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A New Iteration Scheme For Approximating Fixed Points of Nonexpansive Mappings

Balwant Singh Thakur[a], Dipti Thakur[a], Mihai Postolache[b]

[a]*School of Studies in Mathematics, Pt.Ravishankar Shukla University Raipur - 492010 (C.G.), India*
[b]*Department of Mathematics & Informatics, University Politehnica of Bucharest, Bucharest 060042, Romania*

Abstract. In this paper, we introduce a new three-step iteration scheme and establish convergence results for approximation of fixed points of nonexpansive mappings in the framework of Banach space. Further, we show that the new iteration process is faster than a number of existing iteration processes. To support the claim, we consider a numerical example and approximated the fixed point numerically by computer using Matlab.

1. Introduction

Let E be a uniformly convex Banach space, C be a nonempty closed convex subset of E. Throughout this paper, $\mathbb{N}$ denotes the set of all positive integers and $F(T) := \{x : Tx = x\}$. A mapping $T\colon C \to C$ is said to be nonexpansive if $\left\|Tx - Ty\right\| \leq \left\|x - y\right\|$ for all $x, y \in C$ and for all $n \in \mathbb{N}$. For arbitrary chosen $x_1 \in C$, construct a sequence $\{x_n\}$, where x_n is defined iteratively for each positive integer $n \geq 1$ by:

$$x_{n+1} = Tx_n\,, \tag{1}$$

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n Tx_n\,, \tag{2}$$

$$\left.\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n Ty_n\,,\\ y_n &= (1-\beta_n)x_n + \beta_n Tx_n\,. \end{aligned}\right\} \tag{3}$$

The sequences $\{x_n\}$ generated by (1), (2) and (3) are called Picard, Mann [8] and Ishikawa [5] iteration sequences respectively.

In 1955, Krasnoselskii [7] showed that the Picard iteration scheme (1) for a nonexpansive mapping T may fail to converge to fixed point of T even if T has a unique fixed point, but the Mann sequence (2) for $\alpha_n = \frac{1}{2}$, $\forall n \geq 1$ converges strongly to the fixed point of T.

Mann and Ishikawa iteration methods have been studied by several authors for approximation fixed points of nonexpansive mapping, see, e.g., [6, 11, 13–15].

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Email addresses: balwantst@gmail.com (Balwant Singh Thakur), dipti.thakur15@gmail.com (Dipti Thakur), emscolar@yahoo.com (Mihai Postolache)

In 2000, Noor [9] defined the following iterative scheme, by $x_1 \in C$ and

$$\left.\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n \\ y_n &= (1-\beta_n)x_n + \beta_n T z_n \\ z_n &= (1-\gamma_n)x_n + \gamma_n T x_n \end{aligned}\right\} \tag{4}$$

for all $n \geq 1$, where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$.

In 2007, Agrawal *et al.* [2] introduced the following iteration process: For arbitrary chosen $x_1 \in C$ construct a sequence $\{x_n\}$ by

$$\left.\begin{aligned} x_{n+1} &= (1-\alpha_n)Tx_n + \alpha_n T y_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T x_n, \quad n \in \mathbb{N} \end{aligned}\right\} \tag{5}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are in $(0,1)$. They showed that this process converges at a rate same as that of Picard iteration and faster than Mann iteration for contractive mappings.

Recently, Abbas *et al.* [1] introduce the following iteration, where a sequence $\{x_n\}$ is constructed from arbitrary $x_1 \in C$ by

$$\left.\begin{aligned} x_{n+1} &= (1-\alpha_n)Ty_n + \alpha_n T z_n \\ y_n &= (1-\beta_n)Tx_n + \beta_n T z_n \\ z_n &= (1-\gamma_n)x_n + \gamma_n T x_n \end{aligned}\right\} \tag{6}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are in $(0,1)$. They showed that this process converges faster than Agrawal *et al.* [2] iteration process.

Motivated by the previous ones, we introduce a new modified iteration process for finding fixed point of nonexpansive mappings, where the sequence $\{x_n\}$ is generated iteratively by $x_1 \in C$ and

$$\left.\begin{aligned} x_{n+1} &= (1-\alpha_n)Tz_n + \alpha_n T y_n \\ y_n &= (1-\beta_n)z_n + \beta_n T z_n \\ z_n &= (1-\gamma_n)x_n + \gamma_n T x_n \end{aligned}\right\} \tag{7}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $(0,1)$.

