



Finitistic Dimension of Weak Hopf-Galois Extensions

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Abstract. Let H be a finite dimensional weak Hopf algebra and A/B be a right faithfully flat weak H -Galois extension. Then in this note, we first show that if H is semisimple, then the finitistic dimension of A is less than or equal to that of B . Furthermore, using duality theorem, we obtain that if H and its dual H^* are both semisimple, then the finitistic dimension of A is equal to that of B , which means the finitistic dimension conjecture holds for A if and only if it holds for B . Finally, as applications, we obtain these relations for the weak crossed products and weak smash products.

1. Introduction and Preliminaries

Throughout this paper, k denotes a fixed field, and we will always work over k . The tensor product $\otimes = \otimes_k$ and Hom is always assumed to be over k . For an algebra A , denote by $A\text{-Mod}$ and $A\text{-mod}$ the categories of left A -modules and of finitely generated left A -modules, respectively. For an A -module M , let $\text{proj.dim}(M)$ denote the projective dimension of M .

Weak bialgebras and weak Hopf algebras are generalizations of ordinary bialgebras and Hopf algebras in the following sense: the defining axioms are the same, but the multiplicativity of the counit, comultiplicativity of the unit and the antipode conditions are replaced by weaker axioms. Perhaps the easiest example of a weak Hopf algebra is a groupoid algebra; other examples are face algebras [9], quantum groupoids [15] and generalized Kac algebras [19]. The main motivation for studying weak Hopf algebras comes from quantum field theory, operator algebras and representation theory. A purely algebraic approach can be found in [4] and [5].

Galois theory for weak Hopf algebras was developed by [7] using the language of corings which were introduced by Sweedler in [18]. Weak Hopf-Galois extensions were also studied in [3] and [10] using the language of Hopf algebroids.

Recall from Böhm et al. [4] that a weak Hopf algebra H is both a k -algebra (m, μ) and a k -coalgebra (Δ, ε) such that $\Delta(hk) = \Delta(h)\Delta(k)$, and

$$\begin{aligned}\Delta^2(1) &= 1_1 \otimes 1_2 1'_1 \otimes 1'_2 = 1_1 \otimes 1'_1 1_2 \otimes 1'_2, \\ \varepsilon(hkl) &= \varepsilon(hk_1)\varepsilon(k_2l) = \varepsilon(hk_2)\varepsilon(k_1l),\end{aligned}$$

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for all $h, k, l \in H$, together with a k -linear map $S : H \rightarrow H$ (called the antipode) satisfying $S(h_1)h_2 = \Gamma^R(h)$, $h_1S(h_2) = \Gamma^L(h)$, $S(h_1)h_2S(h_3) = S(h)$, where we write $\Delta(h) = h_1 \otimes h_2$ omitting summation symbol and index (see [13]) and $\Gamma^R(h) = 1_1\varepsilon(h1_2)$, $\Gamma^L(h) = \varepsilon(1_1h)1_2$. The antipode S is both an anti-algebra and an anti-coalgebra morphism. If H is a finite dimensional, then S is automatically bijective and the dual $H^* = \text{Hom}(H, k)$ has a natural structure of a weak Hopf algebra. We always assume H is finite dimensional.

Let H be a weak Hopf algebra and A be an algebra. By [6], A is a right weak H -comodule algebra if there is a right H -comodule structure $\rho_A : A \rightarrow A \otimes H$ (with notation $a \mapsto a_0 \otimes a_1$) such that $\rho(ab) = \rho(a)\rho(b)$ for each $a, b \in A$ and $\rho(1) = 1_0 \otimes 1_1 \in A \otimes H^L$, where $H^L = \Gamma^L(H)$ is a separable subalgebra of H .

The coinvariants are defined by

$$B := A^{coH} := \{a \in A \mid \rho_A(a) = 1_0a \otimes 1_1\}$$

is a subalgebra of A . We say the extension A/B is right weak H -Galois if the map $\beta : A \otimes_B A \rightarrow A \otimes H$, given by $a \otimes_B b \mapsto ab_0 \otimes b_1$, is bijective, where

$$A \otimes H := (A \otimes H)\rho(1) = \{a1_0 \otimes h1_1 \triangleq a \otimes_t h \mid a \in A, h \in H\}.$$

The finitistic dimension of an algebra A is defined to be

$$\text{fin.dim}(A) = \sup\{\text{proj.dim}(M) \mid M \in A\text{-mod and proj.dim}(M) < \infty\}.$$

H. Bass conjectured that $\text{fin.dim}(A) < \infty$ for any finite-dimensional algebra A , called the finitistic dimension conjecture (see [1]). Since then, much work has been done toward the proof of this conjecture. It remains to be open.

