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The Short-Time Fourier Transform of Distributions of Exponential Type and Tauberian Theorems for Shift-Asymptotics

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Abstract. We study the short-time Fourier transform on the space $\mathcal{K}_1'(\mathbb{R}^n)$ of distributions of exponential type. We give characterizations of $\mathcal{K}_1'(\mathbb{R}^n)$ and some of its subspaces in terms of modulation spaces. We also obtain various Tauberian theorems for the short-time Fourier transform.

1. Introduction

The short-time Fourier transform (STFT) is a very effective device in the study of function spaces. The investigation of major test function spaces and their duals through time-frequency representations has attracted much attention. For example, the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ and the space of tempered distributions $\mathcal{S}'(\mathbb{R}^n)$ were studied in [10] (cf. [9]). Characterizations of Gelfand-Shilov spaces and ultradistribution spaces by means of the short-time Fourier transform and modulation spaces are also known [11, 17, 26] (cf. [4, 5]).

The purpose of this paper is two folded. On the one hand we study the short-time Fourier transform in the context of the space $\mathcal{K}_1'(\mathbb{R}^n)$ of distributions of exponential type, the dual of the space of exponentially rapidly decreasing smooth functions $\mathcal{K}_1(\mathbb{R}^n)$ (see Section 2 for the definition of all spaces employed in this article). We will obtain various characterizations of $\mathcal{K}_1'(\mathbb{R}^n)$ and related spaces via the short-time Fourier transform. The space $\mathcal{K}_1'(\mathbb{R}^n)$ was introduced by Silva [24] and Hasumi [12] in connection with the so-called space of Silva tempered ultradistributions $\mathcal{U}'(\mathbb{C}^n)$. Let us mention that $\mathcal{K}_1'(\mathbb{R}^n)$ and $\mathcal{U}'(\mathbb{R}^n)$ were also studied by Morimoto through the theory of ultra-hyperfunctions [16] (cf. [18]). We refer to [7, 14, 25, 30] for some applications of the Silva spaces. Our second goal is to present a new kind of Tauberian theorems. In such theorems the exponential asymptotics of functions and distributions can be obtained from those of the short-time Fourier transform. Let us state a sample of our results. In the next statement L stands for a

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locally bounded Karamata slowly varying function [2, 15], namely, a positive function that is asymptotically self-similar in the sense:

$$\lim_{x\to\infty} \frac{L(ax)}{L(x)} = 1, \quad \forall a > 0.$$

Theorem 1.1. *Let f be a positive non-decreasing function on $[0, \infty)$ and let ψ be a positive function such that $\psi'' \in L^1_{loc}(\mathbb{R})$ and $\int_{-\infty}^{\infty}(\psi(t) + |\psi'(t)| + |\psi''(t)|)e^{\beta t+\varepsilon|t|}dt < \infty$, where $\beta \geq 0$ and $\varepsilon > 0$. Suppose that the limits*

$$\lim_{x\to\infty} \frac{e^{2\pi i\xi x}}{e^{\beta x}L(e^x)} \int_0^\infty f(t)\psi(t-x)e^{-2\pi i\xi t}\, dt = J(\xi) \tag{1.1}$$

exist for every $\xi \in \mathbb{R}$, then

$$\lim_{x\to\infty} \frac{f(x)}{e^{\beta x}L(e^x)} = \frac{J(0)}{\int_{-\infty}^{\infty} \psi(t)e^{\beta t}dt}\,. \tag{1.2}$$

Furthermore, if L satisfies $L(xy) \leq AL(x)L(y)$ for all $x, y > 0$ and some constant A, the requirements over ψ can be relaxed to $\int_{-\infty}^{\infty}(\psi(t) + |\psi'(t)| + |\psi''(t)|)L(e^{|t|})e^{\beta t}dt < \infty$.

It turns out that Theorem 1.1 can be deduced from a more general type of Tauberian theorems. In Section 6 we shall give precise descriptions of the S-asymptotic properties [21] of a distribution in terms of the asymptotic behavior of its short-time Fourier transform (S-asymptotics stands for shift-asymptotics). The notion of S-asymptotics measures the asymptotic behavior of the translates of a distribution $T_{-h}f$ with respect the parameter h and it is closely related to the extension of Wiener's Tauberian ideas [15, 29] to the context of Schwartz distributions [20]. We would like to point out that there is an extensive literature about Tauberian theorems for Schwartz distributions, see, e.g., the monographs [21, 28] and references therein. We also mention the article [22], where Tauberian theorems for the short-time Fourier transform were also studied, though with a different approach.

The plan of this article is as follows. In Section 3 we shall present continuity theorems for the STFT and its adjoint on the test function space $\mathcal{K}_1(\mathbb{R}^n)$ and the topological tensor product $\mathcal{K}_1(\mathbb{R}^n)\widehat{\otimes}\mathcal{U}(\mathbb{C}^n)$, where $\mathcal{U}(\mathbb{C}^n)$ is the space of entire rapidly decreasing functions in any horizontal band of $\mathbb{C}^n$. We then use such continuity results to develop a framework for the STFT on $\mathcal{K}_1'(\mathbb{R}^n)$. We also introduce in this paper the space $\mathcal{B}'_\omega(\mathbb{R}^n)$ of ω-bounded distributions and its subspace $\dot{\mathcal{B}}'_\omega(\mathbb{R}^n)$ with respect to an exponentially moderate weight ω; when $\omega = 1$, these spaces coincide with the well-known Schwartz spaces [23, p. 200] of bounded distributions $\mathcal{B}'(\mathbb{R}^n)$ and $\dot{\mathcal{B}}'(\mathbb{R}^n)$, which are of great importance in the study of convolution and growth properties of distributions. Notice that the distribution space $\mathcal{B}'(\mathbb{R}^n)$ also plays an important role in Tauberian theory; see, for instance, Beurling's theorem [6, p. 230] and the distributional Wiener Tauberian theorem from [20]. The spaces $\mathcal{B}'_\omega(\mathbb{R}^n)$ and $\dot{\mathcal{B}}'_\omega(\mathbb{R}^n)$ will be characterized in Section 4 in terms of the short-time Fourier transform and also in terms properties of the set of translates of their elements. Section 5 is devoted to the characterization of $\mathcal{K}_1'(\mathbb{R}^n)$ and related spaces via modulation spaces. The conclusive Section 6 deals with Tauberian theorems, where in particular we give a proof of Theorem 1.1. Our Tauberian hypotheses are actually in terms of membership to suitable modulation spaces, this allows us to reinterpret the S-asymptotics in the weak* topology of modulation spaces.

2. Preliminaries

2.1. Notation

We use the constants in the Fourier transform as

$$\mathcal{F}(\varphi)(\xi) = \widehat{\varphi}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi}\varphi(x)dx. \tag{2.1}$$

The translation and modulation operators are defined by $T_x f(\,\cdot\,) = f(\,\cdot\, - x)$ and $M_\xi f(\,\cdot\,) = e^{2\pi i \xi \cdot\,} f(\,\cdot\,)$, $x, \xi \in \mathbb{R}^n$. The operators $M_\xi T_x$ and $T_x M_\xi$ are called time-frequency shifts and we have $M_\xi T_x = e^{2\pi i x\cdot\xi} T_x M_\xi$. The notation $\langle f, \varphi\rangle$ means dual pairing whereas $(f, \varphi)_{L^2}$ stands for the L^2 inner product. All dual spaces in this article are equipped with the strong dual topology. We denote by $\check{f}$ the function (or distribution) $\check{f}(t) = f(-t)$.

2.2. The STFT

The short-time Fourier transform (STFT) of a function $f \in L^2(\mathbb{R}^n)$ with respect to a window function $\psi \in L^2(\mathbb{R}^n)$ is defined as

$$V_\psi f(x, \xi) = \langle f, \overline{M_\xi T_x \psi}\rangle = \int_{\mathbb{R}^n} f(t)\overline{\psi(t - x)} e^{-2\pi i \xi \cdot t}\, dt,\ \ x, \xi \in \mathbb{R}^n. \tag{2.2}$$

There holds $\left\|V_\psi f\right\|_2 = \left\|f\right\|_2 \left\|\psi\right\|_2$. The adjoint of V_ψ is given by the mapping

$$V^*_\psi F(t) = \iint_{\mathbb{R}^{2n}} F(x, \xi)\psi(t - x) e^{2\pi i \xi \cdot t} dx d\xi,$$

interpreted as an $L^2(\mathbb{R}^n)$-valued weak integral. If $\psi \neq 0$ and $\gamma \in L^2(\mathbb{R}^n)$ is a synthesis window for ψ, namely, $(\gamma, \psi)_{L^2} \neq 0$, then for any $f \in L^2(\mathbb{R}^n)$,

$$f = \frac{1}{(\gamma, \psi)_{L^2}} \iint_{\mathbb{R}^{2n}} V_\psi f(x, \xi) M_\xi T_x \gamma\, d\xi dx. \tag{2.3}$$

Whenever the dual pairing in (2.2) is well-defined, the definition of $V_\psi f$ can be generalized for f in larger classes than $L^2(\mathbb{R}^n)$, for instance: $f \in \mathcal{D}'(\mathbb{R}^n)$ and $\psi \in \mathcal{D}(\mathbb{R}^n)$. In fact, it is enough to have $\psi \in \mathcal{A}(\mathbb{R}^n)$ and $f \in \mathcal{A}'(\mathbb{R}^n)$, where $\mathcal{A}(\mathbb{R}^n)$ is a time-frequency shift invariant topological vector space. Note also that the inversion formula (2.3) holds pointwisely when f is sufficiently regular, for instance, for function in the Schwartz class $\mathcal{S}(\mathbb{R}^n)$. For a complete account on the STFT, we refer to [9].

