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The Existence and Stability of Solutions for Symmetric Generalized Quasi-variational Inclusion Problems

Lam Quoc Anh[a], Nguyen Van Hung[b]

[a]*Department of Mathematics, Teacher College, Cantho University, Cantho, VietNam*
[b]*Center of Research and Development Duy Tan University K7/25, Quang Trung, Danang, VietNam*
Department of Mathematics, Dong Thap University, Cao Lanh City, Dong Thap, VietNam

Abstract. In this paper, we study the symmetric generalized quasi-variational inclusion problems. Then, we establish some existence theorems of solution sets for these problems. Moreover, the stability of solutions for these problems are also onbtained. Finally, we apply these results to symmetric vector quasi-equilibrium problems. The results presented in this paper improve and extend the main results in the literature. Some examples are given to illustrate our results.

1. Introduction

Let X, Y be real locally convex Hausdorff topological vector spaces and $A \subseteq X, B \subseteq Y$ be nonempty sets. Let $S : A \times B \to 2^A, T : A \times B \to 2^B$ be set-valued mappings and $f, g : A \times B \to \mathbb{R}$ be real functions. Noor and Oettli [39] introduced the following the symmetric scalar quasi-equilibrium problem. Find $(\bar{x}, \bar{y}) \in A \times B$ such that $\bar{x} \in S(\bar{x}, \bar{y}), \bar{y} \in T(\bar{x}, \bar{y})$ and

$$f(x, \bar{y}) \geq f(\bar{x}, \bar{y}), \text{ for all } x \in S(\bar{x}, \bar{y}),$$

$$g(\bar{x}, y) \geq g(\bar{x}, \bar{y}), \text{ for all } y \in T(\bar{x}, \bar{y}).$$

In 2003, Fu [22] introduced and studied the symmetric vector quasi-equilibrium problem (in short, (SVQEP)). Let X, Y and Z be real locally convex Hausdorff topological vector spaces, and let $A \subseteq X, B \subseteq Y$ be nonempty sets and $C \subset Z$ be a closed convex point cone with intC $\notin \emptyset$, where intC denotes the interior of C. Let $S : A \times B \to 2^A, T : A \times B \to 2^B$ be set-valued mappings and $f, g : A \times B \to Z$ be vector functions. Find $(\bar{x}, \bar{y}) \in A \times B$ such that $\bar{x} \in S(\bar{x}, \bar{y}), \bar{y} \in T(\bar{x}, \bar{y})$ and

$$f(x, \bar{y}) - f(\bar{x}, \bar{y}) \notin -\text{int}C, \text{ for all } x \in S(\bar{x}, \bar{y}),$$

$$g(\bar{x}, y) - g(\bar{x}, \bar{y}) \notin -\text{int}C, \text{ for all } y \in T(\bar{x}, \bar{y}).$$

The problem is a generalization of the symmetric scalar quasi-equilibrium problem studied in Noor and Oettli [39]. Latter, many authors have investigated the symmetric vector quasi-equilibrium problem for set-valued functions, see [5, 6] and the references therein.

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Email addresses: quocanh@ctu.edu.vn (Lam Quoc Anh), ngvhungdhdt@yahoo.com (Nguyen Van Hung)

Recently, Chen *et al*. [15] considered the symmetric of generalized strong vector quasi-equilibrium problems (in short, (GSSVQEP)). Then, the authors studied existence and stability of solutions for these problems. Let X, Y and Z be real locally convex Hausdorff topological vector spaces, and let $A \subseteq X, B \subseteq Y$ be nonempty sets, $C \subset Z$ be a nonempty closed convex cone. Let $S : A \times B \to 2^A$, $T : A \times B \to 2^B$, $F : A \times B \times A \to 2^Z$ and $G : B \times A \times B \to 2^Z$ be set-valued mappings.

Find $(\bar{x}, \bar{y}) \in A \times B$ such that $\bar{x} \in S(\bar{x}, \bar{y})$, $\bar{y} \in T(\bar{x}, \bar{y})$ and

$$F(\bar{x}, \bar{y}, x) \subset C, \text{ for all } x \in S(\bar{x}, \bar{y}),$$
$$G(\bar{y}, \bar{x}, y) \subset C, \text{ for all } y \in T(\bar{x}, \bar{y}).$$

Motivated by the research works mentioned above, in this paper, we introduce the symmetric generalized quasi-variational inclution problems. Then, we establish some existence theorems of solution sets for these problems. Moreover, we also study the stability of solutions for symmetric generalized quasi-variational inclution problems. Apply these results to symmetric vector quasi-equilibrium problems also obtained.

Now, we pass to our problem setting. Let X, Y, Z be real locally convex Hausdorff topological vector spaces, $A \subset X, B \subset Y$ are nonempty compact subsets. Let $K : A \times B \to 2^A$, $T : A \times B \to 2^B$ be multifunctions and $F : A \times B \times A \to 2^Z$ and $G : B \times A \times B \to 2^Z$ and $P : A \times B \times A \to 2^Z$ and $Q : B \times A \times B \to 2^Z$.

We consider the following two symmetric quasi-variational inclusion problems (in short, (SQIP_1) and (SQIP_2)):

(SQIP_1): Find $(\bar{x}, \bar{y}) \in A \times B$ such that $\bar{x} \in K(\bar{x}, \bar{y})$, $\bar{y} \in T(\bar{x}, \bar{y})$ and

$$F(\bar{x}, \bar{y}, x^*) \cap P(\bar{x}, \bar{y}, x^*) \neq \emptyset, \ \forall x^* \in K(\bar{x}, \bar{y}),$$
$$G(\bar{y}, \bar{x}, y^*) \cap Q(\bar{y}, \bar{x}, y^*) \neq \emptyset, \ \forall y^* \in T(\bar{x}, \bar{y}).$$

(SQIP_2): Find $(\bar{x}, \bar{y}) \in A \times B$ such that $\bar{x} \in K(\bar{x}, \bar{y})$, $\bar{y} \in T(\bar{x}, \bar{y})$ and

$$F(\bar{x}, \bar{y}, x^*) \subseteq P(\bar{x}, \bar{y}, x^*), \ \forall x^* \in K(\bar{x}, \bar{y}),$$
$$G(\bar{y}, \bar{x}, y^*) \subseteq Q(\bar{y}, \bar{x}, y^*), \ \forall y^* \in T(\bar{x}, \bar{y}).$$

We denote that $\Sigma_1(F, G)$ and $\Sigma_2(F, G)$ are the solution sets of (SQIP_1) and (SQIP_2), respectively.

Note that the symmetric quasi-variational inclusion problems encompass many optimization - related models like symmetric generalized vector quasi-equilibrium problems, symmetric vector quasi-variational inequality problems, vector quasi-equilibrium problems, variational inequality problems, Nash equilibria problems, fixed point problems, coincidence-point problems and complementarity problems, *etc*. In recent years, a lot of results for existence of solutions and stability of solutions for symmetric vector quasi-equilibrium problems, vector quasi-equilibrium problems, vector quasi-variational inequality problems and optimization problems have been established by many authors in different ways. For example, equilibrium problems [1–10, 15–17, 19, 23–25, 29, 34, 35, 37, 40, 41], variational inequality problems [25, 30, 31, 36, 42, 43], optimization problems [35, 42, 43], variational relation problems [12, 13, 27, 28, 32, 33] and the references therein.

The structure of our paper is as follows. In the first part of this article, we introduce the model symmetric generalized quasi-variational inclusion problems. In Section 2, we recall some basic definitions for later uses. In Section 3, we establish some existence and closedness theorems by using fixed-point theorem for symmetric generalized quasi-variational relation problem. The stability of the solutions for these problems are also obtained. Applications to symmetric vector quasi-equilibrium problems are presented in Section 5.

2. Preliminaries

In this section, we recall some basic definitions and some of their properties.

Definition 2.1. *([14])* Let X, Y be two topological vector spaces, A be a nonempty subset of X and $F : A \to 2^Z$ be a multifunction.

(i) F is said to be *lower semicontinuous (lsc) at* $x_0 \in A$ if $F(x_0) \cap U \neq \emptyset$ for some open set $U \subseteq Y$ implies the existence of a neighborhood N of x_0 such that $F(x) \cap U \neq \emptyset, \forall x \in N$. F is said to be lower semicontinuous in A if it is lower semicontinuous at all $x_0 \in A$.

(ii) F is said to be *upper semicontinuous (usc) at* $x_0 \in A$ if for each open set $U \supseteq F(x_0)$, there is a neighborhood N of x_0 such that $U \supseteq F(x), \forall x \in N$. F is said to be upper semicontinuous in A if it is upper semicontinuous at all $x_0 \in A$.

(iii) F is said to be *continuous in* A if it is both lsc and usc in A.

(iv) F is said to be *closed* if $\mathrm{Graph(F)} = \{(\mathrm{x}, \mathrm{y}) : \mathrm{x} \in \mathrm{A}, \mathrm{y} \in \mathrm{F(x)}\}$ is a closed subset in $A \times Y$.

Definition 2.2. *([11])* Let X, Y be two topological vector spaces, A is a nonempty subset of X and $F : A \to 2^Y$ be a multifunction and $C \subset Y$ is a nonempty closed convex cone. F is called *upper C-continuous at* $x_0 \in A$, if for any neighborhood U of the origin in Y, there is a neighborhood V of x_0 such that

$$F(x) \subset F(x_0) + U + C, \forall x \in V.$$

Definition 2.3. *([11])* Let X and Y be two topological vector spaces and A is a nonempty convex subset of X. A set-valued mapping $F : A \to 2^Y$ is said to be *properly C-quasiconvex* if for any $x, y \in A$ and $t \in [0, 1]$, we have

either $F(x) \subset F(tx + (1 - t)y) + C$

or $F(y) \subset F(tx + (1 - t)y) + C$.

Definition 2.4. *([15])* Let X, Y be two topological vector spaces, A is a nonempty subset of X and $F : A \to 2^Y$ be a multifunction and $C \subset Y$ is a nonempty closed convex cone.

