with

$$|y_k - y| \le \eta_p \quad \text{for all } k \ge n^*.$$

Choose $k \ge \max\{n_0, n^*\}$. Then for all $j \in \mathbb{N}$ we have

$$\begin{aligned} |x_j - y| \quad & \le \quad |x_j - x_j^{(k)}| + |x_j^{(k)} - y_k| + |y_k - y| \\ & \le \quad \tau_p + \eta_p + |x_j^{(k)} - y_k|. \end{aligned} \tag{17}$$

Observe that, if E_k is as in (16) and $j \in E_k$, then $|x_j - y| \le w_p$, where $w_p = \sigma_p + \tau_p + \eta_p$. So $(w_p)_p$ is an (O)-sequence, and

$$\overline{E} := \{j \in \mathbb{N} : |x_j - y| \not\le w_p\} \subset \mathbb{N} \setminus E_k.$$

Since by hypothesis $\delta_A^{\alpha}(\mathbb{N} \setminus E_k) = 0$, we get also $\delta_A^{\alpha}(\overline{E}) = 0$, which completes the proof. □

We now recall a condition, under which it is possible to replace a countable family of (O)-sequences with one (O)-sequence ([5, Lemma 2.8]).

Lemma 3.8. *Let R be a super Dedekind complete and weakly σ-distributive (ℓ)-group. If $(\sigma_p^{(k)})_p$ is an equibounded family of (O)-sequences, then there is an (O)-sequence $(b_r)_r$ with the property that for every $k, r \in \mathbb{N}$ there exists $\overline{p} = \overline{p}(k, r)$, with $\sigma_{\overline{p}}^{(k)} \le b_r$.*

Remark 3.9. Observe that, if R is a super Dedekind complete and weakly σ-distributive (ℓ)-group and $(x_j^{(k)})_j$, $k \in \mathbb{N}$, are as in Theorem 3.7, A^α-convergent for every k and order equibounded in R (namely with a positive element $u \in R$ with $|x_j^{(k)}| \leq u$ for all $j, k \in \mathbb{N}$), then the $(x_j^{(k)})_j$'s are A^α-convergent with respect to a same (O)-sequence $(b_r)_r$, independent of k. Indeed, let $(y_k)_k$ be a sequence in R such that for all $k \in \mathbb{N}$ there exists an (O)-sequence $(\sigma_p^{(k)})_p$ with $\delta_A^\alpha(B_p^{(k)}) = 0$ for all $k, p \in \mathbb{N}$, where

$$B_p^{(k)} := \{j \in \mathbb{N} : |x_j^{(k)} - y_k| \not\leq \sigma_p^{(k)}\}. \tag{18}$$

Note that order equiboundedness of the double sequence $(x_j^{(k)} - y_k)_{j,k}$ is a consequence of order equiboundedness of $(x_j^{(k)})_{j,k}$. So, without loss of generality, we can assume that $\sigma_p^{(k)} \leq 2\,u$ for each $k, p \in \mathbb{N}$.

By Lemma 3.8 there exists an (O)-sequence $(b_r)_r$ such that for every k, $r \in \mathbb{N}$ there exists $\overline{p} \in \mathbb{N}$ with $\sigma_{\overline{p}}^{(k)} \leq b_r$. From this and (18) it follows that $D_r^{(k)} \subset B_{\overline{p}}^{(k)}$, where $D_r^{(k)} := \{j \in \mathbb{N} : |x_j^{(k)} - y_k| \not\leq b_r\}$. Hence we get $\delta_A^\alpha(D_r^{(k)}) = 0$, since $\delta_A^\alpha(B_{\overline{p}}^{(k)}) = 0$. This proves the claim, by virtue of arbitrariness of r.

References

[1] H. Albeyrak and S. Pehlivan, Statistical convergence and statistical continuity on locally solid Riesz spaces, Topology Appl., 159 (2012), 1887 - 1893.