The aim of this note is to study the relationship of finitistic dimensions under weak Hopf-Galois extensions. Let A/B be a right faithfully flat weak H -Galois extension. First, we show that if H is semisimple, then the finitistic dimension of A is less than or equal to that of B , which generalizes the main result in [11]. Furthermore, using duality theorem, we obtain that if H and its dual H^* are both semisimple, then the finitistic dimension of A is equal to that of B , which generalizes the main result in [12]. As applications, we obtain these relations for the weak crossed products and weak smash products.

2. The Main Results, their Proofs and Corollaries

Consider the following two functors

$$\begin{aligned} A \otimes_B - : B\text{-Mod} &\rightarrow A\text{-Mod}, & M &\mapsto A \otimes_B M, \\ {}_B(-) : A\text{-Mod} &\rightarrow B\text{-Mod}, & M &\mapsto M, \end{aligned}$$

where ${}_B(-)$ is the restriction functor. By adjoint isomorphism theorem, $(A \otimes_B -, {}_B(-))$ is an adjoint pair.

Lemma 2.1. *Let A/B be a right weak H -Galois extension for a semisimple weak Hopf algebra H . Then for any A -module M , M is an A -direct summand of $A \otimes_B M$.*

Proof. Since H is a semisimple weak Hopf algebra, by [20, Proposition 3.1], A/B is a separable extension, which is equivalent to the restriction functor ${}_B(-)$ is separable. Consider the adjoint pair $(A \otimes_B -, {}_B(-))$. If the functor ${}_B(-)$ is separable, then we obtain by [8, Proposition 5] that the natural map $\varepsilon_M : A \otimes_B {}_B M \rightarrow M$ is a split epimorphism for every $M \in A\text{-Mod}$. \square

Proposition 2.2. *Let A/B be a right weak H -Galois extension for a semisimple weak Hopf algebra H . Then for any A -module M , $\text{proj.dim}({}_B M) = \text{proj.dim}({}_A M)$.*

Proof. First, by [3, Corollary 4.3], the natural modules ${}_B A$ and A_B are both finitely generated projective. It follows that any projective resolution of M as an A -module is also a projective resolution of M as a B -module. Thus $\text{proj.dim}({}_B M) \leq \text{proj.dim}({}_A M)$.

Conversely, consider the adjoint pair $(A \otimes_B -, {}_B(-))$. Since ${}_B(-)$ is exact, the functor $A \otimes_B -$ preserves projective objects. It follows that $A \otimes_B P$ is a projective A -module for each projective B -module P . We may assume that $\text{proj.dim}({}_B M) = n < \infty$, and let \mathcal{P} be a projective resolution of M as a B -module of length n . Then $A \otimes_B \mathcal{P}$ is a projective resolution of $A \otimes_B M$ as an A -module. The exactness of this sequence is determined by the projectiveness of A as a right B -module. It implies $\text{proj.dim}({}_A(A \otimes_B M)) \leq \text{proj.dim}({}_B M)$. Also by Lemma 2.1, M is an A -direct summand of $A \otimes_B M$, it follows that $\text{proj.dim}({}_A M) \leq \text{proj.dim}({}_A(A \otimes_B M))$. Thus $\text{proj.dim}({}_A M) \leq \text{proj.dim}({}_B M)$. The proof is completed. \square

Following from Proposition 2.2, we immediately obtain the following result generalizing the main result in [11].