(i) F is called *C-upper semicontinuous at* $x_0 \in A$, if for any neighborhood U of the origin in Y, there is a neighborhood $V(x_0)$ of x_0 such that

$$F(x) \subset F(x_0) + U + C, \forall x \in V(x_0) \cap A.$$

(ii) F is called *C-lower semicontinuous at* $x_0 \in A$, if for each $z \in F(x_0)$, and any neighborhood U of the origin in Y, there is a neighborhood $V(x_0)$ of x_0 such that

$$F(x) \cap (z + U - \mathrm{C}) \neq \emptyset, \forall x \in V(x_0) \cap A.$$

Definition 2.5. *([38])* Let X and Z be two topological vector spaces and $A \subseteq X$ be nonempty convex set, $C \subset Z$ is a nonempty closed convex cone. A mapping $f : A \to Z$ is said to be *C-continuous at* $x_0 \in A$ if, for any open neighborhood V of 0 in Z, there exists an open neighborhood U of x_0 in A such that

$$f(x) \in f(x_0) + V + C, \forall x \in U \cap A,$$

and C-continuous in A if it is C-continuous at every point of A.

Lemma 2.1. *([38]) Let X, Y be two Hausdorff topological vector spaces, A be a nonempty convex subset of X and $F : A \to 2^Y$ be a multifunction.*

(i) If F is upper semicontinuous at $x_0 \in A$ with closed values, then F is closed at $x_0 \in A$;

(ii) If F is closed at $x_0 \in A$ and Y is compact, then F is upper semicontinuous at $x_0 \in A$.

(iii) If F has compact values, then F is usc at $x_0 \in A$ if and only if, for each net $\{x_\alpha\} \subseteq A$ which converges to $x_0 \in A$ and for each net $\{y_\alpha\} \subseteq F(x_\alpha)$, there are $y_0 \in F(x_0)$ and a subnet $\{y_\beta\}$ of $\{y_\alpha\}$ such that $y_\beta \to y_0$.

Lemma 2.2. *[20] Let A be a nonempty convex compact subset of Hausdorff topological vector space X and M be a subset of $A \times A$ such that*

(i) for each at $x \in A, (x, x) \notin M$;
(ii) for each at $y \in A$, the set $\{x \in A : (x, y) \in M\}$ is open in A;
(iii) for each at $x \in A$, the set $\{y \in A : (x, y) \in M\}$ is convex or empty.

Then, there exists $x_0 \in A$ such that $(x_0, y) \notin M$ for all $y \in A$.

Lemma 2.3. *([26]) Let A be a nonempty compact convex subset of a locally convex Hausdorff vector topological space X. If $F : A \longrightarrow 2^A$ is upper semicontinuous and for any $x \in A, F(x)$ is nonempty, convex and closed, then there exists an $x^* \in A$ such that $x^* \in F(x^*)$.*

Now we recall some notions, see [11, 14, 16]. Let (X, d) be a metric space. Denote $\mathcal{K}(X), BC(X)$ and $C\mathcal{K}(X)$ all nonempty compact subsets of X, all nonempty bounded closed subsets of X, and all nonempty convex compact subsets of X (if X is a linear metric space), respectively. Let $E_1, E_2 \subset X$ and define

$$H(E_1, E_2) := \max\{H^*(E_1, E_2), H^*(E_2, E_1)\},$$

where $H^*(E_1, E_2) := \sup_{e_1 \in E_1} d(e_1, E_2)$ and $d(e_1, E_2) := \inf_{e_2 \in E_2} \|e_1 - e_2\|$. It is obvious that H is a Hausdorff metric in $\mathcal{K}(X), BC(X)$ and $C\mathcal{K}(X)$, respectively.

Lemma 2.4. *Let A be a nonempty compact subset of $(X, \|.\|_X)$, and B be a nonempty compact subset of $(Y, \|.\|_Y)$. Let $K : A \times B \to 2^A, T : A \times B \to 2^B$ be continuous set-valued mappings. Assume that for each $(x, y) \in A \times B$, $K(x, y), T(x, y)$ are nonempty compact subsets. Then in $A \times B$, when $(x, y) \to (x^*, y^*)$, we have*

$$K(x, y) \xrightarrow{H} K(x^*, y^*), \text{ and } T(x, y) \xrightarrow{H'} T(x^*, y^*),$$

where H is a Hausdorff metric in $\mathcal{K}(A)$, and H' is a Hausdorff metric in $\mathcal{K}(B)$.

Lemma 2.5. *Let A be a nonempty compact subset of $(X, \|.\|_X)$, and B be a nonempty compact subset of $(Y, \|.\|_Y)$. Let $K : A \times B \to 2^A, T : A \times B \to 2^B$ be continuous set-valued mappings with nonempty compact valued. Then K is continuous if and if, for any $(x^*, y^*) \in A \times B, (x, y) \to (x^*, y^*)$ implies $K(x, y) \xrightarrow{H} K(x^*, y^*)$, and so is it for T.*

Lemma 2.6. *Let A be a nonempty compact subset of $(X, \|.\|_X)$, and B be a nonempty compact subset of $(Y, \|.\|_Y)$. Let $K : A \times B \to 2^A, T : A \times B \to 2^B$ be continuous set-valued mappings with nonempty compact valued. Then K is continuous if and if, for any $(x^*, y^*) \in A \times B, (x, y) \to (x^*, y^*)$ implies $K(x, y) \xrightarrow{H} K(x^*, y^*)$, and so is it for T.*

Lemma 2.7. *Let (X, d) be a metric space and H is a Hausdorff metric in X. Then:*

(i) $(BC(X), H)$ is complete if and if (X, d) is complete.
(ii) $(\mathcal{K}(X), H)$ is complete if and if (X, d) is complete.
(iii) If X is a linear metric space, then $(C\mathcal{K}(X), H)$ is complete if and if (X, d) is complete.

Lemma 2.8. *([43]) Let Z be a metric space and let $M, M_n (n = 1, 2, ...)$ be compact sets in Z. Suppose that for any open set $O \supset M$, there exists n_0 such that $M_n \subset O, \forall n \geq n_0$. Then, any sequence $\{x_n\}$ satisfying $x_n \in M_n$ has a convergent subsequence with limit in M.*

3. Existence of Solutions

In this section, we establish some existence theorems of solution sets for the symmetric generalized quasi-variational inclusion problems (SQIP$_1$) and (SQIP$_2$).

Definition 3.1. Let X, Y, Z be Hausdorff topological vector spaces. Suppose $F, P : X \times Y \times X \to 2^Z$ be two multifunctions.

(i) F is said to be *generalized type I P-quasiconvex*(with respect to the first variable) in a set $A \subset X$, if for each $y \in Y, z \in X$ and $\forall x_1, x_2 \in A, \forall \lambda \in [0,1]$, $F(x_1, y, z) \cap P(x_1, y, z) \neq \emptyset$ and $F(x_2, y, z) \cap P(x_2, y, z) \neq \emptyset$. Then, it follows that

$$F(\lambda x_1 + (1-\lambda)x_2, y, z) \cap P(\lambda x_1 + (1-\lambda)x_2, y, z) \neq \emptyset.$$

(ii) F is said to be *generalized type II P-quasiconvex*(with respect to the first variable) in a set $A \subset X$, if for each $y \in Y, z \in X$ and $\forall x_1, x_2 \in A, \forall \lambda \in [0,1]$, $F(x_1, y, z) \subseteq P(x_1, y, z)$ and $F(x_2, y, z) \subseteq P(x_2, y, z)$. Then, it follows that

$$F(\lambda x_1 + (1-\lambda)x_2, y, z) \subseteq P(\lambda x_1 + (1-\lambda)x_2, y, z).$$

Theorem 3.2. *Assume for the problem* (SQIP$_1$) *that*

(i) K and T are continuous in $A \times B$ with nonempty compact convex values;
(ii) for all $(x, y) \in A \times B$, $F(x, y, x) \cap P(x, y, x) \neq \emptyset$ and $G(y, x, y) \cap Q(y, x, y) \neq \emptyset$;
(iii) the set $\{(y, x^*) \in B \times A : F(., y, x^*) \cap P(., y, x^*) = \emptyset\}$ is convex in A and the set $\{(x, y^*) \in A \times B : G(., x, y^*) \cap Q(., x, y^*) = \emptyset\}$ is convex in B;
(iv) for all $(y, x^*) \in B \times A$, $F(., y, x^*)$ is generalized type I $P(., y, x^*)$-quasiconvex in A and for all $(x, y^*) \in A \times B$, $G(., x, y^*)$ is generalized type I $Q(., x, y^*)$-quasiconvex in B;
(v) the set $\{(x, y, x^*) \in A \times B \times A : F(x, y, x^*) \cap P(x, y, x^*) \neq \emptyset\}$ is closed and the set $\{(y, x, y^*) \in B \times A \times B : G(y, x, y^*) \cap Q(y, x, y^*) \neq \emptyset\}$ is closed.