The purpose of this paper is to prove convergence results for nonexpansive mappings using the iteration (7). We also prove that the iteration (7) converges faster than Picard, Mann, Ishikawa, Noor, Agarwal *et al.*, Abbas *et al.* iteration process, for contractive mappings in the sense of Berinde [3]. We also present numerical example to compare the convergence of (7) with Picard, Mann, Ishikawa, Noor, Agarwal *et al.* and Abbas *et al.* iterations.

2. Preliminaries

Let E be a Banach space and $S_E = \{x \in E : \|x\| = 1\}$ unit sphere on E. For all $\lambda \in (0,1)$, and $x, y \in S_E$ with $x \neq y$, if $\left\|(1-\lambda)x + \lambda y\right\| < 1$, then E is called strictly convex. If E is a strictly convex Banach space and $\|x\| = \left\|y\right\| = \left\|\alpha x + (1-\alpha)y\right\|$ for $x, y \in E$ and $\alpha \in (0,1)$, then $x = y$.

The space E is said to be smooth if

$$\lim_{t \to 0} \frac{\left\|x + ty\right\| - \|x\|}{t} \tag{8}$$

exists for each x and y in S_E. In this case, the norm of E is called Gateaux differentiable. For all $y \in S_E$, if the limit (8) is attained uniformly for $x \in S_E$, then the norm is said to be uniformly Gateaux differentiable or Frechet differentiable.

We call the space E satisfies the Opial's condition [10] if for any sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ implies that

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|,$$

for all $y \in E$ with $y \neq x$.

A mapping $T\colon C \to E$ is demiclosed at $y \in E$ if for each sequence $\{x_n\}$ in C and each $x \in E$, $x_n \rightharpoonup x$ and $Tx_n \to y$ imply that $x \in C$ and $Tx = y$.

The following definitions about the rate of convergence are due to Berinde [3].

Definition 2.1. Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers converging to a and b respectively. If $\lim_{n\to\infty} \frac{|a_n - a|}{|b_n - b|} = 0$, then $\{a_n\}$ *converges faster than* $\{b_n\}$.

Definition 2.2. Suppose that for two fixed-point iteration processes $\{x_n\}$ and $\{u_n\}$, both converging to the same fixed point p, the error estimates

$$\|x_n - p\| \leq a_n \qquad \text{for all } n \geq 1,$$
$$\|u_n - p\| \leq b_n \qquad \text{for all } n \geq 1,$$

are available where $\{a_n\}$ and $\{b_n\}$ are two sequences of positive numbers converging to zero. If $\{a_n\}$ converges faster than $\{b_n\}$, then $\{x_n\}$ *converges faster than* $\{u_n\}$ to p.

We state the following lemmas to be used later on.

Lemma 2.3 ([4]). *Let C be a nonempty closed convex subset of a uniformly convex Banach space E, and T a nonexpansive mapping on C. Then, $I - T$ is demiclosed at zero.*

Lemma 2.4 ([12]). *Suppose E is a uniformly convex Banach space and $0 < p \leq t_n \leq q < 1$ for all $n \in \mathbb{N}$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences of E such that $\limsup_{n\to\infty} \|x_n\| \leq r$, $\limsup_{n\to\infty} \|y_n\| \leq r$ and $\limsup_{n\to\infty} \|t_n x_n + (1 - t_n) y_n\| = r$ hold for some $r \geq 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Lemma 2.5 ([2]). *Let E be a reflexive Banach space satisfying the Opial condition, C a nonempty and convex subset of E, and $T\colon C \to X$ an operator such that $I - T$ demiclosed at zero and $F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence in C such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$ and $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F(T)$. Then $\{x_n\}$ converges weakly to a fixed point of T.*

3. Rate of Convergence

In this section, we show that the iteration process (7) converges faster than the iteration of Abbas *et al.* (6).