2.3. Spaces

The Hasumi-Silva [12, 24] test function space $\mathcal{K}_1(\mathbb{R}^n)$ consists of those $\varphi \in C^\infty(\mathbb{R}^n)$ for which all norms

$$\nu_k(\varphi) := \sup_{t\in\mathbb{R}^n,\ |\alpha|\leq k} e^{k|t|}|\varphi^{(\alpha)}(t)|, \quad k \in \mathbb{N}_0,$$

are finite. The elements of $\mathcal{K}_1(\mathbb{R}^n)$ are called exponentially rapidly decreasing smooth functions. It is easy to see that $\mathcal{K}_1(\mathbb{R}^n)$ is an FS-space and therefore Montel and reflexive. The space $\mathcal{K}_1(\mathbb{R}^n)$ is also nuclear [12].

Note that if $\varphi \in \mathcal{K}_1(\mathbb{R}^n)$, then the Fourier transform (2.1) extends to an entire function. In fact, the Fourier transform is a topological isomorphism from $\mathcal{K}_1(\mathbb{R}^n)$ onto $\mathcal{U}(\mathbb{C}^n)$, the space of entire functions which decrease faster than any polynomial in bands. More precisely, a entire function $\phi \in \mathcal{U}(\mathbb{C}^n)$ if and only if

$$\dot{\nu}_k(\phi) := \sup_{z\in\Pi_k}(1 + |z|^2)^{k/2}|\phi(z)| < \infty, \ \ \forall k \in \mathbb{N}_0,$$

where Π_k is the tube $\Pi_k = \mathbb{R}^n + i[-k, k]^n$.

The dual space $\mathcal{K}'_1(\mathbb{R}^n)$ consists of all distributions f of exponential type, i.e., those of the form $f = \sum_{|\alpha|\leq l}(e^{s|\,\cdot\,|} f_\alpha)^{(\alpha)}$, where $f_\alpha \in L^\infty(\mathbb{R}^n)$ [12]. The Fourier transform extends to a topological isomorphism $\mathcal{F} : \mathcal{K}'_1(\mathbb{R}^n) \to \mathcal{U}'(\mathbb{C}^n)$, the latter space is known as the space of Silva tempered ultradistributions [12] (also called the space of tempered ultra-hyperfunctions [16]). The space $\mathcal{U}'(\mathbb{C}^n)$ contains the space of analytic functionals. See also the textbook [14] for more information about these spaces.

We introduce a generalization of the Schwartz space of bounded distributions $\mathcal{B}'(\mathbb{R}^n)$ [23, p. 200]. Let $\omega : \mathbb{R}^n \to (0, \infty)$ be an exponentially moderate weight, namely, ω is measurable and satisfies the estimate

$$\omega(x + y) \leq A\omega(y)e^{a|x|}, \quad x, y \in \mathbb{R}^n, \tag{2.4}$$

for some constants $A > 0$ and $a \geq 0$. For instance, any positive measurable function ω which is submultiplicative, i.e., $\omega(x+y) \leq \omega(x)\omega(y)$, and integrable near the origin must necessarily satisfy (2.4), as follows from the standard results about subadditive functions [1, 13]. Extending the Schwartz space $\mathcal{D}_{L^1}(\mathbb{R}^n)$, we define the Fréchet space $\mathcal{D}_{L^1_\omega}(\mathbb{R}^n) = \{\varphi \in C^\infty(\mathbb{R}^n) : \varphi^{(\alpha)} \in L^1_\omega(\mathbb{R}^n), \forall \alpha \in \mathbb{N}_0^n\}$, provided with the family of norms

$$\|\varphi\|_{1,\omega,k} := \sup_{|\alpha|\leq k} \int_{\mathbb{R}^n} |\varphi^{(\alpha)}(t)|\omega(t)dt, \quad k \in \mathbb{N}_0.$$

Then, $\mathcal{B}'_\omega(\mathbb{R}^n)$ stands for the strong dual of $\mathcal{D}_{L^1_\omega}(\mathbb{R}^n)$, i.e., $\mathcal{B}'_\omega(\mathbb{R}^n) = (\mathcal{D}_{L^1_\omega}(\mathbb{R}^n))'$. Since we have the dense embedding $\mathcal{K}_1(\mathbb{R}^n) \hookrightarrow \mathcal{D}_{L^1_\omega}(\mathbb{R}^n)$, we have $\mathcal{B}'_\omega(\mathbb{R}^n) \subset \mathcal{K}'_1(\mathbb{R}^n)$. We call $\mathcal{B}'_\omega(\mathbb{R}^n)$ the space of ω-bounded distributions. We also define $\dot{\mathcal{B}}'_\omega(\mathbb{R}^n)$ as the closure of $\mathcal{D}(\mathbb{R}^n)$ in $\mathcal{B}'_\omega(\mathbb{R}^n)$.

Next, we shall consider $\mathcal{K}_1(\mathbb{R}^n)\widehat{\otimes}\mathcal{U}(\mathbb{C}^n)$, the topological tensor product space obtained as the completion of $\mathcal{K}_1(\mathbb{R}^n) \otimes \mathcal{U}(\mathbb{C}^n)$ in, say, the π- or the ε- topology [27]. Explicitly, the nuclearity of $\mathcal{K}_1(\mathbb{R}^n)$ implies that $\mathcal{K}_1(\mathbb{R}^n)\widehat{\otimes}\mathcal{U}(\mathbb{C}^n) = \mathcal{K}_1(\mathbb{R}^n)\widehat{\otimes}_\pi\mathcal{U}(\mathbb{C}^n) = \mathcal{K}_1(\mathbb{R}^n)\widehat{\otimes}_\varepsilon\mathcal{U}(\mathbb{C}^n)$. Thus, the topology of $\mathcal{K}_1(\mathbb{R}^n)\widehat{\otimes}\mathcal{U}(\mathbb{C}^n)$ is given by the family of the norms

$$\rho_k(\Phi) := \sup_{(x,z)\in\mathbb{R}^n\times\Pi_k,\ |\alpha|\leq k} e^{k|x|}(1+|z|^2)^{k/2}\left|\frac{\partial^\alpha}{\partial x^\alpha}\Phi(x,z)\right|, \quad k \in \mathbb{N}_0,$$

and we also obtain $(\mathcal{K}_1(\mathbb{R}^n)\widehat{\otimes}\mathcal{U}(\mathbb{C}^n))' = \mathcal{K}'_1(\mathbb{R}^n)\widehat{\otimes}\mathcal{U}'(\mathbb{C}^n)$.

Finally, let m be a weight on $\mathbb{R}^{2n}$, that is, $m : \mathbb{R}^{2n} \to (0,\infty)$ is measurable and locally bounded. Then, if $p, q \in [1,\infty]$, the weighted Banach space $L^{p,q}_m(\mathbb{R}^{2n})$ consists of all measurable functions F such that

$$\|F\|_{L^{p,q}_m} := \left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}|F(x,\xi)|^p m(x,\xi)^p dx\right)^{q/p} d\xi\right)^{1/q} < \infty.$$

(With the obvious modification when $p = \infty$ or $q = \infty$.)

3. Short-Time Fourier Transform of Distributions of Exponential Type

In this section we study the mapping properties of the STFT on the space of distributions of exponential type. Note that the STFT extends to the sesquilinear mapping $(f,\psi) \mapsto V_\psi f$ and its adjoint induces the bilinear mapping $(F,\psi) \mapsto V^*_\psi F$.

We start with the test function space $\mathcal{K}_1(\mathbb{R}^n)$. If $f, \psi \in \mathcal{K}_1(\mathbb{R}^n)$, then we immediately get that (2.2) extends to a holomorphic function in the second variable, namely, $V_\psi f(x,z)$ is entire in $z \in \mathbb{C}^n$. We write in the sequel $z = \xi + i\eta$ with $\xi, \eta \in \mathbb{R}^n$. Observe also that an application of the Cauchy theorem shows that if $\Phi \in \mathcal{K}_1(\mathbb{R}^n)\widehat{\otimes}\mathcal{U}(\mathbb{C}^n)$ and $\psi \in \mathcal{K}_1(\mathbb{R}^n)$, then for arbitrary $\eta \in \mathbb{R}^n$ we may write $V^*_\psi\Phi$ as

$$V^*_\psi\Phi(t) = \iint_{\mathbb{R}^{2n}} \Phi(x,\xi+i\eta)\psi(t-x)e^{2\pi i(\xi+i\eta)\cdot t}dxd\xi. \tag{3.1}$$

Our first proposition deals with the range and continuity properties of V and V^* on test function spaces.

Proposition 3.1. *The following mappings are continuous:*

(*i*) $V : \mathcal{K}_1(\mathbb{R}^n) \times \mathcal{K}_1(\mathbb{R}^n) \to \mathcal{K}_1(\mathbb{R}^n)\widehat{\otimes}\mathcal{U}(\mathbb{C}^n)$.

(*ii*) $V^* : (\mathcal{K}_1(\mathbb{R}^n)\widehat{\otimes}\mathcal{U}(\mathbb{C}^n)) \times \mathcal{K}_1(\mathbb{R}^n) \to \mathcal{K}_1(\mathbb{R}^n)$.