Then, the (SQIP$_1$) *has a solution, i.e., there exists $(\bar{x}, \bar{y}) \in A \times B$ such that $\bar{x} \in K(\bar{x}, \bar{y})$, $\bar{y} \in T(\bar{x}, \bar{y})$ and*

$$F(\bar{x}, \bar{y}, x^*) \cap P(\bar{x}, \bar{y}, x^*) \neq \emptyset, \forall x^* \in K(\bar{x}, \bar{y}),$$

$$G(\bar{y}, \bar{x}, y^*) \cap Q(\bar{y}, \bar{x}, y^*) \neq \emptyset, \forall y^* \in T(\bar{x}, \bar{y}).$$

Moreover, the solution set of the (SQIP$_1$) *is closed.*

Proof. For all $(x, y) \in A \times B$, define mappings: $\Psi : A \times B \to 2^A$ and $\Gamma : A \times B \to 2^B$ by

$$\Psi(x, y) = \{a \in K(x, y) : F(a, y, x^*) \cap P(a, y, x^*) \neq \emptyset, \ \forall x^* \in K(x, y)\},$$

and

$$\Gamma(x, y) = \{b \in T(x, y) : G(b, x, y^*) \cap Q(b, x, y^*) \neq \emptyset, \ \forall y^* \in T(x, y)\}.$$

(1) Show that $\Psi(x, y)$ and $\Gamma(x, y)$ are nonempty.
Indeed, for all $(x, y) \in A \times B$, $K(x, y)$ is nonempty compact convex set. Setting

$$M = \{(a, x^*) \in K(x, y) \times K(x, y) : F(a, y, x^*) \cap P(a, y, x^*) = \emptyset\}.$$

(a) The condition (ii) yields that, for any $a \in K(x, y)$, $(a, a) \notin M\}$.
(b) The condition (iii) implies that for any $a \in K(x, y)$, $\{x^* \in K(x, y) : (a, x^*) \in M\}$ is convex in $K(x, y)$.
(c) Applying the condition (v), we conclude that, for any $a \in K(x, y)$, $\{x^* \in K(x, y) : (a, x^*) \in M\}$ is open in $K(x, y)$.
By Lemma 2.2, there exists $a \in K(x, y)$ such that $(a, x^*) \notin M$, for all $x^* \in K(x, y)$, i.e., $F(a, y, x^*) \cap P(a, y, x^*) \neq \emptyset, \forall x^* \in K(x, y)\}$. Thus, $\Psi(x, y) \neq \emptyset$. Similarly, we also have $\Gamma(x, y) \neq \emptyset$.

(2) Show that $\Psi(x, y)$ and $\Gamma(x, y)$ are nonempty convex sets.
Let $a_1, a_2 \in \Psi(x, y)$ and $\alpha \in [0,1]$ and put $a = \alpha a_1 + (1-\alpha)a_2$. Since $a_1, a_2 \in K(x, y)$ and $K(x, y)$ is a convex set, we have $a \in K(x, y)$. Thus, for $a_1, a_2 \in \Psi(x, y)$, it follows that

$$F(a_1, y, x^*) \cap P(a_1, y, x^*) \neq \emptyset, \ \forall x^* \in K(x, y),$$

and

$$G(a_2, y, x^*) \cap Q(a_2, y, x^*) \neq \emptyset, \ \forall x^* \in K(x, y),$$

By (iv), $F(., y, x^*)$ is generalized type I $P(., y, x^*)$-quasiconvex.

$$F(\alpha a_1 + (1-\alpha)a_2, y, x^*) \cap P(\alpha a_1 + (1-\alpha)a_2, y, x^*) \neq \emptyset, \quad \text{for all } \alpha \in [0,1],$$

i.e., $a \in \Psi(x, y)$. Therefore, $\Psi(x, y)$ is convex. Similarly, we have $\Gamma(x, y)$ is convex.

(3) We will prove Ψ and Γ are upper semicontinuous in $A \times B$ with nonempty compact values.
First, we show that Ψ is upper semicontinuous in $A \times B$ with nonempty compact values. Indeed, since A is a compact set, by Lemma 2.1(ii), we need only to show that Ψ is a closed mapping. Let a net $\{(x_n, y_n) : n \in I\} \subset A \times B$ such that $(x_n, y_n) \to (x, y) \in A \times B$, and let $a_n \in \Psi(x_n, y_n)$ such that $a_n \to a_0$. Now we need to show that $a_0 \in \Psi(x, y)$. Since $a_n \in K(x_n, y_n)$ and K is upper semicontinuous with nonempty compact values, hence K is closed, thus, we have $a_0 \in K(x, y)$. Suppose the contrary $a_0 \notin \Psi(x, y)$. Then, $\exists x_0^* \in K(x, y)$ such that

$$F(a_0, y, x_0^*) \cap P(a_0, y, x_0^*) = \emptyset. \tag{3.1}$$

By the lower semicontinuity of K, there is a net $\{x_n^*\}$ such that $x_n^* \in K(x_n, y_n)$, $x_n^* \to x_0^*$. Since $a_n \in \Psi(x_n, y_n)$, we have

$$F(a_n, y_n, x_n^*) \cap P(a_n, y_n, x_n^*) \neq \emptyset. \tag{3.2}$$

By the condition (v) and (3.2), we have

$$F(a_0, y, x_0^*) \cap P(a_0, y, x_0^*) \neq \emptyset. \tag{3.3}$$

There is a contradiction between (3.1) and (3.3). Thus, $a_0 \in \Psi(x, y)$. Hence, Ψ is upper semicontinuous in $A \times B$ with nonempty compact values. Similarly, we also have $\Gamma(x, y)$ is upper semicontinuous in $A \times B$ with nonempty compact values.

(4) Now we need to prove the solutions set $\Sigma_1(F, G) \neq \emptyset$.
Define the set-valued mappings $\Phi, \Xi : A \times B :\to 2^{A\times B}$ by

$$\Phi(x, y) = (\Psi(x, y), K(x, y)), \forall (x, y) \in A \times B$$

and

$$\Xi(x, y) = (\Gamma(x, y), T(x, y)), \forall (x, y) \in A \times B.$$

Then Φ, Ξ are upper semicontinuous and $\forall (x, y) \in A \times B$, $\Phi(x, y)$ and $\Xi(x, y)$ are nonempty compact convex subsets of $A \times B$.
Define the set-valued mapping $H : (A \times B) \times (A \times B) \to 2^{(A\times B)\times(A\times B)}$ by

$$H((x, y), (x, y)) = (\Phi(x, y), \Xi(x, y)), \forall (x, y) \in A \times B.$$

Then H is also upper semicontinuous and $\forall (x, y) \in A \times B$, $H((x, y), (x, y))$ is a nonempty closed convex subset of $(A \times B) \times (A \times B)$.
By Lemma 2.3, there exists a point $((\hat{x}, \hat{y}), (\hat{x}, \hat{y})) \in (A\times B)\times(A\times B)$ such that $((\hat{x}, \hat{y}), (\hat{x}, \hat{y})) \in H((\hat{x}, \hat{y}), (\hat{x}, \hat{y}))$, that is

$$(\hat{x}, \hat{y}) \in \Phi(\hat{x}, \hat{y}), \quad (\hat{x}, \hat{y}) \in \Xi(\hat{x}, \hat{y})$$

which implies that $\hat{x} \in \Psi(\hat{x}, \hat{y})$, $\hat{y} \in K(\hat{x}, \hat{y})$ and $\hat{x} \in \Gamma(\hat{x}, \hat{y})$, $\hat{y} \in T(\hat{x}, \hat{y})$. Hence, $\hat{x} \in K(\hat{x}, \hat{y})$, $\hat{y} \in T(\hat{x}, \hat{y})$ and

$$F_1(\hat{x}, \hat{y}, x^*) \cap G_1(\hat{x}, \hat{y}, x^*) \neq \emptyset, \forall x^* \in K(\hat{x}, \hat{y}),$$

and

$$F_2(\hat{y}, \hat{x}, y^*) \cap G_2(\hat{y}, \hat{x}, y^*) \neq \emptyset, \forall y^* \in T(\hat{x}, \hat{y}),$$

i.e., (SQIP$_1$) has a solution.

(5) Now we prove that $\Sigma_1(F, G)$ is closed. Indeed, let a net $\{(x_n, y_n), n \in I\} \in \Sigma_1(F, G)$: $(x_n, y_n) \to (x_0, y_0)$. We need to prove that $(x_0, y_0) \in \Sigma_1(F, G)$. Indeed, by the lower semicontinuity of K and T, for any $x_0 \in K(x_0, y_0)$, $y_0 \in T(x_0, y_0)$, there exist $x_n \in K(x_n, y_n)$, $y_n \in T(x_n, y_n)$ such that $x_n \to x_0$, $y_n \to y_0$. Since $(x_n, y_n) \in \Sigma_1(F, G)$, we have $x_n \in K(x_n, y_n)$, $y_n \in T(x_n, y_n)$ such that

$$F(x_n, y_n, x_n^*) \cap P(x_n, y_n, x_n^*) \neq \emptyset, \forall x_n^* \in K(x_n, y_n),$$

and

$$G(y_n, x_n, y_n^*) \cap Q(y_n, x_n, y_n^*) \neq \emptyset, \forall y_n^* \in T(x_n, y_n).$$

Since K, T are upper semicontinuous in $A \times B$ with nonempty compact values. There exist $x_0^* \in K(x_0, u_0)$ and $y_0^* \in T(x_0, u_0)$ such that $x_n^* \to x_0^*, y_n^* \to y_0^*$ (taking subnets if necessary). By the condition (v) and $(x_n, y_n, x_n^*, y_n^*) \to (x_0, y_0, x_0^*, y_0^*)$, we have

$$F(x_0, y_0, x_0^*) \cap P(x_0, y_0, x_0^*) \neq \emptyset, \forall x_0^* \in K(x_0, y_0),$$

and

$$G(y_0, x_0, y_0^*) \cap Q(y_0, x_0, y_0^*) \neq \emptyset, \forall y_0^* \in T(x_0, y_0).$$

This means that $(x_0, y_0) \in \Sigma_1(F, G)$. Thus $\Sigma_1(F, G)$ is a closed set.

□

Theorem 3.3. *Assume for the problem* (SQIP$_1$) *assumptions* (i), (ii), (iii) *and* (iv) *as in Theorem 3.2 and the condition* (v) *can be replaced by the following condition:*

(v′) *F, G are upper semicontinuous and have nonempty compact values and P, Q are upper semicontinuous and have nonempty closed values.*

Then, the (SQIP$_1$) *has a solution. Moreover, the solution set of the* (SQIP$_1$) *is closed.*

Proof. We omit the proof since the technique is similar as that for Theorem 3.2 with suitable modifications. □

The following example shows that all assumptions of Theorem 3.2 are fulfilled. However, Theorem 3.3 are not satisfied.