Theorem 3.1. *Let C be a nonempty closed convex subset of a norm space E. Let T be a contraction with a contraction factor $k \in (0, 1)$ and fixed point p. Let $\{u_n\}$ be defined by the iteration process (6) and $\{x_n\}$ by (7), where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are in $[\varepsilon, 1 - \varepsilon]$ for all $n \in \mathbb{N}$ and for some ε in $(0, 1)$. Then $\{x_n\}$ converges faster than $\{u_n\}$. That is, our process (7) converges faster than (6).*

Proof. As proved in Theorem 3 of Abbas and Nazir [1],

$$\|u_{n+1} - p\| \leq k^n [1 - (1 - k)\alpha\beta\gamma]^n \|u_1 - p\|,$$

for all $n \in \mathbb{N}$. Let

$$a_n = k^n [1 - (1 - k)\alpha\beta\gamma]^n \|u_1 - p\|.$$

Now

$$\begin{aligned}\left\|z_n - p\right\| &= \left\|(1-\gamma_n)x_n + \gamma_n T x_n - p\right\| \\ &\le (1-\gamma_n)\left\|x_n - p\right\| + k\gamma_n \left\|x_n - p\right\| \\ &= (1-(1-k)\gamma_n)\left\|x_n - p\right\|,\end{aligned}$$

so that

$$\begin{aligned}\left\|y_n - p\right\| &= \left\|(1-\beta_n)z_n + \beta_n T z_n - p\right\| \\ &\le (1-\beta_n)\left\|z_n - p\right\| + k\beta_n \left\|z_n - p\right\| \\ &\le (1-\beta_n)(1-(1-k)\gamma_n)\left\|x_n - p\right\| + k\beta_n(1-(1-k)\gamma_n)\left\|x_n - p\right\| \\ &= (1-(1-k)\beta_n)(1-(1-k)\gamma_n)\left\|x_n - p\right\|.\end{aligned}$$

Thus

$$\begin{aligned}\left\|x_{n+1} - p\right\| &= \left\|(1-\alpha_n)Tz_n + \alpha_n T y_n - p\right\| \\ &\le (1-\alpha_n)k\left\|z_n - p\right\| + k\alpha_n \left\|y_n - p\right\| \\ &\le (1-\alpha_n)k(1-(1-k)\gamma_n)\left\|x_n - p\right\| \\ &\quad + k\alpha_n(1-(1-k)\beta_n)(1-(1-k)\gamma_n)\left\|x_n - p\right\| \\ &= k[(1-(1-k)\gamma_n)\{1-\alpha_n + \alpha_n(1-(1-k)\beta_n)\}]\left\|x_n - p\right\| \\ &= k[(1-(1-k)\gamma_n)(1-\alpha_n+\alpha_n-(1-k)\alpha_n\beta_n)]\left\|x_n - p\right\| \\ &= k[(1-(1-k)\gamma_n)(1-(1-k)\alpha_n\beta_n)]\left\|x_n - p\right\| \\ &= k[1-(1-k)\gamma_n - (1-k)\alpha_n\beta_n + (1-k)^2\alpha_n\beta_n\gamma_n]\left\|x_n - p\right\| \\ &< k[1-(1-k)\gamma_n - (1-k)\alpha_n\beta_n\gamma_n + (1-k)\alpha_n\beta_n\gamma_n]\left\|x_n - p\right\| \\ &= k(1-(1-k)\gamma_n)\left\|x_n - p\right\|.\end{aligned}$$

Let $b_n = k^n(1-(1-k)\gamma)^n \left\|x_1 - p\right\|$. Then

$$\begin{aligned}\frac{b_n}{a_n} &= \frac{k^n[1-(1-k)\gamma]^n \left\|x_1 - p\right\|}{k^n[1-(1-k)\alpha\beta\gamma]^n \left\|u_1 - p\right\|} \\ &= \frac{[1-(1-k)\gamma]^n}{[1-(1-k)\alpha\beta\gamma]^n}\frac{\left\|x_1 - p\right\|}{\left\|u_1 - p\right\|} \\ &\to 0 \quad as \quad n \to \infty.\end{aligned}$$

Consequently $\{x_n\}$ converges faster than $\{u_n\}$. □

Now, we present an example which shows that our iteration process (7) converges at a rate faster than Abbas and Nazir's iteration process (6), Agarwal et al. iteration process (5), Mann iteration (2), Ishikawa iteration (3), Noor iteration (4) and Picard iteration process (1).