Proof. For part (*i*), let $\varphi, \psi \in \mathcal{K}_1(\mathbb{R}^n)$. Let k be an even integer. If $(x,z) \in \mathbb{R}^n \times \Pi_k$ and $|\alpha| \leq k$, then

$$\begin{aligned}
&e^{k|x|}(1+|z|^2)^{k/2}\left|\frac{\partial^\alpha}{\partial x^\alpha}V_\psi\varphi(x,z)\right| \\
&\le (1+nk^2)^{k/2}e^{k|x|}\left|\int_{\mathbb{R}^n}(1-\Delta_t)^{k/2}(\varphi(t)\overline{\psi^{(\alpha)}(t-x)}e^{2\pi\eta\cdot t})dt\right| \\
&\le \tilde{C}_k\sum_{|\beta_1|+|\beta_2|\le k}e^{k|x|}\int_{\mathbb{R}^n}\left|\varphi^{(\beta_1)}(t)\overline{\psi^{(\alpha+\beta_2)}(t-x)}\right|e^{2\pi k|t|}dt,
\end{aligned}$$

which shows that $\rho_k(V_\psi\varphi)\le C_k\nu_{8k}(\varphi)\nu_k(\psi)$. For (*ii*), if $\Phi\in\mathcal{K}_1(\mathbb{R}^n)\widehat{\otimes}\mathcal{U}(\mathbb{C}^n)$, $\psi\in\mathcal{K}_1(\mathbb{R}^n)$, and $|\alpha|\le k$, we obtain

$$\begin{aligned}
e^{k|t|}\left|\frac{\partial^\alpha}{\partial t^\alpha}V_\psi^*\Phi(t)\right| &\le (2\pi)^{|\alpha|}\sum_{\beta\le\alpha}\binom{\alpha}{\beta}e^{k|t|}\iint_{\mathbb{R}^{2n}}|\xi|^k|\Phi(x,\xi)||\psi^{(\beta)}(t-x)|dxd\xi \\
&\le (4\pi)^{|\alpha|}\nu_k(\psi)\iint_{\mathbb{R}^{2n}}|\xi|^k e^{k|x|}|\Phi(x,\xi)|dxd\xi \\
&\le A_{k,n}\nu_k(\psi)\rho_{k+n+1}(\Phi);
\end{aligned}$$

hence $\rho_k(V_\psi^*\Phi)\le A_{k,n}\nu_k(\psi)\rho_{k+n+1}(\Phi)$. □

Observe that if the window $\psi\in\mathcal{K}_1(\mathbb{R}^n)\setminus\{0\}$ and $\gamma\in\mathcal{K}_1(\mathbb{R}^n)$ is a synthesis window, the reconstruction formula (2.3) reads as:

$$\frac{1}{(\gamma,\psi)_{L^2}}V_\gamma^*V_\psi=\mathrm{id}_{\mathcal{K}_1(\mathbb{R}^n)}. \tag{3.2}$$

We now study the STFT on $\mathcal{K}_1'(\mathbb{R}^n)$. Notice that the modulation operators M_z operate continuously on $\mathcal{K}_1(\mathbb{R}^n)$ even when $z\in\mathbb{C}^n$. Thus, if $f\in\mathcal{K}_1'(\mathbb{R}^n)$ and $\psi\in\mathcal{K}_1(\mathbb{R}^n)$ then $V_\psi f$, defined by the dual pairing in (2.2), also extends in the second variable as an entire function $V_\psi f(x,z)$ in $z\in\mathbb{C}^n$. Furthermore, it is clear that $V_\psi f(x,z)$ is C^∞ in $x\in\mathbb{R}^n$. We begin with a lemma.

Lemma 3.2. *Let* $\psi\in\mathcal{K}_1(\mathbb{R}^n)$*.*

(*a*) *Let* $B'\subset\mathcal{K}_1'(\mathbb{R}^n)$ *be a bounded set. There is* $k=k_{B'}\in\mathbb{N}_0$ *such that*

$$\sup_{f\in B',\,(x,z)\in\mathbb{R}^n\times\Pi_\lambda}e^{-k|x|-2\pi x\cdot\Im m\, z}(1+|z|)^{-k}|V_\psi f(x,z)|<\infty,\quad\forall\lambda\ge 0. \tag{3.3}$$

(*b*) *For every* $f\in\mathcal{K}_1'(\mathbb{R}^n)$ *and* $\Phi\in\mathcal{K}_1(\mathbb{R}^n)\widehat{\otimes}\mathcal{U}(\mathbb{C}^n)$*,*

$$\langle V_\psi f,\Phi\rangle=\left\langle f,\overline{V_\psi^*\overline{\Phi}}\right\rangle. \tag{3.4}$$

Proof. Part (*a*). By the Banach-Steinhaus theorem, B' is equicontinuous, so that there are $C>0$ and $k\in\mathbb{N}_0$ such that $|\langle f,\varphi\rangle|\le C\nu_k(\varphi)$, $\forall f\in B',\forall\varphi\in\mathcal{K}_1(\mathbb{R}^n)$. Hence, for all $f\in B'$ ($z=\xi+i\eta$),

$$\begin{aligned}
\left|V_\psi f(x,z)\right| &\le C\sup_{t\in\mathbb{R}^n,|\alpha|\le k}e^{k|t|}\left|\frac{\partial^\alpha}{\partial t^\alpha}\left(e^{-2\pi iz\cdot t}\overline{\psi(t-x)}\right)\right| \\
&\le (2\pi)^kC(1+|z|^2)^{k/2}\sup_{t\in\mathbb{R}^n,|\alpha|\le k}e^{k|t|+2\pi\eta\cdot t}\sum_{\beta\le\alpha}\binom{\alpha}{\beta}\left|\psi^{(\beta)}(t-x)\right| \\
&\le (4\pi)^kC(1+|z|^2)^{k/2}e^{k|x|+2\pi\eta\cdot x}\nu_{k+1+\lfloor 2\pi|\eta|\rfloor}(\psi),
\end{aligned}$$

where $\lfloor 2\pi|\eta|\rfloor$ stands for the integral part of $2\pi|\eta|$.

Part (*b*). We first remark that the left hand side of (3.4) is well defined because of part (*a*). To show (3.4), notice that the integral in (3.1), with $\eta = 0$, can be approximated by a sequence of convergent Riemann sums in the topology of $\mathcal{K}_1(\mathbb{R}^n)$; this justifies the exchange of integral and dual pairing in

$$\left\langle f(t), \iint_{\mathbb{R}^{2n}} \Phi(x,\xi)e^{-2\pi i\xi\cdot t}\overline{\psi(t-x)}dxd\xi \right\rangle_t = \iint_{\mathbb{R}^{2n}} \Phi(x,\xi)\langle f, \overline{M_\xi T_x \psi}\rangle dxd\xi,$$

which is the same as (3.4). □

In particular, if B' is a singleton, part (a) of Lemma 3.2 gives the growth order of the function $V_\psi f$ on every set $\mathbb{R}^n \times \Pi_\lambda$.

Let us define the adjoint STFT on $\in \mathcal{K}_1'(\mathbb{R}^n)\widehat{\otimes}\mathcal{U}'(\mathbb{C}^n)$.

Definition 3.3. *Let $\psi \in \mathcal{K}_1(\mathbb{R}^n)$. The adjoint STFT V_ψ^* of $F \in \mathcal{K}_1'(\mathbb{R}^n)\widehat{\otimes}\mathcal{U}'(\mathbb{C}^n)$ is the distribution $V_\psi^* F \in \mathcal{K}_1'(\mathbb{R}^n)$ whose action on test functions is given by*

$$\langle V_\psi^* F, \varphi\rangle := \left\langle F, \overline{V_\psi \overline{\varphi}}\right\rangle, \quad \varphi \in \mathcal{K}_1(\mathbb{R}^n). \tag{3.5}$$

The next theorem summarizes our results.

Theorem 3.4. *The two STFT mappings*

(i) $V : \mathcal{K}_1'(\mathbb{R}^n) \times \mathcal{K}_1(\mathbb{R}^n) \to \mathcal{K}_1'(\mathbb{R}^n)\widehat{\otimes}\mathcal{U}'(\mathbb{C}^n)$

(ii) $V^* : (\mathcal{K}_1'(\mathbb{R}^n)\widehat{\otimes}\mathcal{U}'(\mathbb{C}^n)) \times \mathcal{K}_1(\mathbb{R}^n) \to \mathcal{K}_1'(\mathbb{R}^n)$

are hypocontinuous. Let $\psi \in \mathcal{K}_1(\mathbb{R}^n) \setminus \{0\}$ and let $\gamma \in \mathcal{K}_1(\mathbb{R}^n)$ be a synthesis window for it. The following inversion and desingularization formulas hold:

$$\frac{1}{(\gamma,\psi)_{L^2}} V_\gamma^* V_\psi = \mathrm{id}_{\mathcal{K}_1'(\mathbb{R}^n)}, \tag{3.6}$$

and, for all $f \in \mathcal{K}_1'(\mathbb{R}^n)$, $\varphi \in \mathcal{K}_1(\mathbb{R}^n)$, and $\eta \in \mathbb{R}^n$,

$$\langle f, \varphi\rangle = \frac{1}{(\gamma,\psi)_{L^2}} \iint_{\mathbb{R}^{2n}} V_\psi f(x, \xi + i\eta) V_{\overline{\gamma}}\varphi(x, -\xi - i\eta) dxd\xi. \tag{3.7}$$

Proof. That V and V^* are hypocontinuous on these spaces follows from Proposition 3.1 and the formula (3.4) from Lemma 3.2; we leave the details to the reader. By the Cauchy theorem, it is enough to show (3.7) for $\eta = 0$. Using (3.5), (3.4), and (3.2), we have $\langle V_\gamma^* V_\psi f, \varphi\rangle = \langle V_\psi f, \overline{V_\gamma \overline{\varphi}}\rangle = \langle f, \overline{V_\psi^* V_\gamma \overline{\varphi}}\rangle = (\gamma,\psi)_{L^2}\langle f, \varphi\rangle$, namely, (3.6) and (3.7). □

The next corollary gives the converse to part (*a*) of Lemma 3.2 under a weaker inequality than (3.3), namely, a characterization of bounded sets in $\mathcal{K}_1'(\mathbb{R}^n)$ in terms of the STFT.