Example 3.4. Let $X = Y = Z = R, A = B = [0, 1]$ and let $S_1(x, u) = S_2(x, u) = [0, \frac{1}{2}], T_1(x, u) = T_2(x, u) = [0, 1], P(x, y, x^*) = Q(y, x, y^*) = [0, +\infty)$ and

$$F(x, y, x^*) = G(y, x, y^*) = F(x) = \begin{cases} [\frac{1}{9}, \frac{1}{3}] & \text{if} \quad x_0 = \frac{1}{3}, \\ [\frac{1}{6}, \frac{2}{3}] & \text{otherwise.} \end{cases}$$

We show that all assumptions of Theorem 3.2 are satisfied. However, F is not upper semicontinuous at $x_0 = \frac{1}{2}$. Also, Theorem 3.3 is not satisfied.

Passing to problem (SQIP$_2$) we have.

Theorem 3.5. *Assume for the problem* (SQIP$_2$) *that*

(i) K and T are continuous in $A \times B$ with nonempty compact convex values;
(ii) for all $(x, y) \in A \times B$, $F(x, y, x) \subseteq P(x, y, x)$ and $G(y, x, y) \subseteq Q(y, x, y)$;
(iii) the set $\{(y, x^*) \in B \times A : F(., y, x^*) \not\subseteq P(., y, x^*)\}$ is convex in A and the set $\{(x, y^*) \in A \times B : G(., x, y^*) \not\subseteq Q(., x, y^*)\}$ is convex in B;
(iv) for all $(y, x^*) \in B \times A$, $F(., y, x^*)$ is generalized type II $P(., y, x^*)$-quasiconvex in A and for all $(x, y^*) \in A \times B$, $G(., x, y^*)$ is generalized type II $Q(., x, y^*)$-quasiconvex in B;
(v) the set $\{(x, y, x^*) \in A \times B \times A : F(x, y, x^*) \subseteq P(x, y, x^*)\}$ is closed and the set $\{(y, x, y^*) \in B \times A \times B : G(y, x, y^*) \subseteq Q(y, x, y^*)\}$ is closed.

Then, the (SQIP$_2$) *has a solution, i.e., there exists $(\bar{x}, \bar{y}) \in A \times B$ such that $\bar{x} \in K(\bar{x}, \bar{y}), \bar{y} \in T(\bar{x}, \bar{y})$ and*

$$F(\bar{x}, \bar{y}, x^*) \subseteq P(\bar{x}, \bar{y}, x^*), \forall x^* \in K(\bar{x}, \bar{y}),$$

$$G(\bar{y}, \bar{x}, y^*) \subseteq Q(\bar{y}, \bar{x}, y^*), \forall y^* \in T(\bar{x}, \bar{y}).$$

Moreover, the solution set of the (SQIP$_2$) *is closed.*

Proof. We can adopt the same lines of proof as in Theorem 3.2 with new multifunctions $\Pi_1(x, y)$ and $\Pi_2(x, y)$ defined as: $\Pi_1 : A \times B \to 2^A$ and $\Pi_2 : B \times A \to 2^B$ by

$$\Pi_1(x, y) = \{a \in K(x, y) : F(a, y, x^*) \subseteq P(a, y, x^*),\ \forall x^* \in K(x, y)\},$$

and

$$\Pi_2(x, y) = \{b \in T(x, y) : G(b, x, y^*) \subseteq Q(b, x, y^*),\ \forall y^* \in T(x, y)\}.$$

□

Remark 3.6. If let $A, B, X, Y, Z, K(x, y), T(x, y)$ as in (SQIP$_2$) and let $F : A \times B \times A \to 2^Z, G : B \times A \times B \to 2^Z$ be set-valued mappings, $P(x, y, x^*) = Q(y, x, y^*) = C$, with $C \subset Z$ is a nonempty closed convex cone. Then, (SQIP$_2$) becomes the generalized symmetric strong vector quasi-equilibrium problem (in short, (GSSVQEP)) studied in [15].

Remark 3.7. In the special case as in Remark 3.6, Chen et al. [15] also obtained an existence result of (GSSVQEP). However, the assumptions of Theorem 3.1 in [15] are different from the assumptions in Theorem 3.5. The following Example 3.9 shows that in this special case, all the assumptions of Theorem 3.5 are satisfied. But, Theorem 3.1 in [15] does not work. The reason is that F and G are not C-upper semicontinuous.

Example 3.8. Let $A = B = [0, 1], X = Y = Z = \mathbb{R}, C = \mathbb{R}_+$, and let $F : [0, 1] \to 2^{\mathbb{R}}$, $K(x, y) = T(x, y) = [0, 1]$ and

$$F(x, y, x^*) = G(y, x, y^*) = F(x) = \begin{cases} [1, 2] & \text{if } \ x_0 = \frac{1}{2}, \\ [\frac{1}{2}, \frac{3}{2}] & \text{otherwise.} \end{cases}$$

We show that all assumptions of Theorem 3.5 are satisfied. However, F is not C-upper semicontinuous at $x_0 = \frac{1}{2}$. Thus, it gives case where Theorem 3.5 can be applied but Theorem 3.1 in [15] does not work.

The following Example 3.9 also shows that in this special case, all the assumptions of Theorem 3.5 are satisfied. But, Theorem 3.1 in [15] does not work. The reason is that F and G are not C-lower semicontinuous.

Example 3.9. Let A, B, X, Y, Z, C, K, T as in Example 3.8, and let $F : [0, 1] \to 2^{\mathbb{R}}$ and

$$F(x, y, x^*) = G(y, x, y^*) = F(x) = \begin{cases} [0, \frac{1}{2}] & \text{if } \ x_0 = \frac{1}{2}, \\ [1, 2] & \text{otherwise.} \end{cases}$$

We show that all assumptions of Theorem 3.5 are satisfied. However, F, G are not C-lower semicontinuous at $x_0 = \frac{1}{2}$. Thus, it gives case where Theorem 3.5 can be applied but Theorem 3.1 in [15] does not work.

Remark 3.10. If $K(x, y) = K(x), T(x, y) = T(x), P(x, y, x^*) = Q(y, x, y^*) = C$, with $C \subset Z$ is a nonempty closed convex cone, and let $F(x, y, x^*) = G(y, x, y^*) = H(x, z, y)$, with $H : A \times B \times A \to 2^A$. Then (SQIP$_2$) becomes the generalized strong vector quasi-equilibrium problem (in short, (GSVQEP)) studied in [37].

In the special cases as Remark 3.10, Long et al [37] is obtained an existence result of (GSVQEP). However, the assumptions in Theorem 3.1 in [37] are different from the assumptions in Theorem 3.5. The following Example 3.11 shows that all the assumptions of Theorem 3.5 are satisfied. But, Theorem 3.1 in [37] is not fulfilled.

Example 3.11. Let $X = Y = Z = \mathbb{R}, A = B = [0, 1], P(x, y, x^*) = Q(y, x, y^*) = C = [0, +\infty)$ and let $K, T : [0, 1] \to 2^{\mathbb{R}}, F : [0, 1] \to 2^{\mathbb{R}}, K(x, y) = K(x) = [0, 1], T(x, y) = T(x) = [0, 2]$ and

$$F(x, y, x^*) = G(y, x, y^*) = F(x) = \begin{cases} [\frac{1}{2}, 1] & \text{if } \ x_0 = \frac{1}{3}, \\ [\frac{1}{5}, \frac{1}{3}] & \text{otherwise.} \end{cases}$$

We show that all the assumptions in Theorem 3.5 are satisfied. However, Theorem 3.1 in [37] is not satisfied. The reason is that F is neither upper C-continuous nor properly C-quasiconvex at $x_0 = \frac{1}{3}$. Thus, it gives cases whereTheorem 3.5 can be applied but Theorem 3.1 in [37] does not work.

Theorem 3.12. *Assume for the problem* (SQIP$_2$) *assumptions* (i), (ii), (iii) *and* (iv) *as in Theorem 3.5 and the condition* (v) *can be replaced by the following condition:*

(v′) *F, G are lower semicontinuous, P, Q are upper semicontinuous and have nonempty closed values.*

Then, the (SQIP$_2$) *has a solution. Moreover, the solution set of the* (SQIP$_2$) *is closed.*

Proof. We omit the proof since the technique is similar as that for Theorem 3.5 with suitable modifications. □

The following example shows that all assumptions of Theorem 3.5 are fulfilled. However, Theorem 3.12 are not satisfied.

Example 3.13. Let $X = Y = Z = R, A = B = [0,1], C = [0,+\infty)$ and let $S_1(x,u) = S_2(x,u) = T_1(x,u) = T_2(x,u) = [0,1]$ and

$$F(x,y,x^*) = G(y,x,y^*) = F(x) = \begin{cases} [\frac{1}{10}, \frac{2}{3}] & \text{if } \quad x_0 = \frac{1}{3}, \\ [\frac{1}{2}, 1] & \text{otherwise.} \end{cases}$$

We show that all assumptions of Theorem 3.5 are satisfied. However, F is not lower semicontinuous at $x_0 = \frac{1}{3}$. Also, Theorem 3.12 is not satisfied.

4. Stability

Throughout this section, let X, Y be Banach spaces, Z be real locally convex Hausdorff topological vector space and $A \subset X, B \subset Y$ be nonempty subsets. Now, we let

$\Omega_1 := \{(K,T,F,G,P,Q) : K : A \times B \to 2^A, T : A \times B \to 2^B$ be continuous in $A \times B$ with nonempty compact convex values, and $F : A \times B \times A \to 2^Z$ and $G : B \times A \times B \to 2^Z$ and $P : A \times B \times A \to 2^Z$ and $Q : B \times A \times B \to 2^Z$ such that, the sets $\{(x,y,x^*) \in A \times B \times A : F(x,y,x^*) \cap P(x,y,x^*) \neq \emptyset\}$ and $\{(y,x,y^*) \in B \times A \times B : G(y,x,y^*) \cap Q(y,x,y^*) \neq \emptyset\}$ are closed, and for all $(y,x^*) \in B \times A$, $F(.,y,x^*)$ is generalized type I $P(.,y,x^*)$-quasiconvex in A, for all $(x,y^*) \in A \times B$, $G(.,x,y^*)$ is generalized type I $Q(.,x,y^*)$-quasiconvex in $B\}$.