Example 3.2. Let $E = \mathbb{R}$ and $C = [1, 50]$. Let $T\colon C \to C$ be a mapping, which is defined by the formula $T(x) = \sqrt{x^2 - 9x + 54}$ for all $x \in C$. Choose $\alpha_n = \beta_n = \gamma_n = \frac{3}{4}$, with the initial value $x_1 = 30$.

The corresponding our iteration process, Abbas and Nazir iteration process (6), Agarwal et al.iteration process (5), Mann iteration process (2) and Picard iteration process (1) are respectively given below.

Step	Picard	Mann	Ishikawa	Noor	Agarwal	Abbas	New iter.
1	30.0000000000	30.0000000000	30.0000000000	30.0000000000	30.0000000000	30.0000000000	30.0000000000
2	26.1533936612	27.1150452459	25.0119824036	23.4891033190	24.0503308189	22.6107900775	21.3066758526
3	22.4191761010	24.2907437151	20.2547559071	17.4668190633	18.4372719353	15.8281562671	13.5889959660
4	18.8373796516	21.5420343135	15.8509087868	12.3265857284	13.3938203603	10.2582064029	8.1129739580
5	15.4696624163	18.8892775011	12.0133051549	8.7275766163	9.3725555853	7.0018379217	6.2256746270
6	12.4130372403	16.3606498049	9.0688620373	6.9585711603	6.9939357160	6.1191542086	6.0151302207
7	9.8166266286	13.9954171304	7.2820400289	6.3102146269	6.1862067854	6.0112132578	6.0009604944
8	7.8750567432	11.8475686983	6.4668031480	6.0979255677	6.0283693653	6.0010243038	6.0000607496
9	6.7187058292	9.9869851099	6.1600652383	6.0306808428	6.0041338820	6.0000933041	6.0000038414
10	6.2187342407	8.4900396666	6.0537250393	6.0095903071	6.0005981884	6.0000084969	6.0000002429
11	6.0583865336	7.4083030742	6.0179028366	6.0029956076	6.0000864719	6.0000007738	6.0000000154
12	6.0148623083	6.7246651786	6.0059514305	6.0009354914	6.0000124982	6.0000000705	6.0000000010
13	6.0037328233	6.3468134658	6.0019768478	6.0002921220	6.0000018064	6.0000000064	6.0000000001
14	6.0009342942	6.1586728531	6.0006564620	6.0000912177	6.0000002611	6.0000000006	6.0000000000
15	6.0002336418	6.0708846663	6.0002179755	6.0000284834	6.0000000377	6.0000000001	6.0000000000
16	6.0000584147	6.0313055772	6.0000723757	6.0000088941	6.0000000055	6.0000000000	6.0000000000
17	6.0000146039	6.0137535390	6.0000240311	6.0000027772	6.0000000008	6.0000000000	6.0000000000
18	6.0000036510	6.0060282506	6.0000079791	6.0000008672	6.0000000001	6.0000000000	6.0000000000
19	6.0000009128	6.0026394884	6.0000026493	6.0000002708	6.0000000000	6.0000000000	6.0000000000
20	6.0000002282	6.0011551843	6.0000008797	6.0000000846	6.0000000000	6.0000000000	6.0000000000
21	6.0000000570	6.0005054713	6.0000002921	6.0000000264	6.0000000000	6.0000000000	6.0000000000
22	6.0000000143	6.0002211587	6.0000000970	6.0000000082	6.0000000000	6.0000000000	6.0000000000
23	6.0000000036	6.0000967598	6.0000000322	6.0000000026	6.0000000000	6.0000000000	6.0000000000
24	6.0000000009	6.0000423330	6.0000000107	6.0000000008	6.0000000000	6.0000000000	6.0000000000
25	6.0000000002	6.0000185208	6.0000000035	6.0000000003	6.0000000000	6.0000000000	6.0000000000
26	6.0000000001	6.0000081029	6.0000000012	6.0000000001	6.0000000000	6.0000000000	6.0000000000
27	6.0000000000	6.0000035450	6.0000000004	6.0000000000	6.0000000000	6.0000000000	6.0000000000
28	6.0000000000	6.0000015509	6.0000000001	6.0000000000	6.0000000000	6.0000000000	6.0000000000
29	6.0000000000	6.0000006785	6.0000000000	6.0000000000	6.0000000000	6.0000000000	6.0000000000
30	6.0000000000	6.0000002969	6.0000000000	6.0000000000	6.0000000000	6.0000000000	6.0000000000
31	6.0000000000	6.0000001299	6.0000000000	6.0000000000	6.0000000000	6.0000000000	6.0000000000
32	6.0000000000	6.0000000568	6.0000000000	6.0000000000	6.0000000000	6.0000000000	6.0000000000
33	6.0000000000	6.0000000249	6.0000000000	6.0000000000	6.0000000000	6.0000000000	6.0000000000
34	6.0000000000	6.0000000109	6.0000000000	6.0000000000	6.0000000000	6.0000000000	6.0000000000
35	6.0000000000	6.0000000048	6.0000000000	6.0000000000	6.0000000000	6.0000000000	6.0000000000
36	6.0000000000	6.0000000021	6.0000000000	6.0000000000	6.0000000000	6.0000000000	6.0000000000
37	6.0000000000	6.0000000009	6.0000000000	6.0000000000	6.0000000000	6.0000000000	6.0000000000
38	6.0000000000	6.0000000004	6.0000000000	6.0000000000	6.0000000000	6.0000000000	6.0000000000
39	6.0000000000	6.0000000002	6.0000000000	6.0000000000	6.0000000000	6.0000000000	6.0000000000
40	6.0000000000	6.0000000001	6.0000000000	6.0000000000	6.0000000000	6.0000000000	6.0000000000
41	6.0000000000	6.0000000000	6.0000000000	6.0000000000	6.0000000000	6.0000000000	6.0000000000