Corollary 3.5. *Let $B' \subset \mathcal{K}_1'(\mathbb{R}^n)$ and $\psi \in \mathcal{K}_1(\mathbb{R}^n) \setminus \{0\}$. If there are $\eta \in \mathbb{R}^n$ and $k \in \mathbb{N}_0$ such that*

$$\sup_{f\in B', (x,\xi)\in\mathbb{R}^{2n}} e^{-k|x|}(1+|\xi|)^{-k}|V_\psi f(x, \xi + i\eta)| < \infty, \tag{3.8}$$

then the set B' is bounded in $\mathcal{K}_1'(\mathbb{R}^n)$. Conversely, if B' is bounded in $\mathcal{K}_1'(\mathbb{R}^n)$ there is $k \in \mathbb{N}_0$ such that (3.3) holds.

Proof. In view of the Banach-Steinhaus theorem, we only need to show that B' is weakly bounded. Let γ be a synthesis window for ψ and let $\varphi \in \mathcal{K}_1(\mathbb{R}^n)$. Then, by the desingularization formula (3.7), we have

$$\sup_{f\in B'} \left|\langle f, \varphi\rangle\right| \le \frac{C_\eta}{(\gamma,\psi)_{L^2}} \iint_{\mathbb{R}^{2n}} e^{k|x|}(1+|\xi|)^k \left|V_{\overline{\gamma}}\varphi(x, -\xi - i\eta)\right| dxd\xi < \infty,$$

because $V_\gamma \varphi \in \mathcal{K}_1(\mathbb{R}^n)\widehat{\otimes}\mathcal{U}(\mathbb{C}^n)$. The converse was already shown in Lemma 3.2. □

4. Characterizations of $\mathcal{B}'_\omega(\mathbb{R}^n)$ and $\dot{\mathcal{B}}'_\omega(\mathbb{R}^n)$

We now turn our attention to the characterization of the space of ω-bounded distributions $\mathcal{B}'_\omega(\mathbb{R}^n)$ and its subspace $\dot{\mathcal{B}}'_\omega(\mathbb{R}^n)$. Recall that ω stands for an exponentially moderate weight, i.e., a positive and measurable function satisfying (2.4).

Theorem 4.1. *Let $f \in \mathcal{K}'_1(\mathbb{R}^n)$ and $\psi \in \mathcal{K}_1(\mathbb{R}^n) \setminus \{0\}$.*

(*i*) *The following statements are equivalent:*

(*a*) $f \in \mathcal{B}'_\omega(\mathbb{R}^n)$.

(*b*) *The set $\{T_{-h}f/\omega(h) : h \in \mathbb{R}^n\}$ is bounded in $\mathcal{K}'_1(\mathbb{R}^n)$.*

(*c*) *There is $s \in \mathbb{R}$ such that*

$$\sup_{(x,\xi)\in\mathbb{R}^{2n}} (1+|\xi|)^{-s}\frac{|V_\psi f(x,\xi)|}{\omega(x)} < \infty. \tag{4.1}$$

(*ii*) *The next three conditions are equivalent:*

(*a*)′ $f \in \dot{\mathcal{B}}'_\omega(\mathbb{R}^n)$.

(*b*)′ $\lim_{|h|\to\infty} T_{-h}f/\omega(h) = 0$ *in* $\mathcal{K}'_1(\mathbb{R}^n)$.

(*c*)′ *There is $s' \in \mathbb{R}$ such that*

$$\lim_{|(x,\xi)|\to\infty} (1+|\xi|)^{-s'}\frac{|V_\psi f(x,\xi)|}{\omega(x)} = 0. \tag{4.2}$$

Remark 4.2. *Theorem 4.1 remains valid if we replace $\mathcal{K}'_1(\mathbb{R}^n)$ and $\mathcal{K}_1(\mathbb{R}^n)$ by $\mathcal{D}'(\mathbb{R}^n)$ and $\mathcal{D}(\mathbb{R}^n)$ everywhere in the statement. Schwartz has shown in [23, p. 204] the equivalence between* (*a*) *and* (*b*) *for $\omega = 1$ by using a much more complicated method involving a parametrix technique.*

Proof. Part (*i*). (*a*) ⇒ (*b*). Let $f \in \mathcal{B}'_\omega(\mathbb{R}^n)$, since $\mathcal{K}_1(\mathbb{R}^n)$ is barreled, we only need to show that the $f * \varphi$ is bounded by ω for fixed $\varphi \in \mathcal{K}_1(\mathbb{R}^n)$. Let $B := \{\phi \in \mathcal{D}(\mathbb{R}^n) : \int_{\mathbb{R}^n} |\phi(x)|\omega(x)dx \le 1\}$. By the assumption (2.4),

$$\|\check{\varphi} * \phi\|_{1,\omega,k} \le A \max_{|\alpha|\le k} \int_{\mathbb{R}^n} |\varphi^{(\alpha)}(x)|e^{a|x|}dx, \quad \forall k \in \mathbb{N}_0,\ \forall \phi \in B,$$

namely, the set $\check{\varphi} * B$ is bounded in $\mathcal{D}_{L^1_\omega}(\mathbb{R}^n)$. Consequently, $\sup_{\phi\in B} |\langle f*\varphi, \phi\rangle| = \sup_{\phi\in B} |\langle f, \check{\varphi}*\phi\rangle| < \infty$. Since $\mathcal{D}(\mathbb{R}^n)$ is dense in $L^1_\omega(\mathbb{R}^n)$, this implies that $f*\varphi \in (L^1_\omega(\mathbb{R}^n))'$, i.e., $\sup_{h\in\mathbb{R}^n} |(f*\varphi)(h)|/\omega(h) < \infty$, as claimed.

(*b*) ⇒ (*c*). Notice that $(V_\psi T_{-h}f)(x,z) = e^{2\pi i z\cdot h}V_\psi f(x+h,z)$. Fix $\lambda \ge 0$. By Corollary 3.5 (cf. (3.3)), there are $k \in \mathbb{N}_0$ and $C_\lambda > 0$ such that, for all $x, h \in \mathbb{R}^n$ and $z \in \Pi_\lambda$,

$$|e^{2\pi i z\cdot h}V_\psi f(x+h,z)| \le C_\lambda \omega(h)(1+|z|)^k e^{(k+2\pi\lambda)|x|}.$$

Taking $x = 0$ and $\Im m\, z = 0$, one gets (4.1).

(*c*) ⇒ (*a*). Fix a synthesis window $\gamma \in \mathcal{K}_1(\mathbb{R}^n)$. In view of (2.4), one has that if j is any non-negative even integer and $\lambda \ge 0$, then, for all $\varphi \in \mathcal{D}_{L^1_\omega}(\mathbb{R}^n)$,

$$\begin{aligned}
&\sup_{z\in\Pi_\lambda}(1+|z|^2)^{j/2}\int_{\mathbb{R}^n} e^{-2\pi x\cdot \Im m\, z}\omega(x)|V_{\overline{\gamma}}\varphi(x,z)|dx\\
&\le \tilde{C}_j \sum_{|\beta_1|+|\beta_2|\le j} \iint_{\mathbb{R}^{2n}} \omega(x)|\varphi^{(\beta_1)}(t)\gamma^{(\beta_2)}(t-x)|e^{2\pi\lambda|t-x|}dtdx\\
&\le A\tilde{C}_j\|\varphi\|_{1,\omega,j}\max_{|\beta|\le j}\int_{\mathbb{R}^n}|\gamma^{(\beta)}(x)|e^{(2\pi\lambda+a)|x|}dx \le C_{j,\lambda}\|\varphi\|_{1,\omega,j}.
\end{aligned}$$

We may assume that s is an even integer. By (4.1) and the previous estimate, we obtain, for every $\varphi \in \mathcal{K}_1(\mathbb{R}^n)$,

$$|\langle f, \varphi\rangle| \leq \frac{C}{(\psi,\gamma)_{L^2}} \iint_{\mathbb{R}^{2n}} (1+|\xi|)^s \omega(x) \left| V_{\overline{\gamma}}\varphi(x,-\xi)\right| dx d\xi \leq C_s \|\varphi\|_{1,\omega,s+n+1},$$

which yields $f \in \mathcal{B}'_\omega(\mathbb{R}^n)$.