$\Omega_2 := \{(K,T,F,G,P,Q) : K : A \times B \to 2^A, T : A \times B \to 2^B$ be continuous in $A \times B$ with nonempty compact convex values, and $F : A\times B\times A \to 2^Z$ and $G : B\times A\times B \to 2^Z$ and $P : A\times B\times A \to 2^Z$ and $Q : B\times A\times B \to 2^Z$ such that the sets $\{(x,y,x^*) \in A \times B \times A : F(x,y,x^*) \subseteq P(x,y,x^*)\}$ and $\{(y,x,y^*) \in B \times A \times B : G(y,x,y^*) \subseteq Q(y,x,y^*)\}$ are closed, and for all $(y,x^*) \in B \times A$, $F(.,y,x^*)$ is generalized type II $P(.,y,x^*)$-quasiconvex in A, for all $(x,y^*) \in A \times B$, $G(.,x,y^*)$ is generalized type II $Q(.,x,y^*)$-quasiconvex in $B\}$.

For $u_1 = (K_1,T_1,F_1,G_1,P_1,Q_1), u_2 = (K_2,T_2,F_2,G_2,P_2,Q_2), u_1,u_2 \in \Omega_1$ and $u_1,u_2 \in \Omega_2$, define

$$\begin{aligned} \xi(u_1,u_2) := & \sup_{(x,y)\in A\times B} H_K(K_1(x,y),K_2(x,y)) + \sup_{(x,y)\in A\times B} H_T(T_1(x,y),T_2(x,y)) \\ & + \sup_{(x,y,x^*)\in A\times B\times A} H_F(F_1(x,y,x^*),F_2(x,y,x^*)) \\ & + \sup_{(y,x,y^*)\in B\times A\times B} H_G(G_1(y,x,y^*),G_2(y,x,y^*)) \\ & + \sup_{(x,y,x^*)\in A\times B\times A} H_P(P_1(x,y,x^*),P_2(x,y,x^*)) \\ & + \sup_{(y,x,y^*)\in B\times A\times B} H_Q(Q_1(y,x,y^*),Q_2(y,x,y^*)), \end{aligned}$$

where H_K, H_T are Hausdorff metrics in $C\mathcal{K}(A), C\mathcal{K}(B)$ and H_F, H_G, H_P, H_Q are Hausdorff metrics in $C(Z)$. Clearly, (Ω_1,ξ) and (Ω_2,ξ) be two metric spaces.

Theorem 4.1. (Ω_1,ξ) *is a complete metric space.*

Proof. Let $\{u_n\}$ be a Cauchy sequence in Ω_1, with $u_n = (K_n, T_n, F_n, G_n, P_n, Q_n)$, $n = 1, 2, \ldots$. Then, for any $\varepsilon > 0$, there exists $N > 0$ such that

$$\xi(u_n, u_m) < \frac{\varepsilon}{6}, \forall n, m \geq N. \tag{4.1}$$

It follows that, for any $(x, y, x^*, y^*) \in A \times B \times A \times B$,

$$H_K(K_n(x, y), K_m(x, y)) < \frac{\varepsilon}{6}, \quad H_T(T_n(x, y), T_m(x, y)) < \frac{\varepsilon}{6}, \tag{4.2}$$

$$H_F(F_n(x, y, x^*), F_m(x, y, x^*)) < \frac{\varepsilon}{6}, \quad H_G(G_n(y, x, y^*), G_m(y, x, y^*)) < \frac{\varepsilon}{6}, \tag{4.3}$$

and

$$H_P(P_n(x, y, x^*), P_m(x, y, x^*)) < \frac{\varepsilon}{6}, \quad H_Q(Q_n(y, x, y^*), Q_m(y, x, y^*)) < \frac{\varepsilon}{6}. \tag{4.4}$$

Then, for any fixed a point $(x, y, x^*, y^*) \in A \times B \times A \times B$, $\{K_n(x, y)\}$ is a Cauchy sequence in $C\mathcal{K}(A)$, $\{T_n(x, y)\}$ is a Cauchy sequence in $C\mathcal{K}(B)$, and $\{F_n(x, y, x^*)\}$, $\{G_n(y, x, y^*)\}$, $\{P_n(x, y, x^*)\}$, $\{Q_n(y, x, y^*)\}$ are Cauchy sequences in $\mathcal{K}(Z)$. By Lemma 2.7 and assumption, $(C\mathcal{K}(A), H_K)$, $(C\mathcal{K}(B), H_T)$, $(\mathcal{K}(Z), H_F)$, $(\mathcal{K}(Z), H_G)$, $(\mathcal{K}(Z), H_P)$ and $(\mathcal{K}(Z), H_Q)$ are complete spaces. It follows that there exist $K(x, y) \in C\mathcal{K}(A)$, $T(x, y) \in C\mathcal{K}(B)$ and $F(x, y, x^*)$, $G(y, x, y^*)$, $P(x, y, x^*)$, $Q(y, x, y^*) \in \mathcal{K}(Z)$ such that

$$K_n(x, y) \xrightarrow{H_K} K(x, y), \quad T_n(x, y) \xrightarrow{H_T} T(x, y), \tag{4.5}$$

$$F_n(x, y, x^*) \xrightarrow{H_F} F(x, y, x^*), \quad G_n(y, x, y^*) \xrightarrow{H_Q} G(y, x, y^*), \tag{4.6}$$

and

$$P_n(x, y, x^*) \xrightarrow{H_P} P(x, y, x^*), \quad Q_n(y, x, y^*) \xrightarrow{H_Q} Q(y, x, y^*). \tag{4.7}$$

Since $H_K(.,.)$, $H_T(.,.)$, $H_F(.,.)$, $H_G(.,.)$, $H_P(.,.)$ and H_Q are continuous, by (4.1), (4.2) and (4.3), for any fixed $n \geq N$ and any $(x, y, x^*, y^*) \in A \times B \times A \times B$, let $m \to +\infty$, we get

$$H_K(K_n(x, y), K(x, y)) < \frac{\varepsilon}{6}, \quad H_T(T_n(x, y), T(x, y)) < \frac{\varepsilon}{6}, \tag{4.8}$$

$$H_F(F_n(x, y, x^*), F(x, y, x^*)) < \frac{\varepsilon}{6}, \quad H_G(G_n(y, x, y^*), G(y, x, y^*)) < \frac{\varepsilon}{6}, \tag{4.9}$$

and

$$H_P(P_n(x, y, x^*), P(x, y, x^*)) < \frac{\varepsilon}{6}, \quad H_Q(Q_n(y, x, y^*), Q(y, x, y^*)) < \frac{\varepsilon}{6}. \tag{4.10}$$

Now, we will prove that K is continuous.

By Lemma 2.6, we need to prove that, for any fixed a point $(x_0, y_0) \in A \times B$ and any $\varepsilon > 0$, there exists a neighborhood $N(x_0, y_0)$ of (x_0, y_0) in $A \times B$ such that

$$H_K(K(x, y), K(x_0, y_0)) < \varepsilon, \forall (x, y) \in N(x_0, y_0) \cap A \times B.$$

Since

$$\begin{aligned} H_K(K(x, y), K(x_0, y_0)) \leq\ & H_K(K(x, y), K_n(x, y)) \\ & + H_K(K_n(x, y), K_n(x_0, y_0)) + H_K(K_n(x_0, y_0)), K_n(x_0, y_0)), \end{aligned}$$

by (4.8), there exists $N > 0$ such that, for any $n > N$,

$$H_K(K(x,y), K_n(x,y)) < \frac{\varepsilon}{6}, \forall (x,y) \in A \times B.$$

Take a fixed $n > N$, by the continuity of K_n and Lemma 2.6, there exists a neighborhood $N(x_0, y_0)$ of (x_0, y_0) in $A \times B$ such that

$$H_K(K_n(x,y), K_n(x_0,y_0)) < \frac{\varepsilon}{6}, \forall N(x_0,y_0) \cap A \times B.$$

And so, we have

$$\begin{aligned} H_K(K(x,y), K(x_0,y_0)) &\le H_K(K(x,y), K_n(x,y)) + H_K(K_n(x,y), K_n(x_0,y_0)) \\ &\quad + H_K(K_n(x_0,y_0)), K_n(x_0,y_0)) < \varepsilon, \forall N(x_0,y_0) \cap A \times B. \end{aligned}$$

Hence, K is continuous in $A \times B$.

Similarly, we can prove that T is continuous in $A \times B$. It is easy see that the sets $\{(x,y,x^*) \in A \times B \times A : F(x,y,x^*) \cap P(x,y,x^*) \neq \emptyset\}$ and $\{(y,x,y^*) \in B \times A \times B : G(y,x,y^*) \cap Q(y,x,y^*) \neq \emptyset\}$ are closed.

Now, we show that, for all $(y,x^*) \in B \times A$, $F(.,y,x^*)$ is generalized type I $P(.,y,x^*)$-quasiconvex in A, for all $(x,y^*) \in A \times B$, $G(.,x,y^*)$ is generalized type I $Q(.,x,y^*)$-quasiconvex in B.