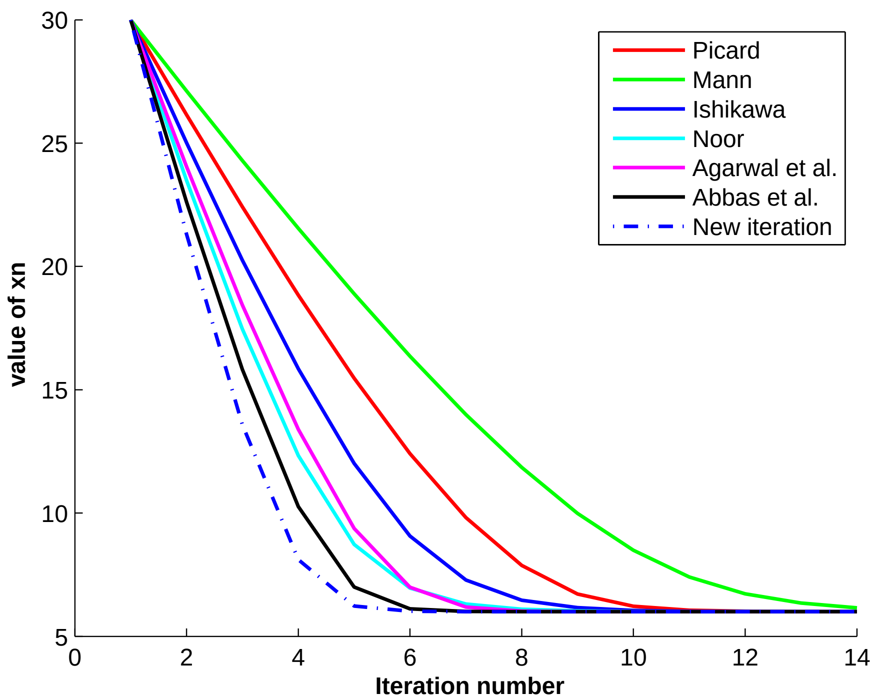
All sequences converges to $x^* = 6$. Comparison shows that our iteration process (7) requires least number of iterations among all the iterations mentioned above.