Part (*ii*). Any of the conditions implies that $f \in \mathcal{B}'_\omega(\mathbb{R}^n)$. $(a)' \Rightarrow (b)'$. Fix $\varphi \in \mathcal{K}_1(\mathbb{R}^n)$. Given fixed $\varepsilon > 0$, we must show that $\limsup_{|h|\to\infty} |\langle T_{-h} f, \varphi\rangle|/\omega(h) \leq \varepsilon$. Notice that $\{T_h\varphi/\omega(h) : h \in \mathbb{R}^n\}$ is a bounded set in $\mathcal{D}_{L^1_\omega}(\mathbb{R}^n)$. Since f is in the closure of $\mathcal{D}(\mathbb{R}^n)$ in $\mathcal{B}'_\omega(\mathbb{R}^n)$, there is $\phi \in \mathcal{D}(\mathbb{R}^n)$ such that $|\langle T_{-h}(f-\phi), \varphi| \leq \varepsilon\omega(h)$ for every $h \in \mathbb{R}^n$. Consequently,

$$\limsup_{|h|\to\infty} \frac{|\langle T_{-h} f, \varphi\rangle|}{\omega(h)} \leq \varepsilon + \lim_{|h|\to\infty} \frac{1}{\omega(h)} \left| \int_{\mathbb{R}^n} \varphi(t-h)\phi(t) dt \right| \leq \varepsilon.$$

$(b)' \Rightarrow (c)'$. If ξ remains on a compact of $K \subset \mathbb{R}^n$, then $\{\overline{M_\xi \psi} : \xi \in K\}$ is compact in $\mathcal{K}_1(\mathbb{R}^n)$, thus, by the Banach-Steinhaus theorem,

$$0 = \lim_{|x|\to\infty} \frac{|\langle T_{-x} f, \overline{M_\xi\psi}\rangle|}{\omega(x)} = \lim_{|x|\to\infty} \frac{|V_\psi f(x,\xi)|}{\omega(x)}, \text{ uniformly in } \xi \in K.$$

There is s such that (4.1) holds. Taking into account that the above limit holds for arbitrary K, we obtain that (4.2) is satisfied for any $s' > s$.

$(c)' \Rightarrow (a)'$. We may assume that s' is a non-negative even integer. Consider the weight $\omega_{s'}(x,\xi) = \omega(x)(1+|\xi|)^{s'}$. The limit relation (4.2) implies that $V_\psi f$ is in the closure of $\mathcal{K}_1(\mathbb{R}^n) \otimes \mathcal{S}(\mathbb{R}^n)$ with respect to the norm $\|\ \|_{L^{\infty,\infty}_{1/\omega_{s'}}}$. Since we have the dense embedding $\mathcal{U}(\mathbb{C}^n) \hookrightarrow \mathcal{S}(\mathbb{R}^n)$, there is a sequence $\{\Phi_j\}_{j=1}^\infty \subset \mathcal{K}_1(\mathbb{R}^n)\widehat{\otimes}\mathcal{U}(\mathbb{C}^n)$ such that $\lim_{j\to\infty} \Phi_j = V_\psi f$ in $L^{\infty,\infty}_{1/\omega_{s'}}(\mathbb{R}^{2n})$. Let $\gamma \in \mathcal{K}_1(\mathbb{R}^n)$ be a synthesis window and set $\phi_j = V^*_\gamma \Phi_j \in \mathcal{K}_1(\mathbb{R}^n)$ (cf. Proposition 3.1). By the relations (3.7) and (3.5), we have for any $\varphi \in \mathcal{K}_1(\mathbb{R}^n)$,

$$|\langle f - \phi_j, \varphi\rangle| \leq \frac{C\|\varphi\|_{1,\omega,s+n+1}}{(\gamma,\psi)_{L^2}} \|V_\psi f - \Phi_j\|_{L^{\infty,\infty}_{1/\omega_{s'}}},$$

where C does not depend on j. Thus, $\phi_j \to f$ in $\mathcal{B}'_\omega(\mathbb{R}^n)$, which in turn implies that $f \in \dot{\mathcal{B}}'_\omega(\mathbb{R}^n)$ because $\mathcal{D}(\mathbb{R}^n) \hookrightarrow \mathcal{K}_1(\mathbb{R}^n)$. $\Box$

We immediately get the ensuing result, a corollary of Theorem 4.1.

Corollary 4.3. $\mathcal{K}'_1(\mathbb{R}^n) = \bigcup_\omega \mathcal{B}'_\omega(\mathbb{R}^n) = \bigcup_\omega \dot{\mathcal{B}}'_\omega(\mathbb{R}^n)$. *In particular,* $f \in \mathcal{D}'(\mathbb{R}^n)$ *belongs to* $\mathcal{K}'_1(\mathbb{R}^n)$ *if and only if there is* $s \in \mathbb{R}$ *such that* $\{e^{-s|h|}T_{-h} f : h \in \mathbb{R}^n\}$ *is bounded in* $\mathcal{D}'(\mathbb{R}^n)$.

5. Characterizations through Modulation Spaces

We present here the characterization of the spaces $\mathcal{K}_1(\mathbb{R}^n)$, $\mathcal{K}'_1(\mathbb{R}^n)$, $\mathcal{B}'_\omega(\mathbb{R}^n)$, $\dot{\mathcal{B}}'_\omega(\mathbb{R}^n)$, $\mathcal{U}(\mathbb{C}^n)$, and $\mathcal{U}'(\mathbb{C}^n)$ in terms of modulation spaces.

Let us recall the definition of the modulation spaces. There are several equivalent ways to introduce them [9]. Here we follow the approach from [4, 5] based on Gelfand-Shilov spaces. We are interested in modulation spaces with respect to weights that are exponentially moderate. We denote by $\mathfrak{M}$ the class of all weight functions m on $\mathbb{R}^{2n}$ that satisfy inequalities (for some constants $A > 0$ and $a \geq 0$):

$$\frac{m(x_1+x_2, \xi_1+\xi_2)}{m(x_1,\xi_1)} \leq Ae^{a(|x_2|+|\xi_2|)}, \quad (x_1,\xi_1), (x_2,\xi_2) \in \mathbb{R}^{2n}.$$

Observe that any so-called v-moderate weight [9] belongs to $\mathfrak{M}$. We also consider the Gelfand-Shilov space $\Sigma_1^1(\mathbb{R}^n)$ of Beurling type (sometimes also denoted as $\mathcal{S}^{(1)}(\mathbb{R}^n)$ or $\mathcal{G}(\mathbb{R}^n)$) and its dual $(\Sigma_1^1)'(\mathbb{R}^n)$. The space $\Sigma_1^1(\mathbb{R}^n)$ consists [3] of all entire functions φ such that

$$\sup_{x\in\mathbb{R}^n} |\varphi(x)|e^{\lambda|x|} < \infty \quad \text{and} \quad \sup_{\xi\in\mathbb{R}^n} |\widehat{\varphi}(\xi)|e^{\lambda|\xi|}d\xi < \infty, \quad \forall\lambda > 0.$$

We refer to [19] for topological properties of $\Sigma_1^1(\mathbb{R}^n)$. The dual space $(\Sigma_1^1)'(\mathbb{R}^n)$ is also known as the space of Silva ultradistributions of exponential type [14, 25] or the space of Fourier ultra-hyperfunctions [18]. If $m \in \mathfrak{M}$, $\psi \in \Sigma_1^1(\mathbb{R}^n) \setminus \{0\}$, and $p, q \in [1,\infty]$, the modulation space $M_m^{p,q}(\mathbb{R}^n)$ is defined as the Banach space

$$M_m^{p,q}(\mathbb{R}^n) = \{f \in (\Sigma_1^1)'(\mathbb{R}^n) : \|f\|_{M_m^{p,q}} := \|V_\psi f\|_{L_m^{p,q}} < \infty\}. \tag{5.1}$$

This definition does not depend on the choice of the window ψ, as different windows lead to equivalent norms. If $p = q$, then we write $M_m^p(\mathbb{R}^n)$ instead of $M_m^{p,q}(\mathbb{R}^n)$. The space $M_m^1(\mathbb{R}^n)$ (for $m = 1$) was original introduced by Feichtinger in [8]. We shall also define $\dot{M}_m^\infty(\mathbb{R}^n)$ as the closed subspace of $M_m^\infty(\mathbb{R}^n)$ given by $\dot{M}_m^\infty(\mathbb{R}^n) = \{f \in (\Sigma_1^1)'(\mathbb{R}^n) : \lim_{|(x,\xi)|\to\infty} m(x,\xi)|V_\psi f(x,\xi)| = 0\}$.

We now connect the space of exponential distributions with the modulation spaces. For it, we consider the weight subclass $\mathfrak{M}_1 \subset \mathfrak{M}$ consisting of all weights m such that (for some $s, a \geq 0$ and $A > 0$)

$$\frac{m(x_1 + x_2, \xi_1 + \xi_2)}{m(x_1, \xi_1)} \leq Ae^{a|x_2|}(1 + |\xi_2|)^s, \quad (x_1, \xi_1), (x_2, \xi_2) \in \mathbb{R}^{2n}. \tag{5.2}$$

Let $m \in \mathfrak{M}_1$. By Proposition 3.1, $\mathcal{K}_1(\mathbb{R}^n) \subset M_m^{p,q}(\mathbb{R}^n)$. Since $\Sigma_1^1(\mathbb{R}^n) \hookrightarrow \mathcal{K}_1(\mathbb{R}^n)$, we obtain that $\mathcal{K}_1(\mathbb{R}^n)$ is dense (weakly* dense if $p = \infty$ or $q = \infty$) in $M_m^{p,q}(\mathbb{R}^n)$ and therefore $M_m^{p,q}(\mathbb{R}^n) \subset \mathcal{K}_1'(\mathbb{R}^n)$. It follows from the results of [9] that we may use $\psi \in \mathcal{K}_1(\mathbb{R}^n) \setminus \{0\}$ in (5.1). Also, if $f \in M_m^{p,q}(\mathbb{R}^n)$ and $\psi \in \mathcal{K}_1(\mathbb{R}^n)$ then $V_\psi f$ is an entire function in the second variable (cf. Section 3); the next proposition describes the norm behavior of $V_\psi f(x,z)$ in the complex variable $z \in \mathbb{C}^n$.