Theorem 2.3. *Let A/B be a right weak H -Galois extension for a semisimple Hopf algebra H . Then $\text{fin.dim}(A) \leq \text{fin.dim}(B)$.*

Now we want to discuss when the finitistic dimension of A is equal to that of B . First we give a duality theorem for weak Hopf-Galois extensions, that is, $A\#H^* \cong \text{End}A_B$. Let H be a finite dimensional weak Hopf algebra. Then a right weak H -comodule algebra A corresponds to a left weak H^* -module algebra A via $f \rightarrow a = a_0 < f, a_1 >$ (see [15]). Thus A and H^* form a weak smash product algebra $A\#H^*$ (see [14]). Now we will give a canonical isomorphism between the weak smash product algebra $A\#H^*$ and the endomorphism algebra $\text{End}A_B$, where the right B -module action on A is the multiplication.

Lemma 2.4. *(The Duality Theorem) Let A/B be a right weak H -Galois extension for a finite dimensional weak Hopf algebra H . Then there is a canonical isomorphism between the algebras $A\#H^*$ and $\text{End}A_B$.*

Proof. It immediately follows from [16, Theorem 2.8], where one only needs to let $M = A$. \square

Proposition 2.5. *Let A/B be a right weak H -Galois extension for a finite dimensional weak Hopf algebra H . If A/B is faithfully flat, then $A\#H^*$ is Morita equivalent to B .*

Proof. By Lemma 2.4, $A\#H^* \cong \text{End}A_B$. Since A/B is right faithfully flat, by the right version of Theorem 2.6 in [7], we obtain that A is a right B -progenerator. Hence $A\#H^*$ is Morita equivalent to B . \square

Now we obtain the main result as follows.

Theorem 2.6. *Let H be a finite dimensional weak Hopf algebra which is semisimple as well as its dual H^* , A/B be a right faithfully flat weak H -Galois extension. Then*

- (1) $\text{fin.dim}(A) = \text{fin.dim}(B)$.
- (2) the finitistic dimension conjecture holds for A if and only if it holds for B .

Proof. (1) First, by Theorem 2.3, $\text{fin.dim}(A) \leq \text{fin.dim}(B)$.

Next, we consider the weak smash product algebra $A\#H^*$. Since $A\#H^*/A$ is a right weak H^* -Galois extension, combining the semisimplicity of H^* , we have $\text{fin.dim}(A\#H^*) \leq \text{fin.dim}(A)$. Since A/B is faithfully flat, by Proposition 2.5, $A\#H^*$ is Morita equivalent to B . It follows that $\text{fin.dim}(B) = \text{fin.dim}(A\#H^*)$. Then

$$\text{fin.dim}(B) = \text{fin.dim}(A\#H^*) \leq \text{fin.dim}(A) \leq \text{fin.dim}(B).$$

Therefore $\text{fin.dim}(A) = \text{fin.dim}(B)$.

- (2) It immediately follows from (1). \square

Note that if H is an ordinary Hopf algebra, then the weak H -Galois extension is just an H -Galois extension. So we get the following corollary.

Corollary 2.7. *Let H be a finite dimensional Hopf algebra which is semisimple as well as its dual H^* , A/B be a right faithfully flat H -Galois extension. Then $\text{fin.dim}(A) = \text{fin.dim}(B)$.*

Let H be a weak Hopf algebra with bijective antipode, $A\#_{\sigma}H$ be a weak crossed product with 2-cocycle σ invertible (see [2] and [17] for detail). Then $A\#_{\sigma}H$ is a right weak H -comodule algebra via $\rho_{A\#_{\sigma}H} = id_A \otimes \Delta_H$ and $(A\#_{\sigma}H)^{coH} = A\#_{\sigma}H_t \cong A$. By [2] and [12], $A\#_{\sigma}H/A$ is a right faithfully flat weak H -Galois extension. If σ is trivial, that is, $\sigma(h, g) = hg \cdot 1_A$, then the weak crossed product $A\#_{\sigma}H$ is just the weak smash product $A\#H$ (see [17]). Thus one can obtain the following corollaries.

Corollary 2.8. ([12]) *Let H be a finite dimensional weak Hopf algebra which is semisimple as well as its dual H^* , and $A\#_{\sigma}H$ be a weak crossed product with σ invertible. Then $\text{fin.dim}(A\#_{\sigma}H) = \text{fin.dim}(A)$.*

Corollary 2.9. *Let H be a finite dimensional weak Hopf algebra which is semisimple as well as its dual H^* , and $A\#H$ be a weak smash product. Then $\text{fin.dim}(A\#H) = \text{fin.dim}(A)$.*

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