We now examine the fastness of the iterations in terms of computation time. All codes were written in Matlab and elapsed time were measured using `tic` and `toc` functions. Since this functions depends upon some random value, we tested programm in two different computers which we call system-1 and system-2, where system-1 is a notebook Intel(R) Core(TM)2 Duo CPU P8700 2.53GHz with 3.00GB RAM and system-2 is a PC Intel(R) Core(TM)2 Duo CPU E8400 3.00GHz with 4.00GB RAM. We tested program 5 times in each system and taken the average elapsed time for each iteration method. The number of iterations and the computational time to obtain the fixed point using different iteration methods up to 15 decimal places of second, are given in the following table.

Iteration Method	Number of iterations	Average elapsed time (sec.)	
		System-1	System-2
1	2	3	4
Picard	27	0.001530001332577	0.001120460080389
Mann	41	0.003159117654725	0.002367597986286
Ishikawa	29	0.001554638207663	0.001367451268890
Noor	27	0.001551315488323	0.001412962226072
Agarwal et al.	19	0.001308025840315	0.001117790973962
Abbas et al.	16	0.001454469725004	0.001337201902983
New iter.	14	0.001234034186288	0.001059824367656

We can see from above mentioned table and the following graph that the new iteration converges faster not only in terms of number of iterations but in elapsed time also.

Convergence behaviour of Picard, Mann, Ishikawa, Noor, Agarwal et al., Abbas et al. and New iteration for the function given in the Example 3.2



4. Convergence Theorems

In this section, we establish some convergence theorems using iteration process (7).

Lemma 4.1. *Let C be a nonempty closed convex subset of a normed linear space E. Let T be a nonexpansive self mapping on C, $\{x_n\}$ be a sequence defined by* (7) *and $F(T) \neq \emptyset$. Then $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F(T)$.*

Proof. Let $p \in F(T)$ for all $n \in \mathbb{N}$. From (7), we have

$$\begin{aligned} \left\|z_n - p\right\| &= \left\|(1-\gamma_n)x_n + \gamma_n T x_n - p\right\| \\ &\le (1-\gamma_n)\left\|x_n - p\right\| + \gamma_n \left\|Tx_n - p\right\| \\ &\le (1-\gamma_n)\left\|x_n - p\right\| + \gamma_n \left\|x_n - p\right\| \\ &= \left\|x_n - p\right\|, \end{aligned} \tag{9}$$

and

$$\begin{aligned} \left\|y_n - p\right\| &= \left\|(1-\beta_n)z_n + \beta_n T z_n - p\right\| \\ &\le (1-\beta_n)\left\|z_n - p\right\| + \beta_n \left\|Tz_n - p\right\| \\ &\le (1-\beta_n)\left\|x_n - p\right\| + \beta_n \left\|x_n - p\right\| \\ &= \left\|x_n - p\right\|, \end{aligned} \tag{10}$$

thus from (9) and (10)

$$\begin{aligned} \left\|x_{n+1} - p\right\| &= \left\|(1-\alpha_n)Tz_n + \alpha_n T y_n - p\right\| \\ &\le (1-\alpha_n)\left\|Tz_n - p\right\| + \alpha_n \left\|Ty_n - p\right\| \\ &\le (1-\alpha_n)\left\|x_n - p\right\| + \alpha_n \left\|x_n - p\right\| \\ &= \left\|x_n - p\right\| \end{aligned}$$

Hence $\lim_{n\to\infty} \left\|x_n - p\right\|$ exists for all $p \in F(T)$. □

Lemma 4.2. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E. Let T be a nonexpansive self mapping on C, $\{x_n\}$ be a sequence given by* (7) *and $F(T) \neq \emptyset$. Then $\lim_{n\to\infty} \| x_n - Tx_n \| = 0$.*

Proof. By Lemma 4.1, it follows that $\lim_{n\to\infty} \left\|x_n - p\right\|$ exists.
Assume that $\lim_{n\to\infty} \left\|x_n - p\right\| = c$.
From (9) and (10) we have

$$\limsup_{n\to\infty} \left\|y_n - p\right\| \le c, \tag{11}$$

and

$$\limsup_{n\to\infty} \left\|z_n - p\right\| \le c. \tag{12}$$

Since T is nonexpansive mapping, we have

$$\left\|Tx_n - p\right\| \le \left\|x_n - p\right\|, \quad \left\|Ty_n - p\right\| \le \left\|y_n - p\right\|, \quad \text{and } \left\|Tz_n - p\right\| \le \left\|z_n - p\right\| .$$