Proposition 5.1. *Let $m \in \mathfrak{M}_1$, $p, q \in [1,\infty]$, and $\psi \in \mathcal{K}_1(\mathbb{R}^n) \setminus \{0\}$. If $f \in M_m^{p,q}(\mathbb{R}^n)$, then* $(\forall\lambda \geq 0)$

$$\sup_{|\eta|\leq\lambda} \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |e^{-2\pi x\cdot\eta} V_\psi f(x, \xi + i\eta) m(x,\xi)|^p dx \right)^{q/p} d\xi \right)^{1/q} < C_\lambda \|f\|_{M_m^{p,q}}. \tag{5.3}$$

(With obvious changes if $p = \infty$ or $q = \infty$.)

Proof. Assume that m satisfies (5.2) and set $v(x,\xi) = (1+|\xi|)^s e^{a|x|}$. Notice first that $e^{-2\pi x\cdot\eta} V_\psi f(x, \xi + i\eta) = V_{\psi_\eta} f(x,\xi)$, where $\psi_\eta(t) = e^{2\pi\eta\cdot t}\psi(t)$. As in the proof of [9, Prop. 11.3.2, p. 234],

$$\|V_{\psi_\eta} f\|_{L_m^{p,q}} = \frac{1}{\|\psi\|_{L^2}^2} \|(V_{\psi_\eta} V_\psi^*) V_\psi f\|_{L_m^{p,q}} \leq C \|V_{\psi_\eta}\psi\|_{L_v^1} \|V_\psi f\|_{L_m^{p,q}}.$$

Since $\{\psi_\eta : |\eta| \leq \lambda\}$ is bounded in $\mathcal{K}_1(\mathbb{R}^n)$, we obtain that $\{V_{\psi_\eta}\psi : |\eta| \leq \lambda\}$ is bounded in $\mathcal{K}_1(\mathbb{R}^n)\widehat{\otimes}\mathcal{U}(\mathbb{C}^n)$; hence $\sup_{|\eta|\leq\lambda} \|V_{\psi_\eta}\psi\|_{L_v^1} < \infty$. □

Using the fundamental identity of time-frequency analysis, i.e. [9, p. 40] $V_\psi f(x,\xi) = e^{-2\pi i x\cdot\xi} V_{\widehat{\psi}} \widehat{f}(\xi, -x)$, we can transfer results from $\mathcal{K}_1'(\mathbb{R}^n)$ into $\mathcal{U}'(\mathbb{R}^n)$ by employing the weight class $\mathfrak{M}_2 = \{m \in \mathfrak{M} : \tilde{m}(x,\xi) = m(\xi, x) \in \mathfrak{M}_1\}$. For $s, a \geq 0$, we employ the following special classes of weights (ω satisfies the conditions imposed in Subsection 2.3):

$$v_{s,a}(x,\xi) := e^{a|x|}(1 + |\xi|)^s \quad \text{and} \quad \omega_s(x,\xi) := \omega(x)(1 + |\xi|)^s.$$

Clearly $v_{s,a}, \omega_s \in \mathfrak{M}_1$. Obviously, for every $m \in \mathfrak{M}_1$ there are $s, a \geq 0$ such that $M_{v_{s,a}}^{p,q}(\mathbb{R}^n) \subseteq M_m^{p,q}(\mathbb{R}^n) \subseteq M_{1/v_{s,a}}^{p,q}(\mathbb{R}^n)$.

Proposition 5.2. *Let $p, q \in [1, \infty]$. Then,*

$$\mathcal{K}_1'(\mathbb{R}^n) = \bigcup_{m \in \mathfrak{M}_1} M_m^{p,q}(\mathbb{R}^n), \quad \mathcal{U}'(\mathbb{C}^n) = \bigcup_{m \in \mathfrak{M}_2} M_m^{p,q}(\mathbb{R}^n), \tag{5.4}$$

$$\mathcal{K}_1(\mathbb{R}^n) = \bigcap_{m \in \mathfrak{M}_1} M_m^{p,q}(\mathbb{R}^n), \quad \mathcal{U}(\mathbb{C}^n) = \bigcap_{m \in \mathfrak{M}_2} M_m^{p,q}(\mathbb{R}^n), \tag{5.5}$$

$$\mathcal{B}_\omega'(\mathbb{R}^n) = \bigcup_{s>0} M_{1/\omega_s}^\infty(\mathbb{R}^n), \text{ and } \dot{\mathcal{B}}_\omega'(\mathbb{R}^n) = \bigcup_{s>0} \dot{M}_{1/\omega_s}^\infty(\mathbb{R}^n). \tag{5.6}$$

Proof. The results for $\mathcal{U}(\mathbb{C}^n)$ and $\mathcal{U}'(\mathbb{C}^n)$ follow from those for $\mathcal{K}_1(\mathbb{R}^n)$ and $\mathcal{K}_1'(\mathbb{R}^n)$. The equalities in (5.6) are a reformulation of the equivalences $(a) \Leftrightarrow (c)$ and $(a)' \Leftrightarrow (c)'$ from Theorem 4.1. By (5.2) and [9, Cor. 12.1.10, p. 254], given $m \in \mathfrak{M}_1$, there are $s, a > 0$ such that the embeddings $M_{v_{s+n+1,a+\varepsilon}}^\infty(\mathbb{R}^n) \subseteq M_m^{p,q}(\mathbb{R}^n) \subseteq M_{1/v_{s,a}}^\infty(\mathbb{R}^n)$ hold. Thus, part (a) from Lemma 3.2 gives the equality $\mathcal{K}_1'(\mathbb{R}^n) = \bigcup_{s,a>0} M_{1/v_{s,a}}^\infty(\mathbb{R}^n) = \bigcup_{m \in \mathfrak{M}_1} M_m^{p,q}(\mathbb{R}^n)$. In view of Proposition 3.1, it only remains to show that

$$\bigcap_{m \in \mathfrak{M}_1} M_m^{p,q}(\mathbb{R}^n) = \bigcap_{s,a>0} M_{v_{s,a}}^\infty(\mathbb{R}^n) \subseteq \mathcal{K}_1(\mathbb{R}^n).$$

We show the latter inclusion by proving that if $f \in M_{v_{s,a}}^\infty(\mathbb{R}^n)$ (with $s, a > 0$), then $\widehat{f}$ is holomorphic in the tube $\mathbb{R}^n + i\{\eta \in \mathbb{R}^n : |\eta| < a/(2\pi)\}$ and satisfies

$$\sup_{|\Im m\, z| \leq \lambda} (1 + |z|^2)^{s/2} |\widehat{f}(z)| < \infty, \quad \forall \lambda < \frac{a}{2\pi}. \tag{5.7}$$

In fact, choose a positive window $\psi \in \mathcal{D}(\mathbb{R}^n)$ such that $\sum_{j \in \mathbb{Z}^n} \psi(t - j) = 1$ for all $t \in \mathbb{R}^n$. Since $f = \sum_{j \in \mathbb{Z}^n} f T_j \psi$, we obtain $\widehat{f} = \sum_{j \in \mathbb{Z}^n} V_\psi f(j, \cdot)$, with convergence in $\mathcal{U}'(\mathbb{C}^n)$. In view of Proposition 5.1, each $V_\psi f(j, z)$ is entire in z and satisfies the bounds

$$\sup_{|\Im m\, z| \leq \lambda} |(1 + |z|^2)^{s/2} |V_\psi(j, z)| < C_\lambda e^{-(a - 2\pi\lambda)|j|}.$$

The Weierstrass theorem implies that $\widehat{f}(z) = \sum_{j \in \mathbb{Z}^n} V_\psi f(j, z)$ is holomorphic in the stated tube domain and we also obtain (5.7). Summing up, if $f \in \bigcap_{s,a>0} M_{v_{s,a}}^\infty(\mathbb{R}^n)$, then $\widehat{f} \in \mathcal{U}(\mathbb{C}^n)$, i.e., $f \in \mathcal{K}_1(\mathbb{R}^n)$. □

The following corollary collects what was shown in the proof of Proposition 5.2.

Corollary 5.3. *Let $s, a > 0$. If $f \in M_{v_{s,a}}^\infty(\mathbb{R}^n)$, then $\widehat{f}$ is holomorphic in the tube $\mathbb{R}^n + i\{\eta \in \mathbb{R}^n : |\eta| < a/(2\pi)\}$ and satisfies the bounds (5.7).*

We make a remark concerning Proposition 5.2.