Indeed, for any n and for every $x_1, x_2 \in A$ and $\forall \lambda \in [0,1]$, $F_n(x_1,y,x^*) \cap P_n(x_1,y,x^*) \neq \emptyset$ and $F_n(x_2,y,x^*) \cap P_n(x_2,y,x^*) \neq \emptyset$. By the generalized type I $P(.,y,x^*)$-quasiconvexity, we have

$$F_n(\lambda x_1 + (1-\lambda)x_2, y, x^*) \cap P_n(\lambda x_1 + (1-\lambda)x_2, y, x^*) \neq \emptyset.$$

Since

$$F_n(x_1,y,x^*) \xrightarrow{H_F} F(x_1,y,x^*), \quad P_n(x_1,y,x^*) \xrightarrow{H_P} P(x_1,y,x^*),$$

$$F_n(x_2,y,x^*) \xrightarrow{H_F} F(x_2,y,x^*), \quad P_n(x_2,y,x^*) \xrightarrow{H_P} P(x_2,y,x^*),$$

and

$$F_n(\lambda x_1 + (1-\lambda)x_2, y, x^*) \xrightarrow{H_F} F(\lambda x_1 + (1-\lambda)x_2, y, x^*),$$

$$P_n(\lambda x_1 + (1-\lambda)x_2, y, x^*) \xrightarrow{H_P} P(\lambda x_1 + (1-\lambda)x_2, y, x^*),$$

it follows that

$$F(\lambda x_1 + (1-\lambda)x_2, y, x^*) \cap P(\lambda x_1 + (1-\lambda)x_2, y, x^*) \neq \emptyset.$$

And so, $F(.,y,x^*)$ is generalized type I $P(.,y,x^*)$-quasiconvex. Similarly, $G(.,x,y^*)$ is generalized type I $Q(.,x,y^*)$-quasiconvex.

By (4.8), (4.9) and (4.10), for any fixed $n \ge N$ and any $(x,y) \in A \times B$, we have

$$H_K(K_n(x,y), K(x,y)) < \frac{\varepsilon}{6}, \quad H_T(T_n(x,y), T(x,y)) < \frac{\varepsilon}{6},$$

$$H_F(F_n(x,y,x^*), F(x,y,x^*)) < \frac{\varepsilon}{6}, \quad H_G(G_n(y,x,y^*), G(y,x,y^*)) < \frac{\varepsilon}{6},$$

and

$$H_P(P_n(x,y,x^*), P(x,y,x^*)) < \frac{\varepsilon}{6}, \quad H_Q(Q_n(y,x,y^*), Q(y,x,y^*)) < \frac{\varepsilon}{6}.$$

Hence,

$$\sup_{(x,y)\in A\times B} H_K(K_n(x,y),K(x,y)) < \frac{\varepsilon}{6}, \quad \sup_{(x,y)\in A\times B} H_T(T_n(x,y),T(x,y)) < \frac{\varepsilon}{6},$$

$$\sup_{(x,y,x^*)\in A\times B\times A} H_F(F_n(x,y,x^*),F(x,y,x^*)) < \frac{\varepsilon}{6},$$

$$\sup_{(y,x,y^*)\in B\times A\times B} H_G(G_n(y,x,y^*),G(y,x,y^*)) < \frac{\varepsilon}{6},$$

$$\sup_{(x,y,x^*)\in A\times B\times A} H_P(P_n(x,y,x^*),P(x,y,x^*)) < \frac{\varepsilon}{6},$$

and

$$\sup_{(y,x,y^*)\in B\times A\times B} H_Q(Q_n(y,x,y^*),Q(y,x,y^*)) < \frac{\varepsilon}{6}.$$

Set $u = (K,T,F,G,P,Q)$, we know that $u \in \Omega_1$ and $\xi(u_n,u) \leq \varepsilon, \forall n \geq N$, i.e., $u_n \xrightarrow{\xi} u$. Thus, (Ω_1,ξ) is a complete metric space. □

Theorem 4.2. (Ω_2,ξ) *is a complete metric space.*

Proof. We omit the proof since the technique is similar as that for Theorem 4.1 with suitable modifications. □

Remark 4.3. In the special case as in Remark 3.6, Theorem 4.2 improves and extends Proposition 4.1 in [15]. Moreover, Theorem 4.2 also improves and extends Proposition 3.1 in [16].

Assume that all the conditions of Theorem 3.2 and Theorem 3.5 are satisfied. Then, for each $u = (K,T,F,G,P,Q) \in \Omega_1,\Omega_2$, (SQIP$_1$) and (SQIP$_1$) have solutions.

For $(K,T,F,G,P,Q) \in \Omega_1,\Omega_2$, let

$$\begin{aligned}\Xi_1(K,T,F,G,P,Q) := \{(\bar{x},\bar{y}) \in A\times B \text{ such that } & \bar{x} \in K(\bar{x},\bar{y}), \bar{y} \in T(\bar{x},\bar{y}) \text{ and}\\ & F(\bar{x},\bar{y},x^*) \cap P(\bar{x},\bar{y},x^*) \neq \emptyset, \forall x^* \in K(\bar{x},\bar{y}),\\ & G(\bar{y},\bar{x},y^*) \cap Q(\bar{y},\bar{x},y^*) \neq \emptyset, \forall y^* \in T(\bar{x},\bar{y}).\}\end{aligned}$$

and

$$\begin{aligned}\Xi_2(K,T,F,G,P,Q) := \{(\bar{x},\bar{y}) \in A\times B \text{ such that } & \bar{x} \in K(\bar{x},\bar{y}), \bar{y} \in T(\bar{x},\bar{y}) \text{ and}\\ & F(\bar{x},\bar{y},x^*) \subseteq P(\bar{x},\bar{y},x^*), \forall x^* \in K(\bar{x},\bar{y}),\\ & G(\bar{y},\bar{x},y^*) \subseteq Q(\bar{y},\bar{x},y^*), \forall y^* \in T(\bar{x},\bar{y}).\}\end{aligned}$$

Then $\Xi_1(K,T,F,G,P,Q) \neq \emptyset, \Xi_2(K,T,F,G,P,Q) \neq \emptyset$ and so $\Xi_1(K,T,F,G,P,Q)$ and $\Xi_2(K,T,F,G,P,Q)$ defined two set-valued mappings from Ω_1 into $A\times B$ and Ω_2 into $A\times B$, respectively.

Theorem 4.4. $\Xi_1 : \Omega_1 \to 2^{A\times B}$ *is upper semicontinuous with compact values.*

Proof. Since $A\times B$ is compact, we need only show that Ξ_1 is a closed mapping. Let a sequence $\{(u_n,(x_n,y_n))\}\subset$ Graph(Ξ_1) be given such that $(u_n,(x_n,y_n))\to(u,(x_0,y_0))\in\Omega\times(A\times B)$, where $u_n=(K_n,T_n,F_n,G_n,P_n,Q_n)$, $u=(K,T,F,G,P,Q)$. We now show that $\{(u,(x_0,y_0))\}\subset$ Graph(Ξ_1).

For any n, since $(x_n,y_n)\in\Xi_1(u_n)$, we have $x_n\in K_n(x_n,y_n)$ and $y_n\in T_n(x_n,y_n)$ such that

$$F_n(x_n,y_n,x^*)\cap P_n(x_n,y_n,x^*)\neq\emptyset,\forall x^*\in K_n(x_n,y_n), \tag{4.11}$$

and

$$G_n(y_n,x_n,y^*)\cap Q_n(y_n,x_n,y^*)\neq\emptyset,\forall y^*\in T_n(x_n,y_n). \tag{4.12}$$

For any open set $O\supset K(x_0,y_0)$, since $K(x_0,y_0)$ is a compact set, there exists $\varepsilon>0$ such that

$$\{x\in B: d(x,K(x_0,y_0))<\varepsilon\}\subset O, \tag{4.13}$$

where $d(x,K(x_0,y_0))=\inf_{x'\in K(x_0,y_0)}\|x-x'\|$.

Since $\xi(u_n,u)\to 0,(x_n,y_n)\to(x_0,y_0)$ and K is upper semicontinuous at (x_0,y_0), $\exists n_0$ such that

$$\sup_{(x,y)\in A\times B} H_K(K_n(x,y),K(x,y))<\frac{\varepsilon}{2}, \tag{4.14}$$

and

$$K(x_n,y_n)\subset\{x\in B: d(x,K(x_0,y_0))<\frac{\varepsilon}{2}\},\forall n\geq n_0. \tag{4.15}$$

From (4.13), (4.14) and (4.15), we have

$$K(x_n,y_n)\subset\{x\in B: d(x,K(x_0,y_0))<\frac{\varepsilon}{2}\}\subset\{x\in B: d(x,K(x_0,y_0))<\varepsilon\}\subset O,\forall n\geq n_0. \tag{4.16}$$

Since $K(x_0,y_0)\subset O$ and $x_n\in K_n(x_n,y_n)$, Lemma 2.8 implies that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ convergent to x_0, it follows that $x_0\in K(x_0,y_0)$. By using the same argument as above, we can show that $y_0\in T(x_0,y_0)$.