Taking lim sup on both sides, we obtain

$$\limsup_{n\to\infty} \left\|Tx_n - p\right\| \le c,$$

$$\limsup_{n\to\infty} \left\|Ty_n - p\right\| \le c,$$

and

$$\limsup_{n\to\infty} \left\|Tz_n - p\right\| \le c. \tag{13}$$

Since

$$c = \lim_{n\to\infty} \left\| x_{n+1} - p \right\| = \lim_{n\to\infty} \left\| (1-\alpha_n)(Tz_n - p) + \alpha_n(Ty_n - p) \right\|,$$

by using Lemma 2.4, we have

$$\lim_{n\to\infty} \left\| Tz_n - Ty_n \right\| = 0.$$

Now

$$\left\| x_{n+1} - p \right\| = \left\| (1-\alpha_n)Tz_n + \alpha_n Ty_n - p \right\| \le \left\| Tz_n - p \right\| + \alpha_n \left\| Tz_n - Ty_n \right\|,$$

yields that

$$c \le \liminf_{n\to\infty} \left\| Tz_n - p \right\|, \tag{14}$$

so that (13) and (14) gives

$$\lim_{n\to\infty} \left\| Tz_n - p \right\| = c.$$

On the other hand, we have

$$\left\| Tz_n - p \right\| \le \left\| Tz_n - Ty_n \right\| + \left\| Ty_n - p \right\| \le \left\| Tz_n - Ty_n \right\| + \left\| y_n - p \right\|,$$

and this yields that

$$c \le \liminf_{n\to\infty} \left\| y_n - p \right\|. \tag{15}$$

From (11) and (15) we get

$$\lim_{n\to\infty} \left\| y_n - p \right\| = c.$$

Using Lemma 2.4, from (12) and (13), we get

$$\lim_{n\to\infty} \| z_n - Tz_n \| = 0.$$

Since

$$\left\| y_n - p \right\| \le \left\| z_n - p \right\| + \beta_n \| Tz_n - z_n \|,$$

we write,

$$c \le \limsup_{n\to\infty} \left\| z_n - p \right\|,$$

then,

$$\left\| z_n - p \right\| = c,$$

so

$$\begin{aligned} c &= \lim_{n\to\infty} \left\| z_n - p \right\| \\ &= \lim_{n\to\infty} \left\| (1-\alpha_n)x_n + \alpha_n Tx_n - p \right\| \\ &= \lim_{n\to\infty} \left\| (1-\alpha_n)(x_n - p) + \alpha_n(Tx_n - p) \right\|, \end{aligned}$$

and by Lemma 2.4, we have

$$\lim_{n\to\infty} \| x_n - Tx_n \| = 0.$$

This completes the proof. □

By using Lemma 2.3, Lemma 2.5, Lemma 4.1 and Lemma 4.2, we will establish the following theorem.

Theorem 4.3. *Let E be a real uniformly convex Banach space which satisfies the Opial's condition, C a nonempty closed convex subset of X and $T\colon C \to C$ a nonexpansive mapping with $F(T) \neq \emptyset$. Let $\{x_n\}$ be the sequence defined by iteration process* (7)*. Then $\{x_n\}$ converges weakly to a fixed point of T.*

Proof. Let $p \in F(T)$. Then $\lim_{n\to\infty} \left\|x_n - p\right\|$ exists. We prove that $\{x_n\}$ has a unique weak subsequential limit in $F(T)$. For, let u and v be weak limits of the subsequences $\{x_{n_i}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$, respectively. By $\lim_{n\to\infty} \left\|x_n - Tx_{n_k}\right\| = 0$ and $I - T$ is demiclosed with respect to zero by Lemma 2.3, therefore we obtain $Tu = u$. Again in the same manner, we can prove that $v \in F(T)$. Next, we prove the uniqueness. From Lemma 4.1 the limits $\lim_{n\to\infty} \|x_n - v\|$ exists. For this suppose that $u \neq v$, then by the Opial's condition

$$\begin{aligned}\lim_{n\to\infty} \|x_n - u\| &= \lim_{n_i\to\infty} \left\|x_{n_i} - u\right\| < \lim_{n_i\to\infty} \left\|x_{n_i} - v\right\| = \lim_{n\to\infty} \|x_n - v\| \\ &= \lim_{n_j\to\infty} \left\|x_{n_j} - v\right\| < \lim_{n_j\to\infty} \left\|x_{n_j} - u\right\| = \lim_{n\to\infty} \|x_n - u\|\end{aligned}$$