[2] G. A. Anastassiou and O. Duman, Towards Intelligent Modeling: Statistical Approximation Theory, Intelligent System Reference Library 14, Springer-Verlag, Berlin, Heidelberg, New York, 2011.

[3] M. Balcerzak, K. Dems and A. Komisarski, Statistical convergence and ideal convergence for sequences of functions, J. Math. Anal. Appl. 328 (1) (2007), 715-729.

[4] S. Bhunia, P. Das and S. K. Pal, Restricting statistical convergence, Acta Math. Hungar. 134 (1-2) (2012), 153-161.

[5] A. Boccuto and D. Candeloro, Sobczyk Hammer decomposition and convergence theorems for measures with values in (ℓ)-groups, Real Anal. Exchange, 33 (1) (2007-2008), 91 - 106.

[6] A. Boccuto, X. Dimitriou and N. Papanastassiou , Unconditional convergence in lattice groups with respect to ideals, Comment. Math. Prace Mat. 50 (2) (2010), 161-174.

[7] A. Boccuto, X. Dimitriou and N. Papanastassiou, Some versions of limit and Dieudonné-type theorems with respect to filter convergence for (ℓ)-group-valued measures, Cent. Eur. J. Math. 9 (6) (2011), 1298-1311.

[8] A. Boccuto, X. Dimitriou and N. Papanastassiou, Basic matrix theorems for $\mathcal{I}$-convergence in (ℓ)-groups, Math. Slovaca 62 (5) (2012), 885-908.

[9] A. Boccuto, B. Riečan and M. Vrábelová, Kurzweil-Henstock Integral in Riesz Spaces, Bentham Science Publ., Singapore, 2009.

[10] A. Caserta, G. Di Maio and Lj. D. R. Kočinac, Statistical convergence in function spaces, Abstr. Appl. Anal. **Vol 2011** (2011), Article ID 420419, 11 pages.

[11] R. Colak, Statistical convergence of order α, Modern methods in Analysis and its Applications, New Delhi, India, Anamaya Pub., (2010), 121-129.

[12] P. Das and S. K. Ghosal, Some further results on $\mathcal{I}$-Cauchy sequences and condition (AP), Comput. Math. Appl. 59 (2010), 2597–2600.

[13] G. Di Maio and Lj. D. R. Kočinac, Statistical convergence in topology, Topology Appl. 156 (2008), 28-45.

[14] O. Duman, M.K. Khan and C. Orhan, A-statistical convergence of approximating operators, Math. Inequal. Appl. **6** (2003) 689-699.

[15] I. Farah, Analytic quotients: Theory of liftings for quotients over analytic ideals on the integers, Mem. Amer. Math. Soc. **148** (2000).

[16] H. Fast, Sur la convergence statistique, Colloq. Math. 2 (1951), 41-44.

[17] A. R. Freedman and J. J. Sember, On summing sequences of 0's and 1's, Rocky Mountain J. Math., 11, (1981), 419 - 425.

[18] J. A. Fridy, On statistical convergence, Analysis, 5 (1985), 301-313.

[19] M. Katětov, Product of filters, Comment. Math. Univ. Carol. 9 (1968), 173-189.

[20] P. Kostyrko, T. Šalát and W. Wilczyński, $\mathcal{I}$-convergence, Real Anal. Exchange 26 (2000/2001), 669-685.

[21] G. G. Lorentz, A contribution to the theory of divergent sequences, Acta Math. 80 (1948), 167-190.

[22] W. A. J. Luxemburg and A. C. Zaanen, Riesz Spaces I, North-Holland Publishing Co., Amsterdam, 1971.

[23] T. Šalát, On statistically convergent sequences of real numbers, Math. Slovaca 30 (1980), 139-150.

[24] H. Steinhaus, Sur la convergence ordinaire et la convergence asymptotique, Colloq. Math. 2 (1951), 73-74.