Next, we need only show that

$$F(x_0,y_0,x^*)\cap P(x_0,y_0,x^*)\neq\emptyset,\forall x^*\in K(x_0,y_0), \tag{4.17}$$

and

$$G(y_0,x_0,y^*)\cap Q(y_0,x_0,y^*)\neq\emptyset,\forall y^*\in T(x_0,y_0). \tag{4.18}$$

Since $(x_n,y_n)\to(x_0,y_0)$ and K is lower semicontinuous at (x_0,y_0), for any $x^*\in K(x_0,y_0)$ there exists $x_n^*\in K(x_n,y_n)$ such that $x_n^*\to x^*$. Since $\xi(u_n,u)\to 0$, we can chose a subsequence $\{K_{n_k}\}$ of $\{K_n\}$ such that

$$\sup_{(x,y)\in A\times B} H_K(K_{n_k}(x,y),K(x,y))<\frac{1}{k}. \tag{4.19}$$

Thus, there exists a subsequence $\{(x_{n_k},y_{n_k})\}$ of $\{(x_n,y_n)\}$ such that

$$H_K(K_{n_k}(x_{n_k},y_{n_k}),K(x_{n_k},y_{n_k}))<\frac{1}{k}.$$

This implies that there exists $\hat{x}^*_{n_k}\in K_{n_k}(x_{n_k},y_{n_k})$ such that

$$\|\hat{x}^*_{n_k}-x^*_{n_k}\|<\frac{1}{k}.$$

As

$$\|\hat{x}^*_{n_k} - x^*\| \leq \|\hat{x}^*_{n_k} - x^*_{n_k}\| + \|x^*_{n_k} - x^*\| < \frac{1}{k} + \|x^*_{n_k} - x^*\| \to 0,$$

and so we have $\hat{x}^*_{n_k} \to x^*$. Since $\hat{x}^*_{n_k} \in K_{n_k}(x_{n_k}, y_{n_k}), x_{n_k} \in K_{n_k}(x_{n_k}, y_{n_k})$, $y_{n_k} \in T_{n_k}(x_{n_k}, y_{n_k})$, applying (4.11), we have

$$F_{n_k}(x_{n_k}, y_{n_k}, \hat{x}^*_{n_k}) \cap P_{n_k}(x_{n_k}, y_{n_k}, \hat{x}^*_{n_k}) \neq \emptyset. \tag{4.20}$$

By the assumption (v) in Theorem 3.2 yields that

$$F(x_0, y_0, x^*) \cap P(x_0, y_0, x^*) \neq \emptyset, \forall\ x^* \in K(x_0, y_0). \tag{4.21}$$

Similarly, we can prove that

$$G(y_0, x_0, y^*) \cap Q(y_0, x_0, y^*) \neq \emptyset, \forall\ y^* \in T(x_0, y_0). \tag{4.22}$$

Since $x_0 \in K(x_0, y_0)$, $y_0 \in T(x_0, y_0)$ and (4.21)-(4.22) yields that $(u, (x_0, y_0)) \in$ Graph(Ξ_1) and so Graph(Ξ_1) is closed. Therefore, Ξ_1 is closed. Since $A \times B$ is a compact set and $\Xi_1(u) \subset A \times B$. Hence Ξ_1 is a upper semicontinuous with compact values. □

Theorem 4.5. $\Xi_2 : \Omega_2 \to 2^{A\times B}$ *is upper semicontinuous with compact values.*

Proof. We omit the proof since the technique is similar as that for Theorem 4.1 with suitable modifications. □

Remark 4.6. In the special case as in Remark 3.6, Theorem 4.5 improves and extends Theorem 4.2 in [15]. Moreover, Theorem 4.5 also improves and extends Lemma 3.1 in [16], Theorem 4.1 in [37].

5. Applications

Let X, Y, Z, A, B be as in Section 1, and $C \subset Z$ be a nonempty closed convex cone. Let $K : A \times B \to 2^A, T : A \times B \to 2^B$ be set-valued mappings and $f : A \times B \times A \to Z, g : B \times A \times B \to Z$ be vector functions. We consider the following two symmetric vector quasi-equilibrium problems (in short, (SQVEP$_1$) and (SQVEP$_2$)), respectively.

(SQVEP$_1$): Find $(\bar{x}, \bar{y}) \in A \times B$ such that $\bar{x} \in K(\bar{x}, \bar{y}), \bar{y} \in T(\bar{x}, \bar{y})$ and

$$f(\bar{x}, \bar{y}, x^*) \cap (Z \setminus -\text{int}C) \neq \emptyset, \forall x^* \in K(\bar{x}, \bar{y}),$$

$$g(\bar{y}, \bar{x}, y^*) \cap (Z \setminus -\text{int}C) \neq \emptyset, \forall y^* \in T(\bar{x}, \bar{y}).$$

and

(SQVEP$_2$): Find $(\bar{x}, \bar{y}) \in A \times B$ such that $\bar{x} \in K(\bar{x}, \bar{y}), \bar{y} \in T(\bar{x}, \bar{y})$ and

$$f(\bar{x}, \bar{y}, x^*) \in Z \setminus -\text{int}C, \forall x^* \in K(\bar{x}, \bar{y}),$$

$$g(\bar{y}, \bar{x}, y^*) \in Z \setminus -\text{int}C, \forall y^* \in T(\bar{x}, \bar{y}).$$

We denote that $\Sigma_1(f, g)$ and $\Sigma_2(f, g)$ are the solution sets of (SQVEP$_1$) and (SQVEP$_2$), respectively.

5.1. *Existence of solutions for ($SQVEP_1$) and ($SQVEP_2$)*

In this section, we discuss the existence and closedness of the solution sets for ($SQVEP_1$) and ($SQVEP_2$).

Theorem 5.1. *Assume for the problem* ($SQVEP_1$) *that*

(i) K *and* T *are continuous in* $A \times B$ *with nonempty compact convex values;*
(ii) for all $(x, y) \in A \times B$, $f(x, y, x) \cap (Z \setminus -\mathrm{int}C) \neq \emptyset$ *and* $g(y, x, y) \cap (Z \setminus -\mathrm{int}C) \neq \emptyset$;
(iii) the set $\{(y, x^*) \in B \times A : f(., y, x^*) \cap (Z \setminus -\mathrm{int}C) = \emptyset\}$ *is convex in* A, *and the set* $\{(x, y^*) \in A \times B : g(., x, y^*) \cap (Z \setminus -\mathrm{int}C) = \emptyset\}$ *is convex in* B;
(iv) for all $(y, x^*) \in B \times A$, $f(., y, x^*)$ *is generalized type I* $(Z \setminus -\mathrm{int}C)$*-quasiconvex in* A, *and for all* $(x, y^*) \in A \times B$, $g(., x, y^*)$ *is generalized type I* $(Z \setminus -\mathrm{int}C)$*-quasiconvex in* B;
(v) the set $\{(x, y, x^*) \in A \times B \times A : f(x, y, x^*) \cap (Z \setminus -\mathrm{int}C) \neq \emptyset\}$ *is closed, and the set* $\{(y, x, y^*) \in B \times A \times B : g(y, x, y^*) \cap (Z \setminus -\mathrm{int}C) \neq \emptyset\}$ *is closed.*

Then, the ($SQVEP_1$) *has a solution, i.e., there exists* $(\bar{x}, \bar{y}) \in A \times B$ *such that* $\bar{x} \in K(\bar{x}, \bar{y}), \bar{y} \in T(\bar{x}, \bar{y})$ *and*

$$f(\bar{x}, \bar{y}, x^*) \cap (Z \setminus -\mathrm{int}C) \neq \emptyset, \forall x^* \in K(\bar{x}, \bar{y}),$$

$$g(\bar{y}, \bar{x}, y^*) \cap (Z \setminus -\mathrm{int}C) \neq \emptyset, \forall y^* \in T(\bar{x}, \bar{y}).$$

Moreover, the solution set of the ($SQVEP_1$) *is closed.*

Proof. Setting $F(x, y, x^*) = f(x, y, x^*), G(y, x, y^*) = g(y, x, y^*)$ and $P(x, y, x^*) = Q(y, x, y^*) = Z \setminus -\mathrm{int}C$. Then, the problem ($SQVEP_1$) becomes a particular case of ($SQIP_1$) and the Corollary 5.1 is a direct consequence of Theorem 3.2. □

Theorem 5.2. *Assume for the problem* ($SQVEP_2$) *assumptions* (i), (ii), (iii) *and* (iv) *as in Theorem 5.1 and the condition* (v) *can be replaced by the following condition:*
(v′) f *is continuous in* $A \times B \times A$ *and* g *is continuous in* $B \times A \times B$.
Then, the ($SQVEP_1$) *has a solution. Moreover, the solution set of the* ($SQVEP_1$) *is closed.*

Proof. We omit the proof since the technique is similar as that for Theorem 5.1 with suitable modifications. □

Theorem 5.3. *Assume for the problem* ($SQVEP_2$) *that*

(i) K *and* T *are continuous in* $A \times B$ *with nonempty compact convex values;*
(ii) for all $(x, y) \in A \times B$, $f(x, y, x) \in Z \setminus -\mathrm{int}C$, *and* $g(y, x, y) \in Z \setminus -\mathrm{int}C$;
(iii) the set $\{(y, x^*) \in B \times A : f(., y, x^*) \notin Z \setminus -\mathrm{int}C\}$ *is convex in* A, *and the set* $\{(x, y^*) \in A \times B : g(., x, y^*) \notin Z \setminus -\mathrm{int}C\}$ *is convex in* B;
(iv) for all $(y, x^*) \in B \times A$, $f(., y, x^*)$ *is generalized type II* $(Z \setminus -\mathrm{int}C)$*-quasiconvex in* A, *and for all* $(x, y^*) \in A \times B$, $g(., x, y^*)$ *is generalized type II* $(Z \setminus -\mathrm{int}C)$*-quasiconvex in* B;
(v) the set $\{(x, y, x^*) \in A \times B \times A : f(x, y, x^*) \in Z \setminus -\mathrm{int}C\}$ *is closed, and the set* $\{(y, x, y^*) \in B \times A \times B : g(y, x, y^*) \in Z \setminus -\mathrm{int}C\}$ *is closed.*

Then, the ($SQVEP_2$) *has a solution, i.e., there exists* $(\bar{x}, \bar{y}) \in A \times B$ *such that* $\bar{x} \in K(\bar{x}, \bar{y}), \bar{y} \in T(\bar{x}, \bar{y})$ *and*

$$f(\bar{x}, \bar{y}, x^*) \in Z \setminus -\mathrm{int}C, \forall x^* \in K(\bar{x}, \bar{y}),$$
$$g(\bar{y}, \bar{x}, y^*) \in Z \setminus -\mathrm{int}C, \forall y^* \in T(\bar{x}, \bar{y}).$$

Moreover, the solution set of the ($SQVEP_2$) *is closed.*

Proof. Setting $F(x, y, x^*) = f(x, y, x^*), G(y, x, y^*) = g(y, x, y^*)$ and $P(x, y, x^*) = Q(y, x, y^*) = Z \setminus -\mathrm{int}C$. Then, the problem ($SQVEP_2$) becomes a particular case of ($SQIP_2$) and the Theorem 5.2 is a direct consequence of Theorem 3.5. □

Definition 5.4. *([21])* Let X and Z be two topological vector spaces and $A \subseteq X$ be nonempty convex set, $C \subset Z$ is a nonempty closed convex cone. Suppose $f : A \to Z$ be a vector function. f is called *properly C-quasiconvex in A* iff, for every $x_1, x_2 \in A$ and each $\lambda \in [0, 1]$, we have

either $f(\lambda x_1 + (1 - \lambda)x_2) \leq f(x_1)$,

or $f(\lambda x_1 + (1 - \lambda)x_2) \leq f(x_2)$.