This is a contradiction, so $u = v$. Hence, $\{x_n\}$ converges weakly to a fixed point of $F(T)$ and this completes the proof. □

Theorem 4.4. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E. Let T be a nonexpansive self mapping on C, $\{x_n\}$ defined by* (7) *and $F(T) \neq \emptyset$. Then $\{x_n\}$ converges to a point of $F(T)$ if and only if* $\liminf_{n\to\infty} d(x_n, F(T)) = 0$ *where* $d(x, F(T)) = \inf\{\left\|x - p\right\| : p \in F(T)\}$

Proof. Necessity is obvious.

Suppose that $\liminf_{n\to\infty} d(x_n, F(T)) = 0$. As proved in Lemma 4.2, $\lim_{n\to\infty} \|x_n - w\|$ exists for all $w \in F(T)$, therefore $\lim_{n\to\infty} d(x_n, F(T))$ exists. But by hypothesis, we have $\liminf_{n\to\infty} d(x_n, F(T)) = 0$, therefore we have $\lim_{n\to\infty} d(x_n, F(T)) = 0$. We will show that $\{x_n\}$ is a Cauchy sequence in C. Since $\lim_{n\to\infty} d(x_n, F(T)) = 0$, for given $\varepsilon > 0$, there exists n_0 in $\mathbb{N}$ such that for all $n \geq n_0$,

$$d(x_n, F(T)) < \frac{\varepsilon}{2}.$$

Particularly, $\inf\{\left\|x_{n_0} - p\right\| : p \in F(T)\} < \frac{\varepsilon}{2}$. Hence, there exist $p^* \in F(T)$ such that $\left\|x_{n_0} - p^*\right\| < \frac{\varepsilon}{2}$. Now, for $m, n \geq n_0$,

$$\|x_{n+m} - x_n\| \leq \left\|x_{n+m} - p^*\right\| + \left\|x_n - p^*\right\| \leq 2\left\|x_{n_0} - p^*\right\| < \varepsilon.$$

Hence $\{x_n\}$ is a Cauchy sequence in C. Since C is closed in the Banach space E, so that there exists a point p in C such that $\lim_{n\to\infty} x_n = p$. Now $\lim_{n\to\infty} d(x_n, F(T)) = 0$ gives that $d(p, F(T)) = 0$. Since F is closed, $p \in F(T)$. □

A mapping $T\colon C \to C$, where C is a subset of a normed space E, is said to satisfy Condition (A) [13] if there exists a nondecreasing function$f\colon [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(r) > 0$, for all $r \in (0, 1)$, such that $\|x - Tx\| \geq f(d(x, F(T))$, for all $x \in C$, where $d(x, F(T)) = \inf\{\left\|x - p\right\| : p \in F(T)\}$.

It is to be noted that Condition (A) is weaker than compactness of the domain C.

Applying Theorem 4.4, we obtain a strong convergence of the process (7) under Condition (A) as follows:

Theorem 4.5. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E. Let T be a nonexpansive self mapping on C, $\{x_n\}$ defined by* (7) *and $F(T) \neq \emptyset$. Let T satisfy Condition (A), then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. We proved in Lemma 4.2 that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{16}$$

From Condition (A) and (16), we get

$$\lim_{n\to\infty} f(d(x_n, F(T)) \leq \lim_{n\to\infty} \|x_n - Tx_n\| = 0,$$

i.e., $\lim_{n\to\infty} f(d(x_n, F(T)) = 0$. Since $f : [0, \infty) \to [0, \infty)$ is a nondecreasing function satisfying $f(0) = 0$, $f(r) > 0$ for all $r \in (0, \infty)$, therefore we have

$$\lim_{n\to\infty} d(x_n, F(T)) = 0.$$

By Theorem 4.4 the sequence $\{x_n\}$ converges strongly to a point of $F(T)$. □

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