Remark 5.4. *Employing [26, Thrms. 3.2 and 3.4], Proposition 5.2 can be extended for $p, q \in (0, \infty]$.*

6. Tauberian Theorems for S-Asymptotics of Distributions

In this section we characterize the so-called S-asymptotic behavior of distributions in terms of the STFT. We briefly explain this notion; we refer to [21] for a complete treatment of the subject.

Let $f \in \mathcal{K}_1'(\mathbb{R}^n)$. The idea of the S-asymptotics is to study the asymptotic properties of the translates $T_{-h}f$ with respect to a locally bounded and measurable comparison function $c : \mathbb{R}^n \to (0, \infty)$. It is said that f has S-asymptotic behavior with respect to c if there is $g \in \mathcal{D}'(\mathbb{R}^n)$ such that

$$\lim_{|h|\to\infty} \frac{1}{c(h)} T_{-h} f = g \quad \text{in } \mathcal{D}'(\mathbb{R}^n). \tag{6.1}$$

The distribution g is not arbitrary; in fact, one can show [21] that the relation (6.1) forces it to have the form $g(t) = Ce^{\beta\cdot t}$, for some $C \in \mathbb{R}$ and $\beta \in \mathbb{R}^n$. If $C \neq 0$, one can also prove [21] that c must satisfy the asymptotic relation

$$\lim_{|h|\to\infty} \frac{c(t+h)}{c(h)} = e^{\beta\cdot t}, \quad \text{uniformly for } t \text{ in compact subsets of } \mathbb{R}^n. \tag{6.2}$$

From now on, we shall always assume that c satisfies (6.2). A typical example of such a c is any function of the form $c(t) = e^{\beta\cdot t} L(e^{|t|})$, where L is a Karamata slowly varying function [2]. The assumption (6.2) implies [21] that (6.1) actually holds in the space $\mathcal{K}_1'(\mathbb{R}^n)$. We will use the more suggestive notation

$$f(t+h) \sim c(h) g(t) \quad \text{in } \mathcal{K}_1'(\mathbb{R}^n) \text{ as } |h| \to \infty \tag{6.3}$$

for denoting (6.1), which of course means that $(f * \check{\varphi})(h) \sim c(h) \int_{\mathbb{R}^n} \varphi(t) g(t) dt$ as $|h| \to \infty$, for each $\varphi \in \mathcal{D}(\mathbb{R}^n)$ (or, equivalently, $\varphi \in \mathcal{K}_1(\mathbb{R}^n)$). In order to move further, we give an asymptotic representation formula and Potter type estimates [2] for c:

Lemma 6.1. *The locally bounded measurable function c satisfies (6.2) if and only if there is $b \in C^\infty(\mathbb{R}^n)$ such that $\lim_{|x|\to\infty} b^{(\alpha)}(x) = 0$ for every multi-index $|\alpha| > 0$ and*

$$c(x) \sim \exp\left(\beta \cdot x + b(x)\right) \quad \text{as } |x| \to \infty. \tag{6.4}$$

In particular, for each $\varepsilon > 0$ there are constants $a_\varepsilon, A_\varepsilon > 0$ such that

$$a_\varepsilon \exp(\beta \cdot t - \varepsilon|t|) \leq \frac{c(t+h)}{c(h)} \leq A_\varepsilon \exp(\beta \cdot t + \varepsilon|t|), \quad t, h \in \mathbb{R}^n. \tag{6.5}$$

Proof. By considering $e^{-\beta\cdot t}c(t)$, one may assume that $\beta = 0$. Let $\varphi \in \mathcal{D}(\mathbb{R}^n)$ be such that $\int_{\mathbb{R}^n} \varphi(t) dt = 1$. Set $b(x) = \int_{\mathbb{R}^n} \log c(t+x) \varphi(t) dt$. Clearly, $b \in C^\infty(\mathbb{R}^n)$ and the relation (6.2) implies that $b(x) = \log c(x) + o(1)$ and $b^{(\alpha)}(x) = o(1)$ as $|x| \to \infty$, for each multi-index $|\alpha| > 0$. This gives (6.4). Conversely, since c is locally bounded, we may assume that actually $c(x) = e^{\beta\cdot x + b(x)}$, but $|b(t+h) - b(h)| \leq |t| \max_{\xi\in[h,t+h]} |\nabla b(\xi)|$, which gives (6.2). Using the fact that $|\nabla b|$ is bounded, the same argument yields (6.5). □

Observe that Lemma 6.1 also tells us that the space $\mathcal{B}_c'(\mathbb{R}^n)$ is well-defined for c. We can now characterize (6.3) in terms of the STFT. The direct part of the following theorem is an Abelian result, while the converse may be regarded as a Tauberian theorem.

Theorem 6.2. *Let $f \in \mathcal{K}_1'(\mathbb{R}^n)$ and $\psi \in \mathcal{K}_1(\mathbb{R}^n)\backslash\{0\}$. If $f \in \mathcal{K}_1'(\mathbb{R}^n)$ has the S-asymptotic behavior (6.3) then, for every $\lambda \geq 0$,*

$$\lim_{|h|\to\infty} e^{2\pi i z\cdot h} \frac{V_\psi f(x+h, z)}{c(h)} = V_\psi g(x, z), \tag{6.6}$$

uniformly for $z \in \Pi_\lambda$ and x in compact subsets of $\mathbb{R}^n$.

Conversely, suppose that the limits

$$\lim_{|x|\to\infty} e^{2\pi i \xi\cdot x} \frac{V_\psi f(x, \xi)}{c(x)} = J(\xi) \in \mathbb{C} \tag{6.7}$$

exist for almost every $\xi \in \mathbb{R}^n$. If there is $s \in \mathbb{R}$ such that

$$\sup_{(x,\xi)\in\mathbb{R}^{2n}} (1+|\xi|)^{-s} \frac{|V_\psi f(x,\xi)|}{c(x)} < \infty, \tag{6.8}$$

then f has the S-asymptotic behavior (6.3) with $g(t) = Ce^{\beta\cdot t}$, where the constant is completely determined by the equation $J(\xi) = C\overline{\widehat{\psi}(-\xi + i\beta/(2\pi))}$.

Remark 6.3. *Assume (6.8). Consider a weight of the form $m_\varepsilon(x,\xi) = e^{\beta\cdot x+\varepsilon|x|}(1+|\xi|)^s$ with $\varepsilon > 0$. It will be shown below that the asymptotics (6.3) holds in the weak* topology of $M^\infty_{1/m_\varepsilon}(\mathbb{R}^n)$, i.e., $(f * \check{\varphi})(h) \sim c(h)\langle g, \varphi\rangle$ as $|h| \to \infty$ for every φ in the modulation space $M^1_{m_\varepsilon}(\mathbb{R}^n)$. Furthermore, one may use in (6.7) and (6.8) a window $\psi \in M^1_{m_\varepsilon}(\mathbb{R}^n)\backslash\{0\}$.*

Proof. Fix $\lambda \geq 0$ and a compact $K \subset \mathbb{R}^n$. Note that the set

$$\{\overline{M_z T_x \psi} : (x,z) \in K \times \Pi_\lambda\}$$

is compact in $\mathcal{K}_1(\mathbb{R}^n)$. By the Banach-Steinhaus theorem,

$$\lim_{|h|\to\infty} e^{2\pi i z h} \frac{V_\psi f(x+h,z)}{c(h)} = \lim_{|h|\to\infty} \left\langle \frac{T_{-h} f}{c(h)}, \overline{M_z T_x \psi} \right\rangle = \left\langle g, \overline{M_z T_x \psi} \right\rangle,$$

uniformly with respect to $(x,z) \in K \times \Pi_\lambda$, as asserted in (6.6).

Conversely, assume (6.7) and (6.8). Let $H = \{\xi \in \mathbb{R}^n : \text{(6.7) holds}\}$. In view of Theorem 4.1, we have that $f \in \mathcal{B}'_c(\mathbb{R}^n)$ or, equivalently, $\{T_{-h} f / c(h) : h \in \mathbb{R}^n\}$ is bounded in $\mathcal{K}'_1(\mathbb{R}^n)$. By the Banach-Steinhaus theorem and the Montel property of $\mathcal{K}'_1(\mathbb{R}^n)$, $T_{-h} f/c(h)$ converges strongly to a distribution g in $\mathcal{K}'_1(\mathbb{R}^n)$ if and only if $\lim_{|h|\to\infty} \langle T_{-h} f, \varphi\rangle / c(h)$ exists for φ in a dense subspace of $\mathcal{K}_1(\mathbb{R}^n)$. Let D be the linear span of $\{\overline{M_\xi T_x \psi} : (x,\xi) \in \mathbb{R}^n \times H\}$. By the desingularization formula (3.7) and the Hahn-Banach theorem, we have that D is dense in $\mathcal{K}_1(\mathbb{R}^n)$. Thus, it suffices to verify that $\lim_{|h|\to\infty} \langle T_{-h} f, \overline{M_\xi T_x \psi}\rangle / c(h)$ exists for each $(x,\xi) \in \mathbb{R}^n \times H$. But in this case (6.2) and (6.7) yield

$$\begin{aligned}
\lim_{|h|\to\infty} \frac{\langle T_{-h} f, \overline{M_\xi T_x \psi}\rangle}{c(h)} &= \lim_{|h|\to\infty} e^{2\pi i \xi\cdot h} \frac{V_\psi f(x+h,\xi)}{c(h)} \\
&= e^{(\beta - 2\pi i \xi)\cdot x} \lim_{|h|\to\infty} e^{2\pi i \xi\cdot(x+h)} \frac{V_\psi f(x+h,\xi)}{c(h+x)} \\
&= e^{(\beta - 2\pi i \xi)\cdot x} J(\xi),
\end{aligned}$$

as required. We already know that $g(t) = Ce^{\beta\cdot t}$. Comparison between (6.6) and (6.7) leads to $J(\xi) = V_\psi g(0,\xi) = C\int_{\mathbb{R}^n} \overline{\psi(t)} e^{\beta\cdot t - 2\pi i \xi\cdot t} dt$. To show the assertion from Remark 6.3, note first that, by using (6.5), one readily verifies that

$$\sup_{h\in\mathbb{R}^n} \frac{\|T_{-h} f\|_{M^\infty_{1/m_\varepsilon}}}{c(h)} < \infty.$$