Remark 5.5. (i) If we let $f(x, y, x^*) = f(x^*, y) - f(x, y), g(y, x, y^*) = g(x, y^*) - g(x, y)$ with $x \in A, y \in B, x^* \in A, y^* \in B$. Then, (SQVEP$_2$) becomes symmetric vector quasi-equilibrium problem studied in [22]. Fu [22] is obtained an existence result for symmetric vector quasi-equilibrium problem. However, the assumptions and proof methods of Theorem in [22] are different from the assumptions and proof methods in Theorem 5.3.

(ii) If we let $K(x, y) = K(x), T(x, y) = T(x), g(y, x, y^*) = f(x, y, x^*)$ with $x \in A, y \in B, x^* \in A, y^* \in B$ and replace $Z \setminus -\text{int}C$ by C. Then, (SQVEP$_2$) becomes strong vector quasi-equilibrium problem studied in [25]. Hou et al. [25] also obtained an existence result for strong vector quasi-equilibrium problem. However, the assumptions and proof methods of Theorem 3.1 in [25] are also different from the assumptions and proof methods in Theorem 5.3.

The following Example 5.6 shows that in the special case as in Remark 5.5(ii), all the assumptions of Theorem 5.3 are satisfied. But, Theorem 3.1 in [25] does not work. The reason is that f is not $(-C)$-continuous.

Example 5.6. Let $X = Y = Z = \mathbb{R}, A = B = [0, 1], C = \mathbb{R}_+$, and let $K : A \to 2^A, T : A \to 2^B$ and $f : [0, 1] \to \mathbb{R}$ be defined by

$$K(x) = T(x) = [0, 1],$$

$$f(x, y, x^*) = g(y, x, y^*) = f(x) = \begin{cases} [0, 1] & \text{if } \ x_0 = \frac{1}{2}, \\ [2, 3] & \text{otherwise.} \end{cases}$$

We show that all assumptions of Theorem 5.3 are satisfied. However, f is not $(-C)$-continuous at $x_0 = \frac{1}{2}$. Thus, it gives case where Theorem 5.3 can be applied but Theorem 3.1 in [25] does not work.

The following Example 5.7 shows that in the special case as Remark 5.5, all the assumptions of Theorem 5.3 are satisfied. But, Theorem 3.1 in [25] and Theorem in [22] do not work.

Example 5.7. Let $X = Y = Z = \mathbb{R}, A = B = [0, 2], C = \mathbb{R}_+$, and let $K : A \to 2^A, T : A \to 2^B$ and $f : [0, 2] \to \mathbb{R}$ be defined by

$$K(x) = T(x) = [0, 2],$$

$$f(x, y, x^*) = g(y, x, y^*) = f(x) = \begin{cases} [1, \frac{3}{2}] & \text{if } \ x_0 = \frac{1}{6}, \\ [0, \frac{1}{2}] & \text{otherwise.} \end{cases}$$

We show that all assumptions of Theorem 5.3 are satisfied. However, f is neither C-continuous nor properly C-quasiconvex at $x_0 = \frac{1}{6}$. Thus, it gives case where Theorem 5.3 can be applied but Theorem 3.1 in [25] and Theorem in [22] do not work.

Theorem 5.8. *Assume for the problem* (SQVEP$_2$) *assumptions* (i), (ii), (iii) *and* (iv) *as in Theorem 5.3 and the condition* (v) *can be replaced by the following condition:*

(v′) *f is continuous in $A \times B \times A$ and g is continuous in $B \times A \times B$.*

Then, the (SQVEP$_2$) *has a solution. Moreover, the solution set of the* (SQVEP$_2$) *is closed.*

5.2. Stability of (SQVEP$_1$) and (SQVEP$_2$)

In this section, we also discuss the semicontinuity of the solutions for (SQVEP$_1$) and (SQVEP$_2$).

Let be A, B, X, Y, Z as in Section 4 and let $\Omega_3 := \{(K, T, f, g) : K : A\times B \to 2^A, T : A\times B \to 2^B$ are continuous in $A\times B$ with nonempty compact convex values, $f : A\times B\times A \to Z, g : B\times A\times B \to Z$ are vector functions and the sets $\{(x, y, x^*) \in A\times B\times A : f(x, y, x^*)\cap(Z\setminus -\mathrm{int}C) \neq \emptyset\}$, and $\{(y, x, y^*) \in B\times A\times B : g(y, x, y^*)\cap(Z\setminus -\mathrm{int}C) \neq \emptyset\}$ are closed, and for all $(y, x^*) \in B \times A$, $f(., y, x^*)$ is generalized type I $(Z \setminus -\mathrm{int}C)$-quasiconvex in A, and for all $(x, y^*) \in A \times B$, $g(., x, y^*)$ is generalized type I $(Z \setminus -\mathrm{int}C)$-quasiconvex in B $\}$.

$\Omega_4 := \{(K, T, f, g) : K : A \times B \to 2^A, T : A \times B \to 2^B$ are continuous in $A \times B$ with nonempty compact convex values, $f : A \times B \times A \to Z, g : B \times A \times B \to Z$ are vector functions and the sets $\{(x, y, x^*) \in A \times B \times A : f(x, y, x^*) \subset Z \setminus -\mathrm{int}C\}$ and $\{(y, x, y^*) \in B \times A \times B : g(y, x, y^*) \subset Z \setminus -\mathrm{int}C\}$ are closed, and for all $(y, x^*) \in B \times A$, $f(., y, x^*)$ is generalized type II $(Z \setminus -\mathrm{int}C)$-quasiconvex in A, and for all $(x, y^*) \in A\times B$, $g(., x, y^*)$ is generalized type II $(Z \setminus -\mathrm{int}C)$-quasiconvex in $B\}$.

For $u_1 = (K_1, T_1, f_1, g_1), u_2 = (K_2, T_2, f_2, g_2), u_1, u_2 \in \Omega_1, \Omega_2$, define

$$\begin{aligned}\xi'(u_1, u_2) := & \sup_{(x,y)\in A\times B} H_K(K_1(x, y), K_2(x, y)) + \sup_{(x,y)\in A\times B} H_T(T_1(x, y), T_2(x, y)) \\ & + \sup_{(x,y,x^*)\in A\times B\times A} \|f_1(x, y, x^*) - f_2(x, y, x^*)\| \\ & + \sup_{(y,x,y^*)\in B\times A\times B} \|g_1(y, x, y^*) - g_2(y, x, y^*)\|,\end{aligned}$$

where H_K, H_T are Hausdorff metrics in $C\mathcal{K}(A), C\mathcal{K}(B)$. Obiously, (Ω_3, ξ') and (Ω_4, ξ') are metric spaces.

Theorem 5.9. (Ω_3, ξ') *is a complete metric space.*

Theorem 5.10. (Ω_4, ξ') *is a complete metric space.*

Remark 5.11. In the special case as in Remark 5.5(i), Theorem 5.10 improves and extends Proposition 3.1 in [16].

Assume that all the conditions of Theorem 5.1 and Theorem 5.3 are satisfied. Then, for each $u = (K, T, f, g) \in \Omega_3, \Omega_4$, (SQVEP$_1$) and (SQVEP$_2$) have solutions.

For $(K, T, f, g) \in \Omega_3, \Omega_4$, we let

$$\begin{aligned}\Xi_3(K, T, f, g) := \{(\bar{x}, \bar{y}) \in A \times B \text{ such that } & \bar{x} \in K(\bar{x}, \bar{y}), \bar{y} \in T(\bar{x}, \bar{y}) \text{ and} \\ & f(\bar{x}, \bar{y}, x^*) \cap (Z \setminus -\mathrm{int}C) \neq \emptyset, \forall x^* \in K(\bar{x}, \bar{y}), \\ & g(\bar{y}, \bar{x}, y^*) \cap (Z \setminus -\mathrm{int}C) \neq \emptyset, \forall y^* \in T(\bar{x}, \bar{y})\}.\end{aligned}$$

and

$$\begin{aligned}\Xi_4(K, T, f, g) := \{(\bar{x}, \bar{y}) \in A \times B \text{ such that } & \bar{x} \in K(\bar{x}, \bar{y}), \bar{y} \in T(\bar{x}, \bar{y}) \text{ and} \\ & f(\bar{x}, \bar{y}, x^*) \subset Z \setminus -\mathrm{int}C, \forall x^* \in K(\bar{x}, \bar{y}), \\ & g(\bar{y}, \bar{x}, y^*) \subset Z \setminus -\mathrm{int}C, \forall y^* \in T(\bar{x}, \bar{y})\}.\end{aligned}$$

Then $\Xi_3(K, T, f, g) \neq \emptyset, \Xi_4(K, T, f, g) \neq \emptyset$, and so $\Xi_3(K, T, f, g), \Xi_4(K, T, f, g)$ defined set-valued mappings from Ω_3 into $A \times B$ and from Ω_4 into $A \times B$, respectively.

Apply to the proof of Theorem 4.4 and Theorem 4.5, we obtain the following results:

Theorem 5.12. $\Xi_3 : \Omega_3 \to 2^{A\times B}$ *is upper semicontinuous with compact values.*

Theorem 5.13. $\Xi_4 : \Omega_4 \to 2^{A\times B}$ *is upper semicontinuous with compact values.*

Remark 5.14. In the special case as in Remark 5.5, Theorem 5.13 improves and extends Theorem 5.1 in [25] and Lemma 3.1 in [16].

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