Since we have the dense embedding $\mathcal{K}_1(\mathbb{R}^n) \hookrightarrow M^1_{m_\varepsilon}(\mathbb{R}^n)$, we also have that D is dense in $M^1_{m_\varepsilon}(\mathbb{R}^n)$ and the assertion follows at once. The fact that one may use a window $\psi \in M^1_{m_\varepsilon}(\mathbb{R}^n)\backslash\{0\}$ in (6.7) and (6.8) follows in a similar fashion because in this case the desingularization formula (3.7) still holds. □

Let us make two addenda to Theorem 6.2. The ensuing corollary improves Remark 6.3, provided that c satisfies the extended submultiplicative condition (for some $A > 0$):

$$c(t+h) \leq A c(t) c(h). \tag{6.9}$$

Corollary 6.4. *Assume that c satisfies (6.9) and set $c_s(x,\xi) = c(x)(1+|\xi|)^s$, $s \in \mathbb{R}$. If $f \in M^{\infty}_{1/c_s}(\mathbb{R}^n)$ and there is $\psi \in M^1_{c_s}(\mathbb{R}^n) \setminus \{0\}$ such that the limits (6.7) exist for almost every $\xi \in \mathbb{R}^n$, then, for some g, the S-asymptotic behavior (6.3) holds weakly* in $M^{\infty}_{1/c_s}(\mathbb{R}^n)$, that is, $(f * \check{\varphi})(h) \sim c(h)\langle g, \varphi\rangle$ as $|h| \to \infty$ for every $\varphi \in M^1_{c_s}(\mathbb{R}^n)$.*

Proof. We retain the notation from the proof of Theorem 6.2. The assumption $f \in M^{\infty}_{1/c_s}(\mathbb{R}^n)$ of course tells us that (6.8) holds. Employing the hypothesis (6.9), one readily sees that $\sup_{h\in\mathbb{R}^n} \|T_{-h}f\|_{M^{\infty}_{1/c_s}}/c(h) < \infty$. A similar argument to the one used in the proof of Theorem 6.2 yields that the set D associated to ψ is dense in $M^1_{c_s}(\mathbb{R}^n)$, which as above yields the result. □

In dimension $n = 1$, the next theorem actually obtains the ordinary asymptotic behavior of f in case it is a regular distribution on $(0,\infty)$ satisfying an additional Tauberian condition. We fix m_ε as in Remark 6.3 and c_s as in Corollary 6.4.

Theorem 6.5. *Let $f \in M^{\infty}_{1/c_s}(\mathbb{R})$. Suppose that*

$$\lim_{x\to\infty} e^{2\pi i \xi\cdot x}\frac{V_\psi f(x,\xi)}{c(x)} = J(\xi) \in \mathbb{C}, \tag{6.10}$$

for almost every $\xi \in \mathbb{R}$, where $\psi \in M^1_{m_\varepsilon}(\mathbb{R}) \setminus \{0\}$ (resp. $\psi \in M^1_{c_s}(\mathbb{R}) \setminus \{0\}$ if c satisfies (6.9)). If there is $\alpha \geq 0$ such that $e^{\alpha t} f(t)$ is a positive non-decreasing function on the interval $(0,\infty)$, then

$$\lim_{t\to\infty} \frac{f(t)}{c(t)} = C, \tag{6.11}$$

where C is the constant from Theorem 6.2.

Proof. Using (6.10), the same method from Theorem 6.2 applies to show that $f(t+h) \sim Cg(t)$ in $\mathcal{K}_1'(\mathbb{R})$ as $h \to \infty$, where $g(t) = Ce^{\beta t}$. We may assume that $\alpha \geq -\beta$. Set $\tilde{f}(t) = e^{\alpha t} f(t)$, $b(t) = e^{\alpha t}c(t)$, and $r = \alpha + \beta \geq 0$. It is enough to show that $\tilde{f}(t) \sim Cb(t)$ as $t \to \infty$, whence (6.11) would follow. By (6.3), we have that

$$\tilde{f}(t+h) \sim b(h)Ce^{rt} \quad \text{as } h \to \infty \text{ in } \mathcal{K}_1'(\mathbb{R}),$$

i.e.,

$$\langle \tilde{f}(t+h), \varphi(t)\rangle \sim Cb(h)\int_{-\infty}^{\infty} e^{rt}\varphi(t)dt, \quad \forall \varphi \in \mathcal{K}_1(\mathbb{R}^n). \tag{6.12}$$

Let $\varepsilon > 0$ be arbitrary. Choose a non-negative test function $\varphi \in \mathcal{D}(\mathbb{R})$ such that $\operatorname{supp}\varphi \subseteq (0,\varepsilon)$ and $\int_0^\varepsilon \varphi(t)dt = 1$. Using the fact that $\tilde{f}$ is non-decreasing on $(0,\infty)$ and (6.12), we obtain

$$\begin{aligned}\limsup_{h\to\infty} \frac{\tilde{f}(h)}{b(h)} &= \limsup_{h\to\infty} \frac{\tilde{f}(h)}{b(h)}\int_0^\varepsilon \varphi(t)dt \leq \lim_{h\to\infty}\frac{1}{b(h)}\int_0^\varepsilon \tilde{f}(t+h)\varphi(t)dt \\ &= \lim_{h\to\infty} \frac{\langle \tilde{f}(t+h),\varphi(t)\rangle}{b(h)} = C\int_0^\varepsilon e^{rt}\varphi(t)\mathrm{d}t \leq Ce^{r\varepsilon},\end{aligned}$$

taking $\varepsilon \to 0^+$, we have shown that $\limsup_{h\to\infty} \tilde{f}(h)/b(h) \leq C$. Similarly, choosing in (6.12) a non-negative φ such that $\operatorname{supp}\varphi \subseteq (-\varepsilon, 0)$ and $\int_{-\varepsilon}^0 \varphi(t)dt = 1$, one obtains $\liminf_{h\to\infty} \tilde{f}(h)/b(h) \geq C$. This shows that $\tilde{f}(t) \sim Cb(t)$ as $t \to \infty$, as claimed. □

We conclude this article with a proof of Theorem 1.1.

Proof. [Proof of Theorem 1.1] Set $c(t) = e^{\beta t}L(e^{|t|})$ and, as before (with $s = 0$), $c_0(x,\xi) = c(x)$ and $m_\varepsilon(x,\xi) = e^{\beta x+\varepsilon|x|}$. Note that (1.2) is the same as (6.11). Let us first verify that $\psi \in M^1_{m_\varepsilon}(\mathbb{R})$. In fact, if we take another window $\gamma \in \mathcal{K}_1(\mathbb{R})$, we have

$$\iint_{\mathbb{R}^2} |V_\gamma\psi(x,\xi)|e^{\beta x+\varepsilon|x|}dxd\xi = \iint_{\mathbb{R}^2}(1+|\xi|^3)|V_\gamma\psi(x,\xi)|e^{\beta x+\varepsilon|x|}dx\frac{d\xi}{1+|\xi|^3}$$
$$\leq \tilde{C}\left(\iint_{\mathbb{R}^2}\psi(t-x)|\gamma(t)|e^{\beta x+\varepsilon|x|}dtdx + \sum_{j=0}^{3}\iint_{\mathbb{R}^2}|\psi^{(j)}(t-x)\gamma^{(3-j)}(t)|e^{\beta x+\varepsilon|x|}dtdx\right),$$

which is finite (a similar argument shows that $\psi \in M^1_{c_0}(\mathbb{R})$ if $\int_{-\infty}^{\infty}(\psi(t)+|\psi'(t)|+|\psi''(t)|)L(e^{|t|})e^{\beta t}dt < \infty$). In view of Theorem 6.5, it is enough to establish $f \in M^\infty_{1/c_0}(\mathbb{R})$. Let us first show the crude bound $f(t) = O(c(t))$. Set $A_1 = \int_0^\infty \psi(t)dt < \infty$. Since f is non-decreasing, we have

$$f(x) \leq \frac{1}{A_1}\int_0^\infty f(t+x)\psi(t)dt \leq \frac{1}{A_1}\int_0^\infty f(t)\psi(t-x)dt \leq A_2c(x),$$

because of (1.1) with $\xi = 0$. Thus

$$|V_\psi f(x,\xi)| \leq A_2\int_0^\infty c(t)\psi(t-x)dt \leq c(x)\tilde{A}_\varepsilon\int_{-\infty}^{\infty}e^{\beta t+\varepsilon|t|}\psi(t)dt < A_3c(x), \quad \forall(x,\xi)\in\mathbb{R}^2$$

(likewise in the other case using $L(xy) \leq AL(x)L(y)$), which completes the proof. □

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